Moment Infinitely Divisible Weighted Shifts

(joint work with Chafiq Benhida and George Exner)

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Some Key Words and Phrases

- subnormal
- weighted shift
- Berger measure
- Aluthge transform
- Schur products
- other transformations
- moment problem
- completely monotone functions
- completely alternating functions
- Agler shifts
- infinitely divisible weighted shifts
- Laplace transform methods

Hyponormality and Subnormality

$\mathcal{L}(\mathcal{H})$: algebra of operators on a Hilbert space $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$ (We say that N is a lifting of T, or an extension of T.
- hyponormal if $T^*T \geq TT^*$
- For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.

- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

normal $\Rightarrow$ subnormal $\Rightarrow$ k-hyponormal $\Rightarrow$ hyponormal

(Bram-Halmos)

$$\begin{aligned} T \text{ subnormal} \quad &\Leftrightarrow \quad T \text{ is } k\text{-hyponormal for all } k \geq 1 \\ &\Leftrightarrow \quad (T, T^2, ..., T^k) \text{ is hyponormal for all } k \geq 1. \end{aligned}$$

n-CONTRACTIVITY

For $n = 1, 2, \ldots$, T is said to be *n-contractive* if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \geq 0.$$

Agler-Embry Characterization of Subnormality: Assume that T is a contraction. Then T is subnormal if and only if T is n-contractive for all $n \geq 1$.

Unilateral Weighted Shifts

- Given a bounded sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, \ldots$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0),$$

  where $\{e_k\}$ is the canonical ONB of $\ell^2(Z_+)$.

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift.

- In general,

$$W_\alpha = U_+ D_\alpha$$

  (polar decomposition); here $D_\alpha := diag\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$

- $\|W_\alpha\| = \sup_k \alpha_k$

- $$W_\alpha^n e_k = \alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1} e_{k+n},$$

  so

  $$W_\alpha^n \cong \bigoplus_{i=0}^{n-1} W_{\beta^{(i)}},$$

- The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)

Berger Measures

- (Berger; Gellar-Wallen) W_α is subnormal iff there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \ \ (\text{all } k \geq 0).$$

  ξ is the Berger measure of W_α.

- For $0 < a < 1$ we let $S_a := shift\{a, 1, 1, ...\}$.
- The Berger measure of U_+ is δ_1.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

k-hyponormal and n-contractive U.W.S.

(RC, 1990) W_α is k-hyponormal iff the following Hankel moment matrices are positive for $m = 0, 1, 2, \ldots$:

$$\begin{pmatrix} \gamma_m & \gamma_{m+1} & \gamma_{m+2} & \cdots & \gamma_{m+k} \\ \gamma_{m+1} & \gamma_{m+2} & & \cdots & \gamma_{m+k+1} \\ \gamma_{m+2} & \cdots & & \cdots & \gamma_{m+k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_{m+k} & \gamma_{m+k+1} & & \cdots & \gamma_{m+2k} \end{pmatrix} \geq 0.$$

(An operator matrix condition is replaced by a scalar matrix condition.)

(G. Exner, 2006) W_α is n-contractive iff

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \gamma_{m+j} \geq 0, \qquad m = 0, 1, \ldots .$$

Aluthge Transform

Let T be a Hilbert space operator, let $P := |T|$ be its positive part, and let $T = VP$ denote the canonical polar decomposition of T, with V a partial isometry and $\ker V = \ker T = \ker P$.

We define the Aluthge transform of T as

$$AT(T) := \sqrt{P}V\sqrt{P}.$$

The iterates of AT are

$$AT^{n+1} := AT(AT^{n}(T)) \quad (n \geq 1).$$

The Aluthge transform has been extensive studied, in terms of algebraic, structural and spectral properties.

For instance,

- (i) $T = AT(T) \Leftrightarrow T$ is quasinormal;
- (ii) (Aluthge, 1990) If $0 < p < \frac{1}{2}$ and T is p-hyponormal, then $AT(T)$ is $(p + \frac{1}{2})$-hyponormal;
- (iii) (Jung, Ko & Pearcy, 2000) If $AT(T)$ has a n.i.s., then T has a n.i.s.
- (iv) (Kim-Ko, 2005; Kimura, 2004) T has property (β) if and only if $AT(T)$ has property (β); and
- (v) (Ando, 2005) $\left\|(T-\lambda)^{-1}\right\| \geq \left\|(AT(T)-\lambda)^{-1}\right\|$ $(\lambda \notin \sigma(T))$.

On the other hand,

G. Exner (IWOTA 2006 Lecture): subnormality is not preserved under AT. Concretely, Exner proved that the Aluthge transform of the weighted shift in the following example is not subnormal.

Example

(RC, Y. Poon and J. Yoon, 2005) Let

$$\alpha \equiv \alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n+\frac{1}{2}}{2^n+1}}, & \text{if } n \geq 1 \end{cases},$$

Then W_α is subnormal, with 3-atomic Berger measure

$$\mu = \frac{1}{3}(\delta_0 + \delta_{1/2} + \delta_1).$$

Note that the Aluthge transform of a weighted shift is again a weighted shift.

Concretely, the weights of $AT(W_\alpha)$ are

$$\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_2\alpha_3}, \sqrt{\alpha_3\alpha_4}, \cdots .$$

Define

$$W_{\sqrt{\alpha}} := \text{shift}\,(\sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \cdots).$$

Then $AT(W_\alpha)$ is the Schur product of $W_{\sqrt{\alpha}}$ and its restriction to the subspace $\bigvee\{e_1, e_2, \cdots\}$. Thus, a sufficient condition for the subnormality of $AT(W_\alpha)$ is the subnormality of $W_{\sqrt{\alpha}}$.

To compare W_α and $AT(W_\alpha)$, we assume that $W_{\sqrt{\alpha}}$ is subnormal.

This guarantees that both W_α and $AT(W_\alpha)$ are subnormal. Denote the Berger measures of W_α and $AT(W_\alpha)$ by μ and $AT(\mu)$, resp.

Problem

How is $AT(\mu)$ related to μ? In Exner's example, is the fact that card supp $\mu = 3$ *of significance, since it is an odd number, and the Aluthge transform of W_α is related to W_α^2?*

Other Transformations

Given W_α and $0 < p < 1$, consider the weighted shift W_{α^p} with weights

$$\alpha_i^{(p)} := (\alpha_i)^p \ (i \geq 0).$$

If W_α is subnormal, with Berger measure μ, under what conditions is W_{α^p} subnormal?

If it is, how is the Berger measure of W_{α^p} related to μ ?

When is W_{α^p} subnormal for all $0 < p < 1$?

Agler Shifts

For $j = 2, 3, \ldots$, the j-th Agler shift A_j is given by

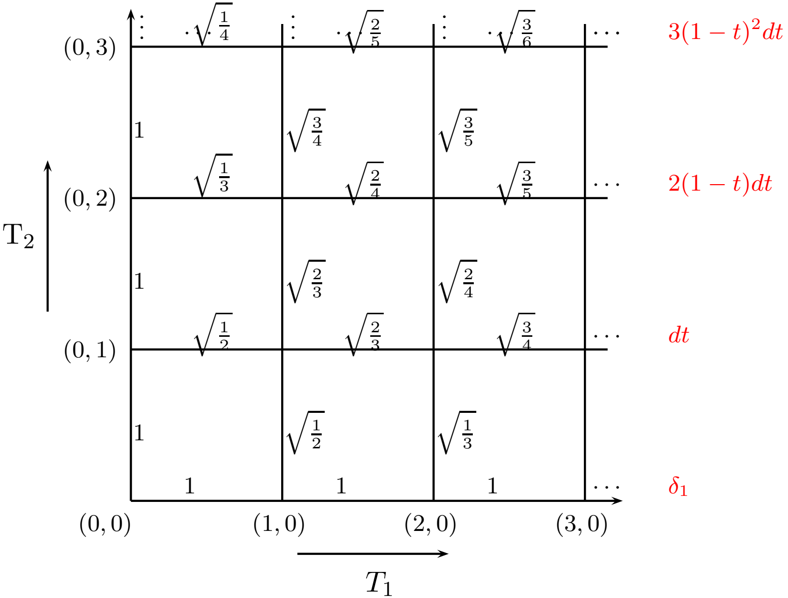
$$\alpha^j := \sqrt{\frac{1}{j}}, \sqrt{\frac{2}{j+1}}, \sqrt{\frac{3}{j+2}}, \ldots .$$

It is well known that A_j is subnormal, with Berger measure

$$d\mu^j(t) = (j-1)(1-t)^{j-2} dt.$$

Clearly, A_2 is the Bergman shift, and the remaining Agler shifts are the upper row shifts of the Drury-Arveson 2-variable weighted shift, which incidentally is a spherical complete hyperexpansion.

Weight Diagram of the Drury-Arveson Shift



Powers and Aluthge transforms of the Agler shifts are again subnormal, via an approach based on completely monotone functions.

Definition

A function $f : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ is *completely monotone* if its derivatives alternate in sign: $f^{(2j)} \geq 0$ for $j \geq 1$, and $f^{(2j+1)} \leq 0$ for $j \geq 0$.

Proposition

f is completely monotone if and only if $f = \mathcal{L}(\mu)$ for some positive measure μ, where $\mathcal{L}$ denotes the Laplace transform.

Fact: If a completely monotone function f interpolates the moments of a shift, i.e., $f(n) = \gamma_n$ for all $n \geq 0$, then the shift is subnormal.

Theorem

(Exner, 2009) For $j = 2, 3, \ldots$, let A_j be the j-th Agler shift. Any p-th power transformation ($p > 0$) of A_j is subnormal, as is any m-th iterated Aluthge transform of A_j.

The proof (which uses monotone function theory) offers no information about the Berger measure of the resulting shift.

Using the difference operator we can rephrase the Agler-Embry conditions for shifts, as follows.

$$\nabla^0 \varphi := \varphi, \qquad (\nabla \varphi)(n) := \varphi(n) - \varphi(n+1),$$

$$\nabla^k \varphi := \nabla \nabla^{k-1} \varphi,$$

for all $k \geq 1$.

For example,

$$(\nabla\varphi)(n) = \varphi(n) - \varphi(n+1),$$

$$(\nabla^2\varphi)(n) = \varphi(n) - 2\varphi(n+1) + \varphi(n+2),$$

$$(\nabla^3\varphi)(n) = \varphi(n) - 3\varphi(n+1) + 3\varphi(n+2) - \varphi(n+3),$$

and so on.

A sequence φ is said to be *completely monotone* if $(\nabla^k\varphi)(n) \geq 0$ for all $k, n \geq 0$.

It follows that the Agler-Embry conditions for subnormality of W_α entail that the moment sequence (γ_n) is completely monotone.

W_α is subn. $\Longleftrightarrow$ (γ_n) is compl. monot. $\Longleftrightarrow$ $(\nabla^k \gamma)(n) \geq 0 \;$ (all $k, n \geq 0$).

A sequence ψ is said to be *completely alternating* if

$$(\nabla^k \psi)(n) \leq 0 \quad (\text{for all } k \geq 1, n \geq 0);$$

equivalently, if the sequence $-\nabla\psi$ is completely monotone.

Similarly, given an integer $k \geq 1$, a sequence ψ is said to be *k-alternating* if $(\nabla^k \psi)(n) \leq 0$ for all $n \geq 0$.

Proposition

ψ is completely alternating if and only if $\varphi_t := e^{-t\psi}$ is completely monotone for all $t > 0$.

Theorem

ψ is completely alternating if and only if it has an associated Lévy-Khintchine measure; that is,

$$\psi(n) = a + bn + \int_0^1 (1 - t^{n+1}) \, d\mu(t),$$

where $\mu \geq 0$. In the case of the Agler shifts A_j, the sequence of **weights** *squared is*

$$\int_0^1 (1 - t^{n+1})(j-1)t^{j-2} \, dt.$$

$\mathcal{CA}$: completely alternating sequences

$\mathcal{CA}_+$: completely alternating sequences with all positive terms

$\mathrm{Log}\,\mathcal{CA}$: positive-term sequences (x_n) such that $(\ln x_n) \in \mathcal{CA}$.

(Studied by A. Athavale, V.M. Sholapurkar, A. Ranjekar, etc.)

$\mathcal{CA}$ and $\mathcal{CA}_+$ are closed under sums, the addition of constants, and (positive) scalar multiples.

$\mathrm{Log}\,\mathcal{CA}$ is closed under (positive) scalar multiplication, products, and positive powers.

$\varphi \in \mathcal{CA}_+ \Rightarrow \ln(1+\varphi) \in \mathcal{CA}_+$

(Athavale-Ranjekar; Berg-Christensen-Ressel) $\varphi \in \mathcal{CA}_+ \Rightarrow \varphi^p \in \mathcal{CA}_+$ (for all $0 < p < 1$).

Since $(\frac{n+1}{n+2}) \in \mathcal{CA}_+$ we have $(\sqrt{\frac{n+1}{n+2}}) \in \mathcal{CA}_+$.

Proposition

(Benhida, RC and Exner, 2015) $\mathcal{CA}_+ \subseteq \mathrm{Log}\,\mathcal{CA}$.

Proposition

(Benhida, RC and Exner, 2015) $\mathrm{Log}\,\mathcal{CA} \not\subseteq \mathcal{CA}$.

Proof.

The sequence $\left((\frac{n+1}{n+2})^3\right)$ is log completely alternating but not completely alternating. For, A_2 (with weight sequence $\left(\sqrt{\frac{n+1}{n+2}}\right)$) has weights squared completely alternating. Therefore the weights squared are log completely alternating, so clearly any power of them is log completely alternating. On the other hand, one checks, using *Mathematica*, that $\left((\frac{n+1}{n+2})^3\right)$ is not completely alternating. $\square$

A Link Between Moments and Weights

Recall that $\gamma_{n+1} = \gamma_n \alpha_n^2$ (all $n \geq 0$). It follows that

$$\operatorname{Log} \gamma_{n+1} - \operatorname{Log} \gamma_n = \operatorname{Log} \alpha_n^2$$

or, equivalently,

$$(\nabla \operatorname{Log} \gamma)(n) = -(\operatorname{Log} \alpha^2)(n) = (\nabla^0 \operatorname{Log} \alpha^2)(n)$$

for all $n \geq 0$. Using mathematical induction one can then prove that

$$(\nabla^{k+1} \operatorname{Log} \gamma)(n) = -(\nabla^k \operatorname{Log} \alpha^2)(n)$$

for all $n \geq 0$.

It follows that if $-\operatorname{Log} \gamma$ is completely alternating, then $\operatorname{Log} \alpha^2$ is completely alternating.

Moment Infinitely Divisible (MID) Weighted Shifts

Definition

W_α is MID if the shift with weight sequence (α_n^t) corresponds to a subnormal weighted shift, for every $t > 0$.

Theorem

(Benhida, RC and Exner, 2015) Let W_α be a weighted shift with (positive) weight sequence $\alpha = (\alpha_n)$. Then W_α is moment infinitely divisible if and only if

(i) $\alpha_n \leq 1$, $n = 0, 1, 2, \ldots$, *and*

(ii) (α_n^2) *is log completely alternating; equivalently,* $(\mathrm{Log}\, \alpha_n^2) \in \mathcal{CA}$.

Sketch of Proof. Recall that ψ is completely alternating if and only if

$$\varphi_t := e^{-t\psi} \qquad (*)$$

is completely monotone for all $t > 0$. Also, W_α is MID if and only if γ^t is completely monotone for all $t > 0$. Since $\gamma^t = e^{-t(-\mathrm{Log}\,\gamma)}$, we can let $\psi := -\mathrm{Log}\,\gamma$ and $\varphi_t \equiv \gamma^t$ in (*), and conclude that W_α is MID if and only if $-\mathrm{Log}\,\gamma$ is completely alternating.

Assume now that W_α is MID. Then $-\mathrm{Log}\,\gamma$ is completely alternating or, equivalently, $\nabla \mathrm{Log}\,\gamma$ is completely monotone. Then $(\nabla^{k+1}\mathrm{Log}\,\gamma) \geq 0$ for all $k \geq 0$, and therefore

$$\nabla^k(-\mathrm{Log}\,\alpha^2) \geq 0$$

for all $k \geq 0$. We thus have

$$\nabla^{k+1}(\operatorname{Log}\alpha^2) \leq 0$$

for all $k \geq 0$ and

$$\operatorname{Log}\alpha^2 \leq 0;$$

that is,

$$\operatorname{Log}\alpha^2 \in \mathcal{CA} \text{ and } \alpha_n \leq 1 \text{ (all } n \geq 0).$$

The proof of the converse is entirely similar. □

Corollary

(i) All Agler shifts are MID

(ii) Also, all $S(a,b,c,d)$ shifts, and their subshifts, are MID. Here $S(a,b,c,d)$ is the shift defined by RC, Y.T. Poon and J. Yoon, with weights $\sqrt{\frac{an+b}{cn+d}}$, where $a,b,c,d>0$ and $ad>bc$.

Corollary

Suppose W_α is a weighted shift.

1. *If W_α is MID, let $W_{\alpha'}$ be the rank-one perturbation consisting only of decreasing the zeroth weight of W_α (while leaving it positive). Then $W_{\alpha'}$ is MID;*
2. *If the weighted shift with weight sequence $e^\alpha=(e^{\alpha_0},e^{\alpha_1},\dots)$ is MID, then W_α is MID.*

We now improve a result of C. Cowen and J. Long, who established the subnormality of a certain weighted shift of importance in considering Toeplitz operators (existence of non-normal, non-analytic subnormal Toeplitz operators).

Corollary

If $0 < p < 1$, the weighted shift with weights $\alpha_n := (1 - p^{2n+2})^{\frac{1}{2}}$ for $n = 0, 1, \ldots$ is subnormal and moment infinitely divisible. Further, if $p_1, p_2, \ldots, p_k$ is a collection of values in $(0, 1)$, the weighted shift with weights $\alpha_n := \prod_{i=1}^{k} (1 - p_i^{2n+2})^{\frac{1}{2}}$ for $n = 0, 1, \ldots$ is subnormal and MID.

Example

The weighted shift with weights

$$\alpha_n := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n+1), \qquad n = 0, 1, \ldots,$$

or the shift with weights the square roots of these, is MID and subnormal. This is because this sequence is completely alternating (and it increases to the Euler constant $\gamma \approx 0.577\ldots$).

Corollary

If W_α is moment infinitely divisible then so is $AT(W_\alpha)$.

Proof.

Since $(\ln \alpha_n)$ is completely alternating, so is $(\ln \sqrt{\alpha_n \alpha_{n+1}})$. □

Mean and Cesàro Transforms

Definition

Let $T \equiv VP$ be the polar decomposition of T. The mean transform is $\widehat{T} := \frac{VP+PV}{2}$.

Theorem

(S.H. Lee, W.Y. Lee and J. Yoon, 2014) The mean transform of $B_+^{(2)}$ is subnormal.

Proposition

Suppose W_α has $(\alpha_n^2) \in \mathcal{CA}$, and let $\alpha'_n := \sqrt{\frac{\alpha_n^2+\alpha_{n+1}^2}{2}}$. Then $W_{\alpha'}$ and $\widehat{W_\alpha}$ are MID and hence subnormal. Moreover, if $(\alpha_n) \in \mathcal{CA}$ then $\widehat{W_\alpha}$ is MID and hence subnormal.

Corollary

The mean transform of B_+ and the mean transform of $B_+^{(2)}$ are MID, hence subnormal.

Proposition

If the sequence (α_n) is completely alternating then its Cesàro transform $(C(\alpha_n))$ is completely alternating. If the sequence (α_n) is log completely alternating, its geometric Cesàro transform $(GC(\alpha_n))$ is log completely alternating.

Other Related Results

Proposition

If (γ_n) is a completely monotone sequence then so is $(e^{\gamma_n - 1})$, and therefore the shift with these moments is subnormal.

Proposition

If (γ_n) is an infinitely divisible sequence then so is $(e^{\gamma_n - 1})$, and therefore the shift with these moments is moment infinitely divisible.

Corollary

If the sequence (α_n) is completely alternating, then the shift with weights e^{α_n} is moment infinitely divisible.

The Square Root Problem

Given a subnormal weighted shift W_α, under what conditions is its "square root" $W_{\sqrt{\alpha}}$ subnormal?

Equivalently, by the uniqueness in the Hamburger moment problem:

$$\int t^n \, d\mu(t) = \left(\int t^n \, d\nu(t) \right)^2, \quad n = 0, 1, 2, \ldots \quad . \tag{1}$$

Thus, the **Square Root Problem** can be stated as follows:

Given a probability measure μ (supported on a compact interval in $\mathbb{R}_+$), does there exist ν satisfying (1) ? If ν exists, can one find it?

A Basic Case

Problem

What is the Berger measure for the square root of the Bergman shift, $\sqrt{B_+}$, with weight sequence

$$\sqrt[4]{\frac{1}{2}}, \sqrt[4]{\frac{2}{3}}, \sqrt[4]{\frac{3}{4}}, \ldots \text{ ?}$$

Equivalently, what is the measure ν with the following moment matching equations:

$$\int t^n dt = \left(\int t^n d\nu(t) \right)^2 \ (n = 0, 1, 2, \ldots)?$$

Associated Moment Problem for Other Transformations

For $0 < p < 1$ consider the moment matching equations

$$\int t^n \, d\mu(t) = \left(\int t^n \, d\nu(t) \right)^p , \quad n = 0, 1, 2, \ldots \quad . \tag{2}$$

One then tries to find one measure, given the other.

A Result About the Support

Let us recast the moment matching equations in terms of product measures:

For each n,

$$\begin{aligned}\int t^n \, d\mu(t) &= \left(\int t^n \, d\nu(t)\right)^2 \\ &= \left(\int s^n \, d\nu(s)\right) \cdot \left(\int t^n \, d\nu(t)\right) \\ &= \iint s^n t^n \, d\nu(s) d\nu(t) \\ &= \iint s^n t^n \, d(\nu \times \nu).\end{aligned}$$

If we define $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $p(x, y) := xy$, we obtain that

$$\mu = (\nu \times \nu) \circ p^{-1}.$$

Theorem

(RC & G. Exner, 2015; J. Stochel & J.B. Stochel, 2012) Suppose that μ and ν are measures supported in some compact subset of $\mathbb{R}_+$ satisfying (1), and let $p(x, y) \equiv xy$ be as above. Then

(1) $\mu = (\nu \times \nu) \circ p^{-1}$

(2) $\text{supp}(\mu) = \overline{(\text{supp}(\nu))^2}$.

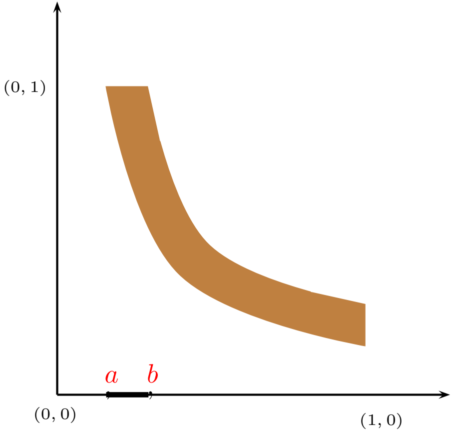
When all of the above happens, we will write $\mu = \nu^2$.

This result has allowed one to decipher the above mentioned case of a subnormal unilateral weighted shift with 3-atomic Berger measure.

Observe that in this situation

$$\mu(E) = (\nu \times \nu)(p^{-1}(E)) = (\nu \times \nu)(\{(s,t) : st \in E\}).$$

One can give a geometrical interpretation, using the following picture (in the case when $E = [a, b]$)



Absolutely Continuous Measures

Assume we wish to solve the Square Problem $\mu = \nu^2$, and we know that ν is absolutely continuous with respect to Lebesgue measure on $[0,1]$. Write $d\nu(t) \equiv g(t)dt$, with g the Radon-Nikodym derivative in L^1. It appears reasonable to hope that μ will also be absolutely continuous, and to pursue its Radon-Nikodym derivative, call it f. Now, for $0 < a < 1$, we have (almost everywhere)

$$f(a) = \lim_{n\to\infty} \frac{\mu([a, a+1/n])}{1/n}.$$

$$\begin{aligned}\mu([a, a+1/n]) &= \int\int_{p^{-1}(E)} 1 \, d\nu \times d\nu \\ &= \int_a^{a+1/n} \int_{a/x}^{1} 1 \; g(y)g(x) \, dy \, dx + \\ &\quad + \int_{a+1/n}^{1} \int_{a/x}^{(a+1/n)/x} 1 \; g(y)g(x) \, dy \, dx \\ &= \int_a^1 \int_{a/x}^{(a+1/n)/x} 1 \; g(y)g(x) \, dy \, dx,\end{aligned}$$

(where we have used that g vanishes outside $[0, 1]$). Then with f the Radon-Nikodym derivative of μ, we have:

$$
\begin{aligned}
f(a) &= \lim_{n\to\infty} \frac{\mu([a, a+1/n])}{1/n} \\
&= \lim_{n\to\infty} \left(\frac{1}{1/n}\right) \int_a^1 \int_{a/x}^{(a+1/n)/x} 1 \; g(y)g(x)\, dy\, dx \\
&= \int_a^1 \left(\lim_{n\to\infty} \frac{\int_{a/x}^{(a+1/n)/x} 1 \; g(y)\, dy}{\frac{1}{nx}} \right) \frac{1}{x} g(x)\, dx \\
&= \int_a^1 g\left(\frac{a}{x}\right) g(x) \frac{1}{x}\, dx \quad \text{(a.e.).}
\end{aligned}
$$

We thus have:

Proposition

(RC & G. Exner, 2015) Let $\mu = \nu^2$ and assume that $d\nu(t) = g(t)dt$, with $g \in L^1([0,1])$. Then $d\mu(t) = f(t)dt$, where

$$f(a) = \int_a^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx \quad \text{(a.e.)}.$$

Remark

For future use, we note that, since g vanishes outside $[0,1]$, one has

$$\int_a^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx = \int_0^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx. \tag{3}$$

This displays the integral as a convolution $g * g$ with respect to Haar measure on the multiplicative semigroup $(0,1)$.

As a consequence, we get a concrete expression for f if g is a polynomial.

Theorem

(RC & G. Exner, 2015) Suppose that g is a polynomial, say $g(x) \equiv \sum_{i=0}^{n} a_i x^i$, positive on $[0,1]$ and inducing a probability measure $d\nu(t) = g(t)\chi_{(0,1)}(t)dt$. Then $\mu = \nu^2$ is absolutely continuous with respect to Lebesgue measure and with Radon-Nikodym derivative $f \cdot \chi_{0,1)}$ where f is given by

$$
\begin{aligned}
f(x) &= \sum_{i=0}^{n-1} \left(\left(\frac{1-x^{i+1}}{i+1} \right) \sum_{j=0}^{n-i-1} a_j a_{j+i+1} x^j \right) + \\
&\quad - \ln x \cdot \sum_{i=0}^{n} a_i^2 x^i \\
&\quad + \sum_{i=-n-1}^{-2} \left(\left(\frac{1-x^{i+1}}{i+1} \right) \sum_{j=-i-1}^{n} a_j a_{j+i+1} x^j \right).
\end{aligned}
$$

Corollary

Let ν be the measure on $[0,1]$ given by $d\nu(t) = 1\,dt$; i.e., ν is Lebesgue measure on $[0,1]$. Then ν^2 is given by $-\ln t\,dt$. In particular, $(1\,dt)^2$ is singular at the origin.

Corollary

The Aluthge transform of the weighted shift associated with the Berger measure $-\ln t\,dt$ is A_3, the third Agler shift; that is,

$$AT(A_2^{(2)}) = A_3.$$

This result does not hold for other Agler shifts.

Laplace Transform Method

Theorem

(A. Athavale, 2013) The square root measure for $1\,dt$ *(on* $[0,1]$*) is* $\frac{1}{\sqrt{\pi}}(-\ln t)^{(-1/2)}\,dt$.

The key to this and allied results is the Laplace transform and the movement of the moment problem from the interval $[0,1]$ to the interval $[0,\infty)$. Recall that the Laplace transform of a function h is $H \equiv \mathcal{L}\{h\}$ where

$$H(s) := \int_0^\infty e^{-st}h(t)\,dt.$$

Now suppose that $F := \mathcal{L}\{g\}$. Then

$$\begin{aligned}
F(s+1) &= \mathcal{L}\{e^{-t}g(t)\}(s) \quad \text{“First Shifting Theorem”} \\
&= \int_0^\infty e^{-st}e^{-t}g(t)\,dt \\
&= \int_0^1 u^{s+1}\frac{1}{u}f(u)\,du \quad (u := e^{-t}, f(u) := g(-\ln u)) \\
&= \int_0^1 u^s f(u)\,du.
\end{aligned}$$

Theorem

(RC & G. Exner, 2015) The q-th power of $1\,dt$ *(on* $[0,1]$*) for* $q>0$ *is*

$$f(u)\,du = \frac{1}{\Gamma(q)}(-\ln u)^{q-1}\,du$$

(where Γ *denotes the classical Gamma function). Alternatively, the Berger measure of the weighted shift whose moment sequence is* $\left(\sqrt[q]{\frac{1}{n+1}}\right)_{n=1}^{\infty}$ *is* $\frac{1}{\Gamma(q)}(-\ln u)^{q-1}\,du$*.*

Corollary

(Special Case: $q=\frac{1}{2}$*)*

$$\sqrt{1\ du} = \frac{1}{\Gamma(\frac{1}{2})}(-\ln u)^{-\frac{1}{2}}\,du = \frac{1}{\sqrt{\pi}}(-\ln u)^{-\frac{1}{2}}\,du.$$