THE LIFTING PROBLEM FOR HYPONORMAL PAIRS OF COMMUTING SUBNORMAL OPERATORS

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Abstract. We construct three different families of commuting pairs of subnormal operators, jointly hyponormal but not admitting commuting normal extensions. Each such family can be used to answer in the negative a 1988 conjecture of RC, P. Muhly and J. Xia. We also obtain a sufficient condition under which joint hyponormality does imply joint subnormality.

Our tools include the use of 2-variable weighted shifts, the six-point test for joint hyponormality, disintegration of measures techniques, the theory of multivariable moment problems, and matrix positivity. We obtain new necessary conditions for the existence of a lifting, and generate new pathology associated with bringing together the Berger measures associated to each individual weighted shift.

For subnormal 2-variable weighted shifts, we then find the precise relationship between the Berger measure of the pair and the Berger measures of the shifts associated to horizontal rows and vertical columns of weights.

Finally, we consider the (multivariable) spectral theory of these hyponormal pairs, and discover some unexpected new phenomena, not present in the single variable theory.

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Contents

1. Preliminaries

From P.R. Halmos, A Hilbert Space Problem Book, Second Edition, Springer-Verlag, 1982:

Problem 203. Give an example of a hyponormal operator that is not subnormal.

This is not easy. The techniques used are almost sufficient to yield...

Halmos eventually gives $\alpha, \beta, 1, 1, 1, \ldots$ as a counterexample.

Problem 209. Give an example of a hyponormal operator whose square is not hyponormal.

This is not easy. It is, in fact, bound to be at least as difficult as the construction of a hyponormal operator that is not subnormal (Problem 203), since any solution of Problem 209 is automatically a solution of Problem 203.

One of the things we'll do today is show a collection of examples of hyponormal pairs of commuting subnormals that are not jointly subnormal.

1.1. Positivity of Block Matrices.

Theorem 1.1. *(Smul'jan, 1959)*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases}.$$

Moreover, rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = rank\ A \Leftrightarrow C = W^*AW.$

Corollary 1.2. $A \geq 0$, *rank* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = rank\ A \Rightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$

1.2. Subnormality and Hyponormality.

$\mathcal{L}(\mathcal{H})$: algebra of bounded operators on a Hilbert space $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$

- For $S, T \in \mathcal{B}(\mathcal{H})$, $[S,T] := ST - TS$.
- An n-tuple $T \equiv (T_1, ..., T_n)$ is jointly hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- T is normal if T is commuting and each T_i is normal
- T is subnormal if T is the restriction of a normal n-tuple to a common invariant subspace
- normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.
- (Bram-Halmos, $n = 1$)

$$\begin{aligned} T \text{ subnormal} \quad &\Leftrightarrow \quad T \text{ is } k\text{-hyponormal for all } k \geq 1 \\ &\Leftrightarrow \quad (T, T^2, ..., T^k) \text{ is hyponormal for all } k \geq 1. \end{aligned}$$

$$\text{subnormal} \Rightarrow \cdots \Rightarrow k\text{-hyponormal} \Rightarrow \cdots \Rightarrow 2\text{-hyponormal}$$

polynomially hyponormal

($p(T)$ hyponormal for all $p \in \mathbb{C}[z]$)

1.3. **Weighted Shifts and Berger's Theorem.** Given a sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, ...$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0)$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot ... \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_{k+1}^2 - \alpha_k^2 \geq 0$ (all $k \geq 0$)

- (Berger's Theorem) W_α is subnormal iff there exists $\xi \geq 0$ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \;\; (\text{all } k \geq 0).$$

Definition 1.3. *W_α is flat (or briefly, α is flat) if $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.*

Theorem 1.4. *(Propagation)*
(i) (Stampfli, 66) Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. In this case, the Berger measure is $\xi \equiv (1 - a_0^2)\delta_0 + a_0^2 \delta_{a_1^2}$, where δ_p denotes the point-mass probability measure with support the singleton $\{p\}$.
(ii) (RC, 88) Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.
(iii) (Choi, 2000) Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.

Application 1.5. *Recall*
Problem 203. *Give an example of a hyponormal operator that is not subnormal.*
Problem 209. *Give an example of a hyponormal operator whose square is not hyponormal.*
(i) $W_\alpha : 0 < \alpha_0 < \alpha_1 < 1 = 1 = 1 = \ldots$ is hyponormal but not subnormal. The square of W_α however, is unitarily equivalent to $(\alpha_0\alpha_1 < 1 = 1 = \ldots) \bigoplus (\alpha_1 < 1 = 1 = \ldots)$, and therefore subnormal.
(ii) $T := U_+^* + 2U_+$ is hyponormal, but T^2 is not hyponormal.

1.4. **Multivariable Weighted Shifts.**

$$\alpha_k, \beta_{\mathbf{k}} \in \ell^\infty(Z_+^2), \quad k \equiv (k_1, k_2) \in Z_+^2 := Z_+ \times Z_+$$

$$\ell^2(Z_+^2) \cong \ell^2(Z_+) \bigotimes \ell^2(Z_+).$$

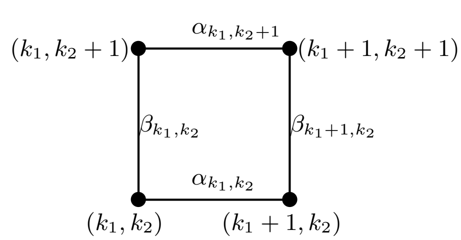
We define the 2-variable weighted shift $T \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \;\; (\text{all } \mathbf{k}). \tag{1.1}$$



- Trivially, a pair of unilateral weighted shifts W_a and W_β gives rise to a 2-variable weighted shift $T \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in Z_+^2$). In this case, T is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(Z_+^2)$ and $\ell^2(Z_+) \bigotimes \ell^2(Z_+)$,

$$T_1 \cong W_a \bigotimes I$$

and

$$T_2 \cong I \bigotimes W_\beta,$$

and T is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed.
- (Jewell-Lubin)

$$\begin{aligned} W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} &:= \prod_{i=0}^{k_1-1} \alpha_{(i,0)}^2 \cdot \prod_{j=0}^{k_2-1} \beta_{(k_1-1,j)}^2 \\ &= \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \ \ (\text{all } \mathbf{k} \geq \mathbf{0}). \end{aligned}$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

2. The Lifting Problem for Commuting Subnormals

- If W_α is subnormal, and if for $h \geq 1$ we let

$$\mathcal{M}_h := \bigvee \{e_n : n \geq h\},$$

then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$.
- $T \equiv (T_1, T_2)$ subnormal $\Rightarrow T_i$ subnormal for $i = 1, 2$. For instance,

$$T_1 \cong \bigoplus_{j=0}^{\infty} W_{\alpha^{(j)}},$$

where $\alpha_i^{(j)} := \alpha_{(i,j)}$, so that $W_{\alpha^{(j)}}$ has associated Berger measure

$$d\nu_j(t_1) := \frac{1}{\gamma_{(0,j)}} \int_{[0,a_2]} t_2^j d\Phi_{t_1}(t_2),$$

where

$$d\mu(t_1, t_2) \equiv d\Phi_{t_1}(t_2) d\eta(t_1)$$

is the canonical disintegration of μ by vertical slices.

Problem 2.1. *(Lifting Problem for Commuting Subnormals) Find necessary and sufficient conditions on T_1 and T_2 to guarantee the subnormality of T.*

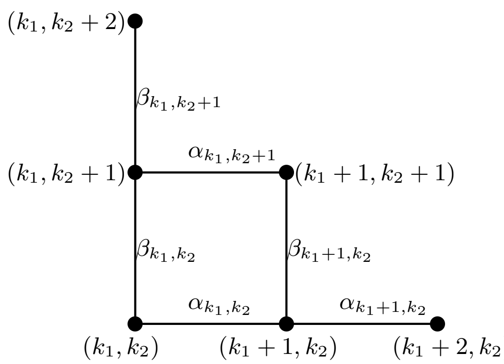
It is well known that the above mentioned necessary conditions do not suffice. In terms of the marginal measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$, the problem can be phrased as a reconstruction-of-measure problem, that is, under what conditions on the measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$ does there exist a 2-variable measure μ correctly interpolating all the powers $t_1^{k_1} t_2^{k_2}$ $(k_1, k_2 \geq 0)$.

2.1. **The Six-Point Test for Joint Hyponormality.**

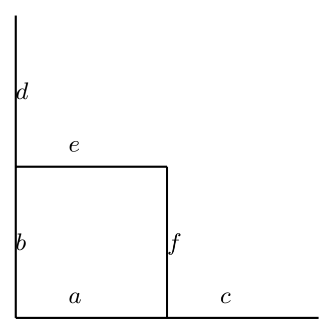
Theorem 2.2. *(RC, 1988) Let $T \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \Leftrightarrow (([T_j^*, T_i] e_{\mathbf{k}+\varepsilon_j}, e_{\mathbf{k}+\varepsilon_i}))_{i,j=1}^2 \geq 0 \ \textit{(all } \mathbf{k} \in \mathbf{Z}_+^2\textit{)}$$

$$\Leftrightarrow \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \ \textit{(all } \mathbf{k} \in \mathbf{Z}_+^2\textit{)}.$$



Application. (Propagation) (RC - Yoon, 2003) If $\alpha_{\mathbf{k}+\varepsilon_1} = \alpha_{\mathbf{k}}$ for some k, then $\alpha_{\mathbf{k}+\varepsilon_2} = \alpha_{\mathbf{k}}$.



$$\begin{pmatrix} c^2-a^2 & ef-ab \\ ef-ab & d^2-b^2 \end{pmatrix} \geq 0 \text{ and } c=a \Rightarrow ef=ab$$
$$\Rightarrow aef=aab \Rightarrow e(af)=a^2b$$
$$\Rightarrow e(eb)=a^2b$$
$$\Rightarrow e^2b=a^2b \Rightarrow e=a$$

Unlike the single-variable case, in which there is a clear separation between hyponormality and subnormality, much less is known about the multivariable case.

Conjecture 2.3. *(RC-Muhly-Xia, 1988) Let $T \equiv (T_1, T_2)$ be a pair of commuting subnormal operators on H. Then T is subnormal if and only if T is hyponormal.*

- M. Dritschel and S. McCullough, working independently, have been able to obtain a separate example. We shall see later that their example is a special case of a general construction that produces nonsubnormal hyponormal pairs with $T_1 \cong T_2$.

Proposition 2.4. *(Subnormal backward extension of a 1-variable weighted shift) (cf RC, 1988) Let T be a weighted shift whose restriction $T_{\mathcal{M}}$ to $M := \vee\{e_1, e_2, \cdots\}$ is subnormal, with associated measure $\mu_{\mathcal{M}}$. Then T is subnormal (with associated measure μ) if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$
(ii) $\alpha_0^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$
In this case, $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) + (1-\alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}) d\delta_0(t)$, *where δ_0 denotes Dirac measure at 0. In particular, T is never subnormal when $\mu_{\mathcal{M}}(\{0\}) > 0$.*

Proof. $\Rightarrow$) The moments of T and $T_{\mathcal{M}}$ are related by the equation

$$\gamma_k(T_{\mathcal{M}}) \equiv \alpha_1^2 \cdots \alpha_k^2 = \frac{\gamma_{k+1}(T)}{\alpha_0^2}$$

so that

$$\frac{1}{\alpha_0^2} \int t^{k+1} d\mu(t) = \int t^k d\mu_{\mathcal{M}}(t) \text{ (all } k \geq 0),$$

that is,

$$t d\mu(t) = \alpha_0^2 d\mu_{\mathcal{M}}(t).$$

It follows at once that

$$d\mu(t) = \lambda d\delta_0(t) + \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(s),$$

where $\lambda \geq 0$. Since $\int d\mu = 1$, we must have $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ and $\alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$. Finally, it is straightforward to verify that $\lambda = (1-\alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})$.

$\Leftarrow$) Let

$$d\mu(t) := \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) + (1-\alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}) d\delta_0(t).$$

By hypotheses, μ is a positive Borel measure on $[0, \|T\|^2]$. Moreover,

$$\int d\mu = \alpha_0^2 \int \frac{1}{t} d\mu_{\mathcal{M}} + (1 - \alpha_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}) \int d\delta_0 = 1,$$

and for $k \geq 1$,

$$\begin{aligned} \int t^k d\mu(t) &= \alpha_0^2 \int t^k \frac{1}{t} d\mu_{\mathcal{M}}(t) + (1 - \alpha_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}) \int t^k d\delta_0(t) \\ &= \alpha_0^2 \int t^{k-1} d\mu_{\mathcal{M}}(t) = \alpha_0^2 \gamma_{k-1}(T_{\mathcal{M}}) = \gamma_k(T). \end{aligned}$$

Therefore, T is subnormal, with Berger measure μ. □

- The maximum possible value for α_0 in Proposition 2.4, namely $(\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})})^{-1}$, will be denoted by $\alpha_{ext} \equiv \alpha_{ext}(\mu_{\mathcal{M}})$. More generally, given a (1-variable) subnormal weighted shift W_η with weight sequence $\eta_1 \leq \eta_2 \leq ...$ and Berger measure ν, we let

$$\eta_{ext} := \begin{cases} 0 & \text{if } \frac{1}{t} \notin L^1(\nu) \\ (\|\frac{1}{t}\|_{L^1(\nu)})^{-1} & \text{if } \frac{1}{t} \in L^1(\nu) \end{cases}.$$

  Observe that when the weight sequence η is strictly increasing and $\frac{1}{t} \in L^1(\nu)$, we must necessarily have $\eta_{ext} < \eta_1$, by a result of J. Stampfli.
- $shift(\alpha_0, \alpha_1, ...)$ denotes the weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$.
- $U_+ := shift(1, 1, ...)$
- For $0 < a < 1$ we let $S_a := shift\{a, 1, 1, ...\}$.
- The Berger measures of U_+ and S_a are δ_1 and $(1 - a^2)\delta_0 + a^2\delta_1$, respectively.
- B_+ denotes the Bergman shift, whose Berger measure is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are given by the formula $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.
- (RC, 1990) $W_\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, ...$ is quadratically hyponormal.
- An important class of one-variable subnormal weighted shifts is obtained by considering measures ξ with exactly two atoms t_0 and t_1. These shifts arise naturally in the Subnormal Completion Problem and in the theory of truncated moment problems.
  For $t_0, t_1 \in R_+$, $t_0 < t_1$, and $\rho_0, \rho_1 > 0$, the moments of the 2-atomic measure $\xi := \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1}$ satisfy the 2-step recursive relation

$$\gamma_{n+2} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} \ (n \geq 0);$$

  at the weight level, this can be written as

$$\alpha_{n+1}^2 = \frac{\varphi_0}{\alpha_n^2} + \varphi_1 \ (n \geq 0).$$

  The coefficients of recursion are given by

$$\varphi_0 = -\frac{\alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \text{ and } \varphi_1 = \frac{\alpha_1^2 (\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}, \tag{2.1}$$

the atoms t_0 and t_1 are the roots of the equation

$$t^2 - (\varphi_0 + \varphi_1 t) = 0, \tag{2.2}$$

and the densities ρ_0 and ρ_1 uniquely solve the 2×2 system of equations

$$\begin{cases} \rho_0 + \rho_1 & = & 1 \\ \rho_0 t_0 + \rho_1 t_1 & = & \alpha_0^2 \end{cases}. \tag{2.3}$$

Lemma 2.5. *(RC-Fialkow, 1994) For $0 < \alpha_0 < \alpha_1 < \alpha_2$, let $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ be the weighted shift described by (2.1), (2.2) and (2.3), and let $W_\eta := shift(\alpha_1, \alpha_2, ...)$, that is, $W_\eta \equiv W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}|_{\mathcal{M}}$. Then*

$$\eta_{ext} = \alpha_0.$$

Theorem 2.6. *(RC-IB Jung, 2000) Let $\alpha : 1, (1, \sqrt{1+h}, \sqrt{1+h+k})^\wedge$ with $h, k > 0$. Then W_α is positively quadratically hyponormal if and only if $(h,k) \in \mathcal{U}_1$, with $\mathcal{U}_1$ as shown below.*

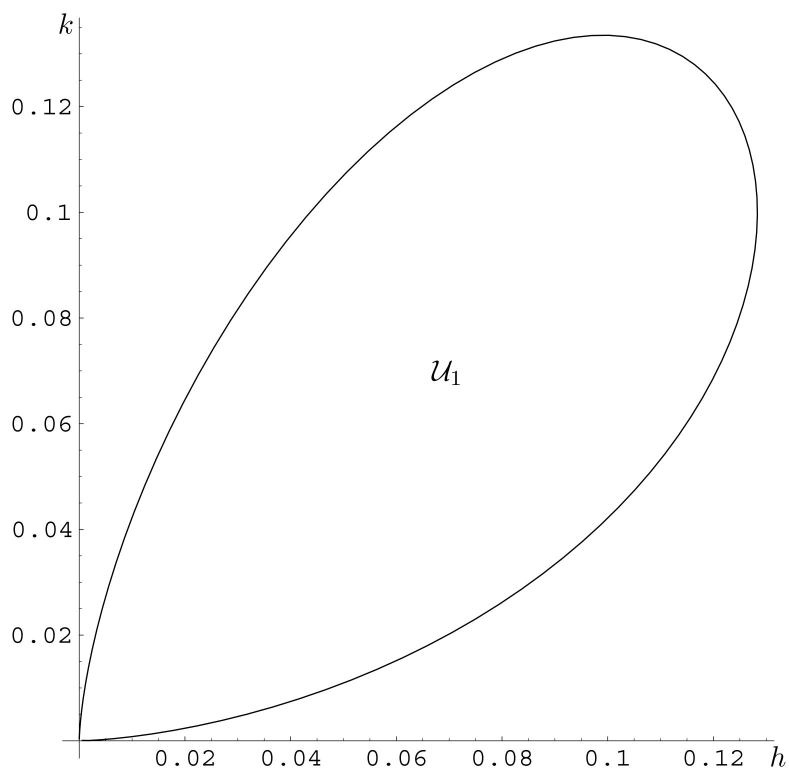


FIGURE 1. For $\alpha : 1, (1, \sqrt{1+h}, \sqrt{1+h+k})^\wedge$, W_α is positively quadratically hyponormal $\Longleftrightarrow$ $(h,k) \in \mathcal{U}_1$.

3. The First Family of Counterexamples

Construction of the family. Let $0 < a, b < 1$ and let $\{\xi_k\}_{k=0}^{\infty}$ and $\{\eta_k\}_{k=0}^{\infty}$ be two strictly increasing weight sequences. Consider the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ on $\ell^2(Z_+^2)$ given by the double-indexed weight sequences

$$\alpha(\mathbf{k}) := \begin{cases} \xi_{k_1} & \text{if } k_1 \geq 1 \text{ or } k_2 \geq 1 \\ a & \text{if } k_1 = 0 \text{ and } k_2 = 0 \end{cases} \tag{3.1}$$

and

$$\beta(\mathbf{k}) := \begin{cases} \eta_{k_2} & \text{if } k_1 \geq 1 \text{ or } k_2 \geq 1 \\ b & \text{if } k_1 = 0 \text{ and } k_2 = 0 \end{cases}. \tag{3.2}$$

where W_ξ and W_η are two single-variable subnormal weighted shifts with Berger measures ν and ω, resp., and

$$a\eta_0 = b\xi_0 \tag{3.3}$$

(to guarantee the commutativity of T_1 and T_2 (1.1)).

- T_1 and T_2 are subnormal provided $a \leq \xi_{ext}(\nu_{\mathcal{M}})$ and $b \leq \eta_{ext}(\omega_{\mathcal{M}})$; in particular, $a < \xi_1$ and $b < \eta_1$.

Proposition 3.1. $\mathbf{T}$ *is subnormal only if* $a \leq s$, *where* $s := \sqrt{\frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2+\xi_0^2\eta_1^2-\xi_0^2\eta_0^2}}$.

Proposition 3.2. $\mathbf{T}$ *is hyponormal if and only if* $a \leq h$, *where* $h := \xi_0\sqrt{\frac{\xi_1^2\eta_1^2-\xi_0^2\eta_0^2}{\xi_0^2\eta_1^2+\xi_1^2\eta_0^2-2\xi_0^2\eta_0^2}}$.

Thus, to ascertain the existence of a nonsubnormal, hyponormal 2-variable weighted shift $\mathbf{T}$ (with T_1 and T_2 subnormal), it suffices to show that for appropriate choices of ξ_0, ξ_1, η_0 and η_1, it is possible to obtain $s < h$, while keeping $a \leq \xi_{ext}(\nu_{\mathcal{M}})$ and $b \equiv \frac{a\eta_0}{\xi_0} \leq \eta_{ext}(\omega_{\mathcal{M}})$. Now,

$$h^2 - s^2 = \frac{\xi_0^4\eta_0^2(\xi_1^2 - \xi_0^2)(\eta_1^2 - \eta_0^2)}{(\xi_0^2\eta_1^2 + \xi_1^2\eta_0^2 - 2\xi_0^2\eta_0^2)(\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2)} > 0.$$

Therefore, it suffices to prove the existence of strictly increasing weight sequences $\{\xi_i\}$ and $\{\eta_j\}$ such that

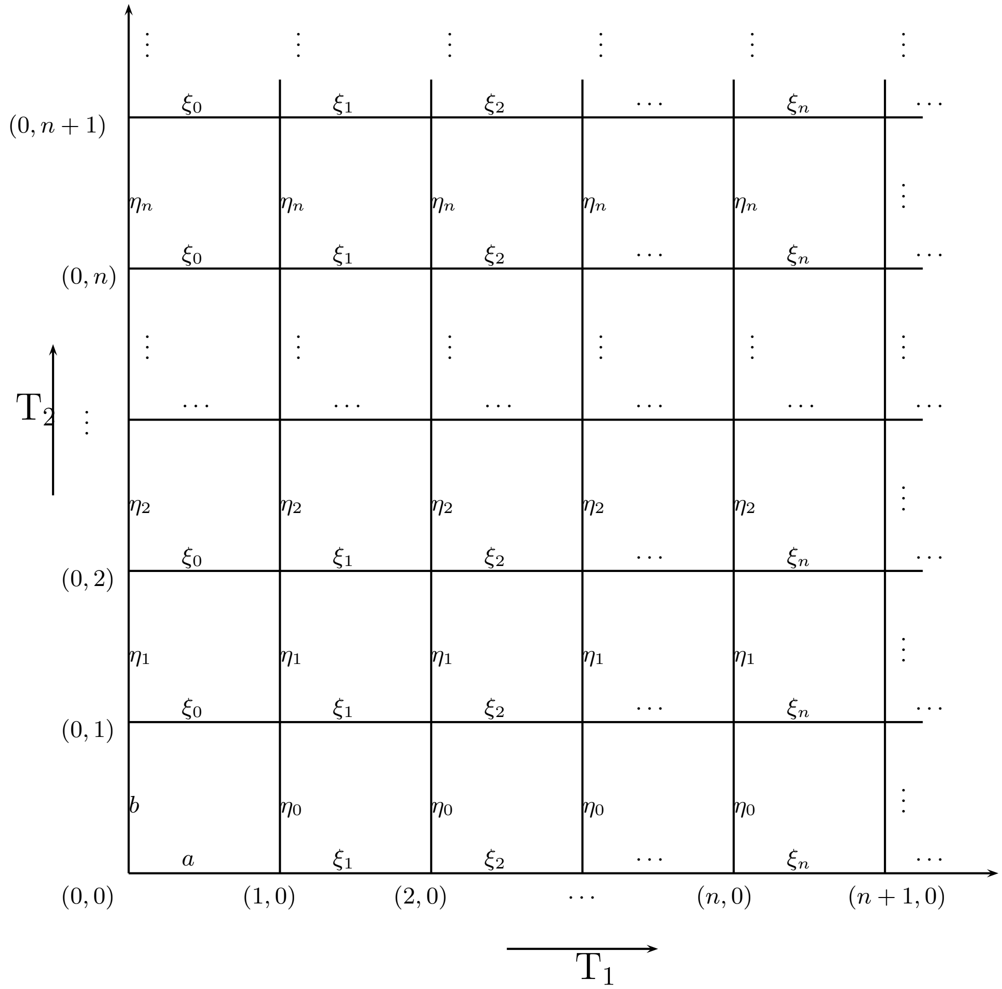
(i) $a \leq h$ (hyponormality of $\mathbf{T}$)
(ii) $a > s$ (nonsubnormality of $\mathbf{T}$)
(iii) $a \leq \xi_{ext}(\nu_{\mathcal{M}})$ (subnormality of T_1)
(iv) $a \leq s_2 := \frac{\xi_0}{\eta_0}\eta_{ext}(\omega_{\mathcal{M}})$ (subnormality of T_2).

We now seek to determine the relative positions of h, s, s_2, ξ_0, $\xi_{ext}(\nu_{\mathcal{M}})$ and ξ_1 in the positive real axis.

Claim 1: $\xi_0 \leq \xi_{ext}(\nu_{\mathcal{M}})$, because $shift\{\xi_0, \xi_1, ...\}$ is subnormal.

Claim 2: $\xi_0 < s$. For,

$$s^2 - \xi_0^2 = \frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2} - \xi_0^2 = \frac{\xi_0^2(\xi_1^2 - \xi_0^2)(\eta_1^2 - \eta_0^2)}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2} > 0.$$



Claim 3: $s < \xi_1$. For,

$$\xi_1^2 - s^2 = \xi_1^2 - \frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2} = \frac{\xi_1^2\eta_0^2(\xi_1^2 - \xi_0^2)}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2} > 0.$$

Claim 4: $h < \xi_1$.
Claim 5: $s < s_2$ whenever $\eta_0 < u := \frac{\xi_0^2\eta_e^2\eta_1^2}{\xi_1^2(\eta_1^2-\eta_e^2)+\xi_0^2\eta_e^2}$, where $\eta_e \equiv \eta_{ext}(\omega_{\mathcal{M}})$.
Claim 6: $h \leq s_2$ whenever $\eta_0 \leq v := \frac{\xi_1^2(\eta_1^2-\eta_e^2)+2\xi_0^2\eta_e^2-\sqrt{(\eta_1^2-\eta_e^2)(\xi_1^4(\eta_1^2-\eta_e^2)+4\xi_0^2\eta_e^2(\xi_1^2-\xi_0^2))}}{2\xi_0^2}$.

We now summarize what we have so far. For $\eta_0 < \min\{u, v\}$ we have

$$(\text{Have}) \begin{cases} \xi_0 < s < h \leq s_2 \\ \\ h < \xi_1 \\ \\ \xi_{ext}(\nu_{\mathcal{M}}) < \xi_1. \end{cases} \qquad (\text{Need}) \begin{cases} s < a \leq h \\ \\ a \leq \xi_{ext}(\nu_{\mathcal{M}}) \\ \\ a \leq s_2. \end{cases}$$

Thus, if we can ensure that $h \leq \xi_{ext}(\nu_{\mathcal{M}})$, the construction of the example will be complete by taking a such that $s < a \leq h$.

Since $h \leq s_2$, it suffices to build $shift(\xi_0, \xi_1, ...)$ in such a way that $\xi_{ext}(\nu_{\mathcal{M}}) = s_2$. Recall

Lemma 3.3. *(RC-Fialkow, 1994) For $0 < \alpha_0 < \alpha_1 < \alpha_2$, let $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ be the weighted shift described by (2.1), (2.2) and (2.3), and let $W_\eta := shift(\alpha_1, \alpha_2, ...)$, that is, $W_\eta \equiv W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}|_{\mathcal{M}}$. Then $\eta_{ext} = \alpha_0$.*

We first build a 2-step recursively generated weighted shift whose first three weights are s_2, ξ_1 and ξ_2, and we then consider the shift $W_{\xi_0(\xi_1,\xi_2,\xi_3)^\wedge}$, where ξ_3 is given by $\xi_3 := \frac{\varphi_0}{\xi_2^2} + \varphi_1$ obtained from the equation $\gamma_4 = \varphi_0\gamma_2 + \varphi_1\gamma_3$.

The extremal value of $W_{(\xi_1,\xi_2,\xi_3)^\wedge}$ is s_2, and $\xi_0 < s_2$, so the subnormality of $W_{\xi_0(\xi_1,\xi_2,\xi_3)^\wedge}$ is guaranteed. This completes the construction of the example.

Theorem 3.4. *Let $\mathbf{T} \equiv (T_1, T_2)$ be the 2-variable weighted shift defined by (3.1) and (3.2), let*

$$\begin{cases} h := \xi_0\sqrt{\frac{\xi_1^2\eta_1^2 - \xi_0^2\eta_0^2}{\xi_0^2\eta_1^2 + \xi_1^2\eta_0^2 - 2\xi_0^2\eta_0^2}}, \\ \\ s := \sqrt{\frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^2\eta_0^2 + \xi_0^2\eta_1^2 - \xi_0^2\eta_0^2}}, \\ \\ s_2 := \frac{\xi_0}{\eta_0}\eta_e, \text{ where } \eta_e \equiv \eta_{ext}(\omega_{\mathcal{M}}), \\ \\ u := \frac{\xi_0^2\eta_e^2\eta_1^2}{\xi_1^2(\eta_1^2 - \eta_e^2) + \xi_0^2\eta_e^2}, \text{ and} \\ \\ v := \frac{\xi_1^2(\eta_1^2 - \eta_e^2) + 2\xi_0^2\eta_e^2 - \sqrt{(\eta_1^2 - \eta_e^2)(\xi_1^4(\eta_1^2 - \eta_e^2) + 4\xi_0^2\eta_e^2(\xi_1^2 - \xi_0^2))}}{2\xi_0^2}. \end{cases}$$

Assume further that, as above, $s_2 = \xi_{ext}(\nu_{\mathcal{M}})$ and $\eta_0 \leq \min\{u, v\}$. Finally, choose a such that $s < a \leq h$. Then

(i) $T_1T_2 = T_2T_1$*;*

(ii) T_1 *is subnormal;*

(iii) T_2 *is subnormal;*

(iv) $\mathbf{T}$ *is hyponormal; and*

(v) $\mathbf{T}$ *is not subnormal.*

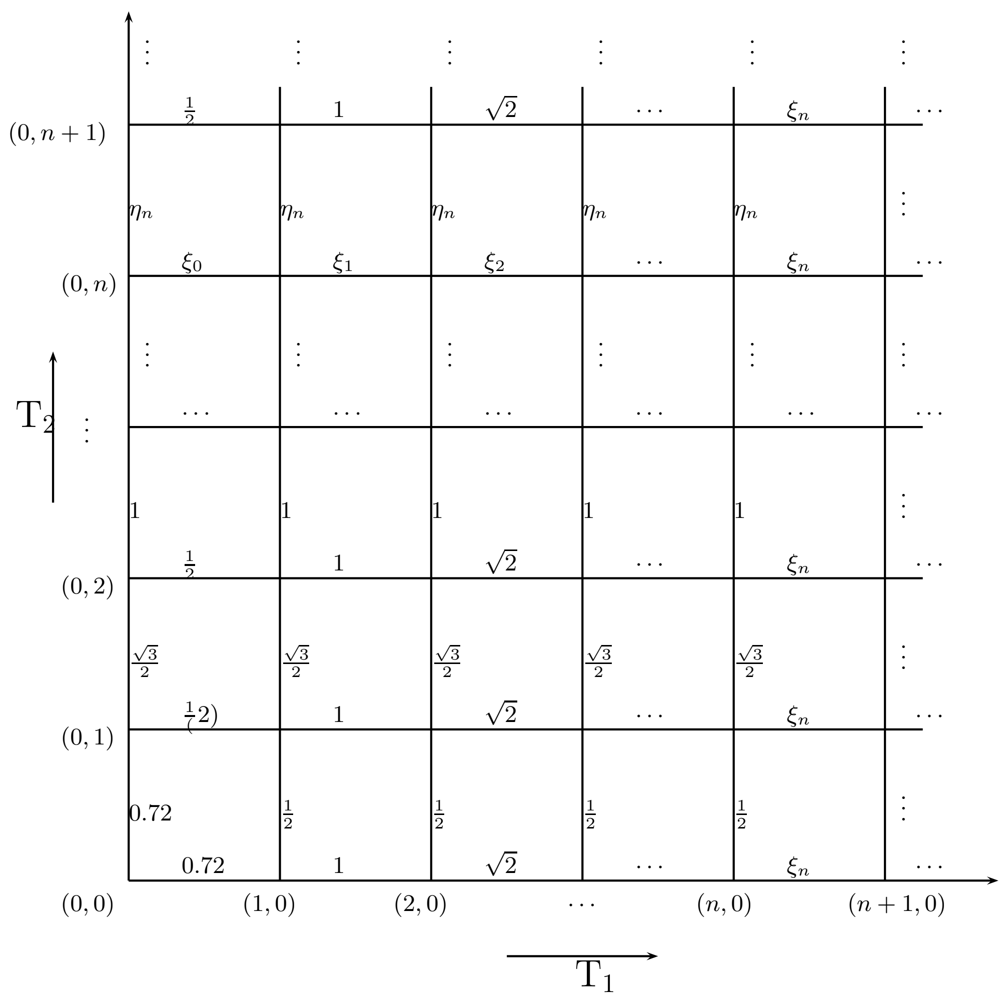
Example 3.5. For a concrete numerical example, let

$$d\omega_{\mathcal{M}}(t) := 2dt \ on \ [\frac{1}{2}, 1],$$

so that $\left\|\frac{1}{t}\right\|_{L^1(\omega_{\mathcal{M}})} = 2\ln 2$. It follows that

$$\eta_e \equiv \eta_{ext}(\omega_{\mathcal{M}}) = \frac{1}{\sqrt{2\ln 2}}$$

and $\eta_1 = \frac{\sqrt{3}}{2}$. Now take $\xi_0 := \frac{1}{2}$ and $\xi_1 := 1$, and $\eta_0 := \frac{1}{2}$.



With this choice of η_0 we obtain

$$s = \frac{\sqrt{2}}{2} \cong 0.707, \ h = \frac{1}{2}\sqrt{\frac{11}{5}} \cong 0.742$$

and

$$s_2 = \eta_e = \frac{1}{\sqrt{2\ln 2}} \cong 0.849.$$

We can then take $a \in (s,h]$, for instance $a := 0.72$.
To build the weighted shift W_ξ we start with s_2, ξ_1 and $\xi_2 := \sqrt{2}$ to obtain $\varphi_0 = \frac{1}{1-2\ln 2}$ and $\varphi_1 = \frac{1-4\ln 2}{1-2\ln 2}$. It follows that the measure associated to $shift(\xi_0, \xi_1, \xi_2, ...)$ is

$$d\nu(t) = \frac{1}{4t}(\rho_0 d\delta_{t_0}(t) + \rho_1 d\delta_{t_1}(t)) + (1 - \frac{1}{4}\left\|\frac{1}{t}\right\|_{L^1(\nu_{\mathcal{M}})})d\delta_0(t).$$

4. The Second Family of Counterexamples

Recall that W_α is subnormal if and only if there exists a probability measure $\xi \equiv \xi_\alpha$ supported in $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k(\alpha) := \alpha_0^2 \cdot ... \cdot \alpha_{k-1}^2 = \int t^k \, d\xi(t) \ \ (k \geq 1).$$

Example 4.1. If $W_\alpha \equiv shift(\alpha_0, 1, 1, ...)$, we have $\xi_\alpha = (1-\alpha_0^2)\delta_0 + \alpha_0^2 \delta_1$.

Lemma 4.2. *Given two 1-variable weight sequences α and β, the 2-variable weighted shift*

$$(W_\alpha \bigotimes I, I \bigotimes W_\beta)$$

is always subnormal, with Berger measure

$$\mu := \xi_\alpha \times \xi_\beta.$$

Definition 4.3. *Let μ and ν be two positive measures on R_+. We say that $\mu \leq \nu$ on $X := R_+$, if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq R_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on R_+.*

Definition 4.4. *μ probability measure on $X \times Y$, with $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (also a probability measure) on $X \times Y$ is given by*

$$d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t).$$

Example 4.5. B_+ : Bergman shift on $\ell^2(Z_+)$, $M \equiv M_1 := \bigvee\{e_1, e_2, ...\}$.
$B_+|_{\mathcal{M}}$ is subnormal, with Berger measure $d\mu(t) := tdt$ on $[0,1]$.
Then $d\mu_{ext}(t) = dt$, so μ_{ext} is the Berger measure of B_+.

Definition 4.6. *For μ on $X \times Y$, the marginal measure μ^X is given by*

$$\mu^X := \mu \circ \pi_X^{-1},$$

where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus,

$$\mu^X(E) = \mu(E \times Y),$$

for every $E \subseteq X$. If μ is a probability measure, then so is μ^X.

Lemma 4.7. *μ : Berger measure of 2-variable weighted shift T*
ν : Berger measure of $shift(\alpha_{00}, \alpha_{10}, ...)$.
Then $\nu = \mu^X$. As a consequence,

$$\iint f(s)\, d\mu(s,t) = \int f(s)\, d\mu^X(s) \quad (\textit{all } f \in C(X)).$$

Corollary 4.8. *μ : Berger measure of 2-variable weighted shift T.*
For $j \geq 1$, let $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$.
Then the Berger measure of $shift(\alpha_{0j}, \alpha_{1j}, ...)$ is $\nu_j \equiv \mu_j^X$.

Example 4.9. $(\xi \times \eta)^X = \xi$.

Lemma 4.10. *Let μ and ω be two measures on $X \times Y$, and assume that $\mu \leq \omega$. Then $\mu^X \leq \omega^X$.*

Proposition 4.11. *(Subnormal backward extension of a 2-variable weighted shift)*
M : subspace of $\ell^2(Z_+^2)$ associated to indices k with $k_2 \geq 1$.
Assume $T_{\mathcal{M}}$ subnormal with measure $\mu_{\mathcal{M}}$ and $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ subnormal with measure ν.
Then T is subnormal if and only if
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;

(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu$.
Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \nu$.
When T is subnormal, its Berger measure is

$$d\mu(s,t) = \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + (d\nu(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)) d\delta_0(t).$$

Key Ideas.

$$\gamma_{\mathbf{k}+\varepsilon_2}(\mathbf{T}) = \beta_{00}^2 \gamma_{\mathbf{k}}(\mathbf{T}_{\mathcal{M}}) \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2),$$

so

$$t d\mu(s,t) = \beta_{00}^2 d\mu_{\mathcal{M}}(s,t)$$

and

$$\mu_{\mathcal{M}}(E \times \{0\}) = 0 \quad (\text{for all } E \subseteq X).$$

It follows at once that

$$\begin{aligned} \iint \frac{1}{t}\, d\mu_{\mathcal{M}}(s,t) &= \iint_{(t>0)} \frac{1}{t}\, d\mu_{\mathcal{M}}(s,t) = \frac{1}{\beta_{00}^2} \iint_{(t>0)} \frac{1}{t}\, t d\mu(s,t) \\ &= \frac{1}{\beta_{00}^2} \mu((t>0)) \leq \frac{1}{\beta_{00}^2}. \end{aligned}$$

□

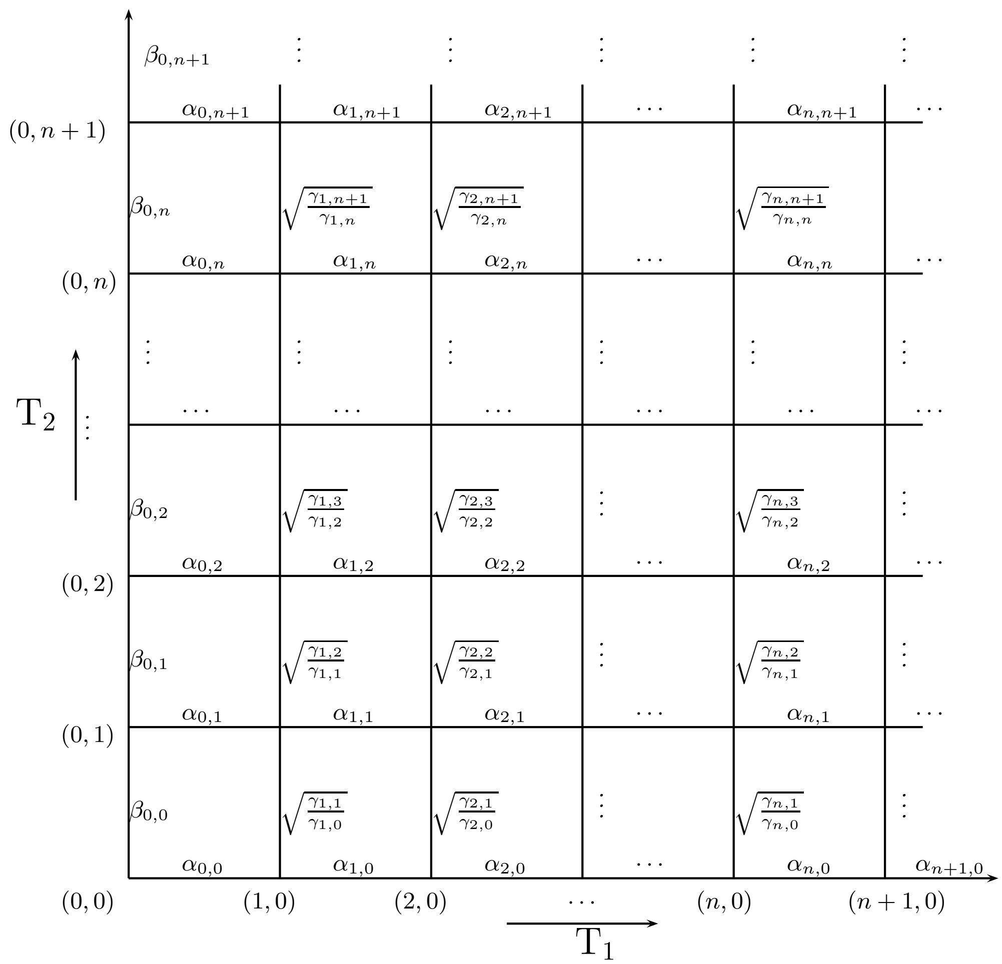


FIGURE 2

Consider now the 2-variable weighted shift given by Figure 3, where $\max\{y, x, \frac{ay}{x}\} < 1$.

Proposition 4.12. **T** *is hyponormal if and only if* $y \le \min\{\frac{x}{a}, x\sqrt{\frac{1-x^2}{x^2-2a^2x^2+a^4}}\}$.

Proof. By the Six-point Test, it is enough to check that

$$H := \begin{pmatrix} 1-x^2 & \frac{a^2y}{x} - yx \\ \frac{a^2y}{x} - yx & 1-y^2 \end{pmatrix} \ge 0.$$

Since $x < 1$, the positivity of H is equivalent to $\det H \ge 0$, i.e.,

$$(1-x^2)(1-y^2) \ge (\frac{a^2y}{x} - yx)^2,$$

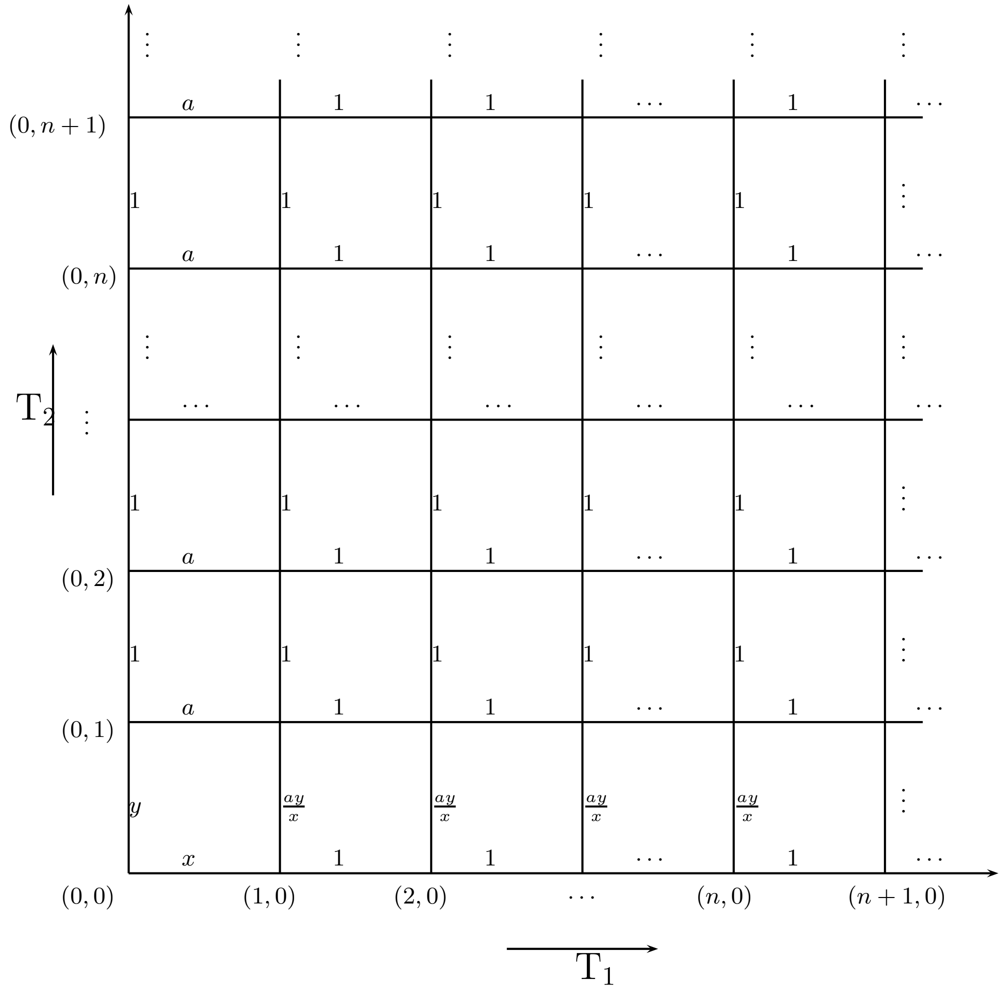


FIGURE 3

which in turn is equivalent to

$$y \leq x\sqrt{\frac{1-x^2}{x^2-2a^2x^2+a^4}}$$

(observe that $x^2 - 2a^2x^2 + a^4 = x^2(1 - x^2) + (x^2 - a^2)^2 > 0$). □

Proposition 4.13. **T** *is subnormal if and only if* $y \leq \sqrt{\frac{1-x^2}{1-a^2}}$.

Proof.

$$\mathbf{T}_{\mathcal{M}} \cong (S_a \otimes I, I \otimes U_+)$$

(where $S_a := shift(a, 1, 1, ...)$ and $U_+ := shift(1, 1, 1, ...)$.

$T_{\mathcal{M}}$ is subnormal, with Berger measure $\mu_{\mathcal{M}} := [(1-a^2)\delta_0 + a^2\delta_1] \times \delta_1$. Then

$$\begin{aligned}
\mathbf{T} \text{ is subnormal } &\Leftrightarrow \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu \\
&\Leftrightarrow y^2[(1-a^2)\delta_0 + a^2\delta_1] \leq (1-x^2)\delta_0 + x^2\delta_1 \\
&\Leftrightarrow y^2(1-a^2) \leq 1-x^2 \text{ and } ay \leq x \\
&\Leftrightarrow y \leq \min\{\frac{x}{a}, \sqrt{\frac{1-x^2}{1-a^2}}\}.
\end{aligned}$$

□

Theorem 4.14. **T** *is hyponormal and not subnormal if and only if* $x > a$ *and* $\sqrt{\frac{1-x^2}{1-a^2}} < y \leq x\sqrt{\frac{(1-x^2)}{x^2+a^4-2a^2x^2}}$

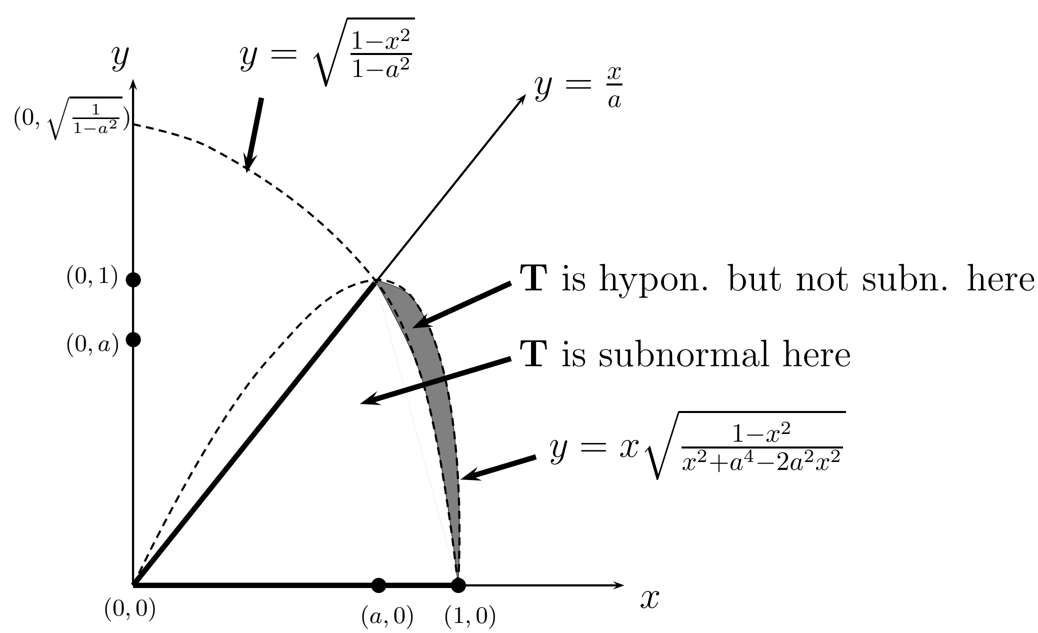


FIGURE 4

5. The Third Family of Counterexamples

Construction of the family. Let us consider the following 2-variable weighted shift (see Figure 5), where

$$\begin{cases} \text{(i)} \ \ 0<\xi_1<\xi_2<...<\xi_n \nearrow 1; \\ \\ \text{(ii)} \ \ W_\xi := shift(\xi_1,\xi_2,...) \text{ is subnormal with Berger measure } \nu; \\ \\ \text{(iii)} \ \ \frac{1}{s^2} \in L^1(\nu) \text{ (this implies that } \frac{1}{s} \in L^1(\nu)\text{, by Jensen's inequality);} \\ \\ \text{(iv)} \ \ \xi_e \equiv \xi_{ext} := (\int \frac{1}{s} d\nu(s))^{-1/2}; \\ \\ \text{(v)} \ \ a \le \frac{1}{\xi_e}(\int \frac{1}{s^2} d\nu(s))^{-1/2}; \\ \\ \text{(vi)} \ \ b \le \xi_e^2 \text{ (this implies the condition } b<\xi_e\text{); and} \\ \\ \text{(vii)} \ \ a^2 \le \frac{b^2+\xi_e^2}{2}. \end{cases}$$

(Recall that ξ_e is the maximum possible value for ξ_0 in Proposition 2.4.)

- $T_1 \cong T_2$ and that $T_1T_2 = T_2T_1$.
- T_1 (and therefore T_2) is subnormal. For, the choice of ξ_e immediately implies that $shift(\xi_e,\xi_1,\xi_2,...)$ is subnormal, with Berger measure $d\nu_e(s) := \frac{\xi_e^2}{s} d\nu(s)$.
- $shift(a,\xi_e,\xi_1,...)$ is subnormal if and only if $\frac{1}{s} \in L^1(\nu_e)$ (i.e., $\frac{1}{s^2} \in L^1(\nu)$
- $T_1|_{\bigvee\{e_{(i,0)}:i\ge 0\}}$ is subnormal. Moreover, the subnormality of T_1 when restricted to $\bigvee\{e_{(i,j)}: i \ge 0\}$ $(j>0)$ requires that $b \le \xi_e$.

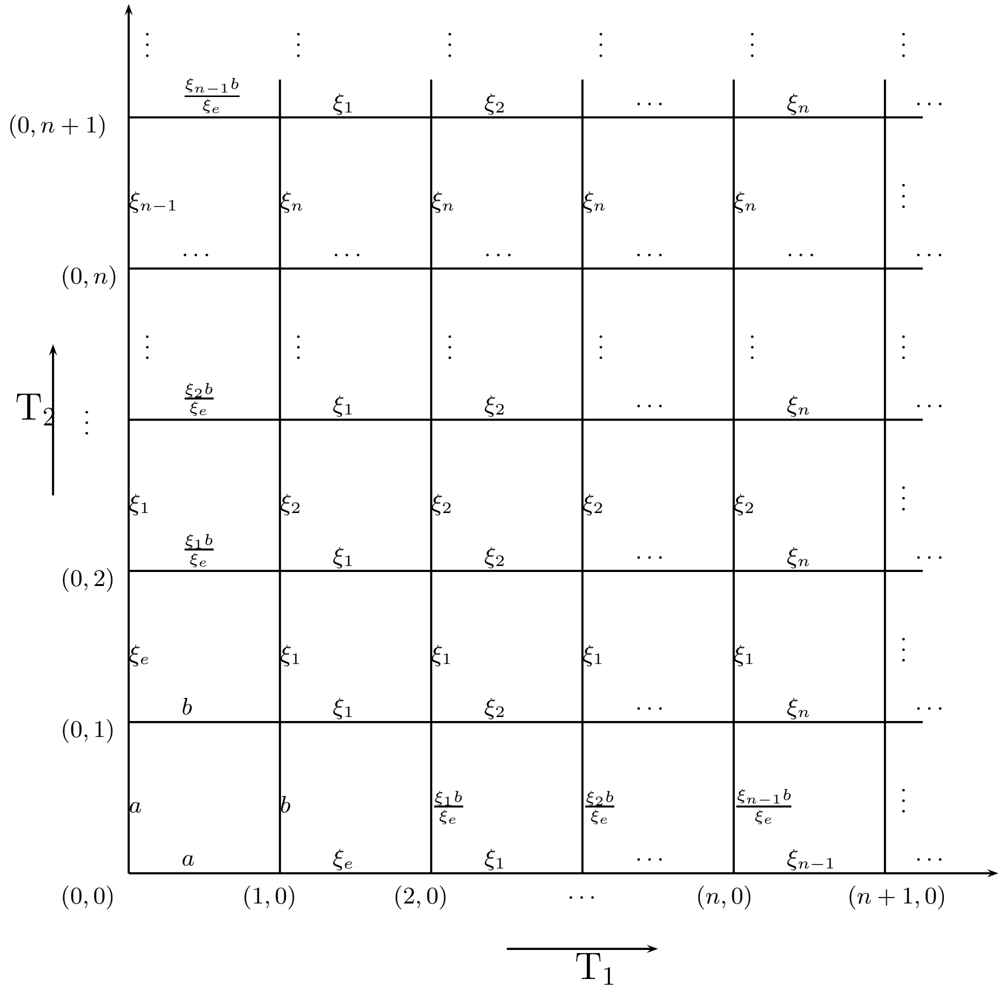
For a concrete numerical example, consider the probability measure $d\nu(s) := 3s^2 ds$ on the interval $[0,1]$. The measure ν corresponds to a subnormal weighted shift with weights $\xi_1 = \sqrt{\frac{3}{4}}$, $\xi_2 = \sqrt{\frac{4}{5}}$, $\xi_3 = \sqrt{\frac{5}{6}}$, ... ; indeed, in this case W_ξ is the restriction of the Bergman shift B_+ to the invariant subspace M_2 obtained by removing the first two basis vectors in the canonical orthonormal basis of $\ell^2(Z_+)$. Clearly $\frac{1}{s^2} \in L^1(\nu)$, and $\int \frac{1}{s^2} d\nu(s) = 3$; moreover, $\int \frac{1}{s} d\nu(s) = \frac{3}{2}$, so in this case $\xi_e = \sqrt{\frac{2}{3}}$. Choosing $a = \sqrt{\frac{1}{2}}$ and $b = \sqrt{\frac{1}{3}}$ we see that conditions (i) - (vii) are satisfied.

Proposition 5.1. $\mathbf{T}$ *is hyponormal.*

Proposition 5.2. $\mathbf{T}$ *is not subnormal if* $p<0$*, where*

$$p := \xi_e^2\xi_1^4 + 4a^2b^2\xi_1^2 - b^2\xi_1^4 - a^2b^2\xi_e^2 - a^2b^4 - 2a^2\xi_1^4.$$

Proof. Assume that $\mathbf{T}$ is subnormal, and consider the moment matrix associated to the monomials 1, x, y and yx, that is,



$$M := \begin{pmatrix} 1 & a^2 & a^2 & a^2b^2 \\ a^2 & a^2\xi_e^2 & a^2b^2 & a^2b^2\xi_1^2 \\ a^2 & a^2b^2 & a^2\xi_e^2 & a^2b^2\xi_1^2 \\ a^2b^2 & a^2b^2\xi_1^2 & a^2b^2\xi_1^2 & a^2b^2\xi_1^4 \end{pmatrix}.$$

In the presence of a representing measure, it is well known that M must be positive semi-definite, so in particular $\det M \geq 0$. Now, a straightforward calculation shows that

$$\det M = a^6b^2\left(\xi_e^2 - b^2\right)\left(\xi_e^2\xi_1^4 - \xi_e^2a^2b^2 - 2a^2\xi_1^4 - b^2\xi_1^4 + 4a^2b^2\xi_1^2 - b^4a^2\right) = a^6b^2\left(\xi_e^2 - b^2\right)p.$$

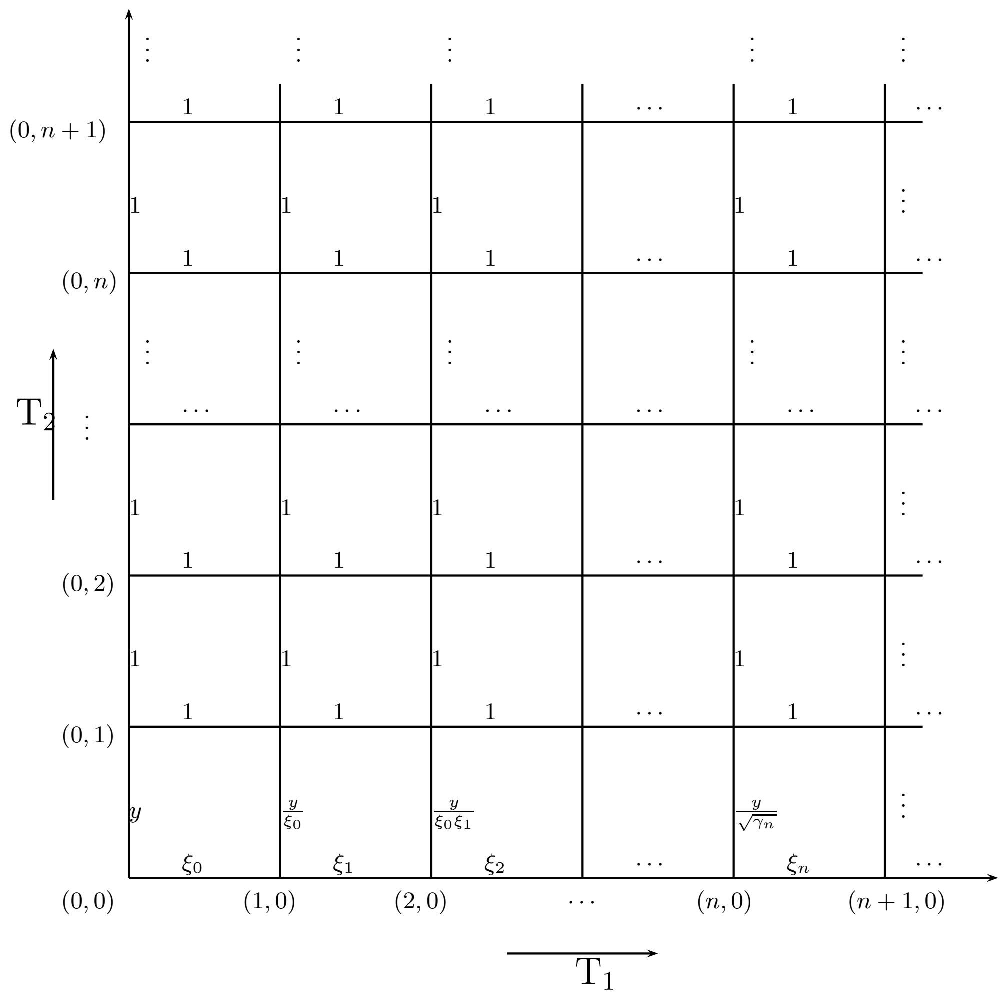
It follows that $p \geq 0$. Therefore, $\mathbf{T}$ is not subnormal whenever $p < 0$, as desired. □

Theorem 5.3. *Let $a > 0$ be such that $\sqrt{\frac{\xi_e^2}{2}} < a \leq \sqrt{\frac{\xi_e^2+\xi_e^4}{2}}$ and $a \leq \frac{1}{\xi_e}(\int \frac{1}{s^2}d\nu(s))^{-1/2}$, and define $b := \sqrt{2a^2 - \xi_e^2}$. Then $\mathbf{T} \equiv (T_1, T_2)$ satisfies conditions (i)-(vii), is hyponormal, and is not subnormal.*

Corollary 5.4. *(Dritschel-McCullough) Let* $dv(s) := 3t^2 ds$ *on* $[0,1]$ *and choose* $a = \sqrt{\frac{1}{2}}$ *and* $b = \sqrt{\frac{1}{3}}$. *Then* $\mathbf{T}$ *is commuting, has subnormal components, is hyponormal, but is not subnormal.*

6. An Instance When Hyponormality Suffices

Lemma 6.1. *Let* ν *be a probab. measure on* $[0,1]$, *and* $\gamma_n \equiv \gamma_n(\nu) := \int s^n d\nu(s)$ $(n \geq 0)$. *The sequence* $\{\gamma_n\}_{n=0}^{\infty}$ *is bounded below iff* ν *has an atom at* $\{1\}$.



$shift(\xi_0, \xi_1, \cdots)$ subnormal contraction with Berger measure ν, $y \leq 1$

- $T_1 T_2 = T_2 T_1$
- T_1 is subnormal
- T_2 subn. $\Leftrightarrow \frac{y}{\sqrt{\gamma_n}} \leq 1$ (all $n \geq 0$) $\Leftrightarrow y^2 \leq \gamma_n$ (all $n \geq 0$) $\Rightarrow \nu(\{1\}) > 0$.

Theorem 6.2. $\mathbf{T}$ *is hyponormal* $\Leftrightarrow$ $\mathbf{T}$ *is subnormal.*

7. A Measure-Theoretic Perspective

For 2-variable weighted shifts, the existence of a normal lifting is equivalent to the existence of a positive regular Borel probability measure on $\mathbb{R}_+^2$ (the so-called Berger measure for the pair), interpolating the moments generated by the weight sequences. In turn, each of the subnormal components of the pair comes equipped with a countable family of canonical probability measures supported in the nonnegative real axis $\mathbb{R}_+$, obtained as the solution of power moment problems with data directly linked to the actual weights. From this perspective, the Lifting Problem consists of "compatibly gluing together" the individual measures on $\mathbb{R}_+$ to produce a measure μ on $\mathbb{R}_+^2$ such that μ satisfies the required properties to be the Berger measure of the pair.

Using techniques from the theory of disintegration of measures, we have generated interesting pathology associated with bringing together the Berger measures associated to each individual weighted shift. Our study reveals some significant obstructions to the Lifting Problem for Commuting Subnormals, while at the same time producing new necessary conditions for the existence of the lifting.

We will first establish a new necessary condition for the existence of a (joint) Berger measure μ: for each $j \geq 0$, $\xi_{j+1} \ll \xi_j$ and, similarly, for each $i \geq 0$, $\eta_{i+1} \ll \eta_i$, where ξ_j (resp. η_i) is the Berger measure of the j-th horizontal slice of T_1 (resp. the i-th vertical slice of T_2). We do this by properly identifying these 1-variable measures as the marginal components of a family of 2-variable measures associated with μ. We then obtain several families of counterexamples to Conjecture 2.3, by showing that the new necessary condition is not sufficient. These new counterexamples are structurally different from those presented before, in that they emphasize the measure-theoretical aspects of the Lifting Problem.

8. A Brief Account of Some Basic Results in the Theory of Disintegration of Measures

Lemma 8.1. *Given two* 1*-variable weight sequences* α *and* β*, the* 2*-variable weighted shift* $(W_\alpha \bigotimes I, I \bigotimes W_\beta)$ *is always subnormal, with Berger measure* $\mu := \xi_\alpha \times \xi_\beta$.

Definition 8.2. *Given a measure* μ *on* $X \times Y$*, the marginal measure* μ^X *is given by* $\mu^X := \mu \circ \pi_X^{-1}$*, where* $\pi_X : X \times Y \to X$ *is the canonical projection onto* X*. Thus,* $\mu^X(E) = \mu(E \times Y)$*, for every* $E \subseteq X$*. Observe that if* μ *is a probability measure, then so is* μ^X.

Lemma 8.3. *Let* μ *be the Berger measure of a* 2*-variable weighted shift* $T \equiv (T_1, T_2)$*, and let* ξ *be the Berger measure of* $shift(\alpha_{00}, \alpha_{10}, ...)$*. Then* $\xi = \mu^X$*. As a consequence,* $\iint f(s)\ d\mu(s,t) = \int f(s)\ d\mu^X(s)$ *for all* $f \in C(X)$.

Corollary 8.4. *Let μ be the Berger measure of a 2-variable weighted shift $T \equiv (T_1, T_2)$. For $j \geq 0$, let $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$. Then the Berger measure of $shift(\alpha_{0j}, \alpha_{1j}, ...)$ is $\xi_j \equiv \mu_j^X$. In particular, the Berger measure of $shift(\alpha_{00}, \alpha_{10}, ...)$ is μ^X.*

Example 8.5. Let $\mu := \xi \times \eta$ be a probability product measure on $X \times Y$. Then $\mu^X = \xi$.

Lemma 8.6. *Let μ and ω be two measures on $X \times Y$, and assume that $\mu \ll \omega$. Then $\mu^X \ll \omega^X$.*

Let μ be the Berger measure of a 2-variable weighted shift $T \equiv (T_1, T_2)$. Although Corollary 8.4 indicates how to obtain the Berger measure ξ_j of the horizontal j-th slice of T_1 in terms of μ, the description is not completely satisfactory, in that it may not be easy to find the marginal measures μ_j ($j \geq 0$). We will now employ disintegration of measure techniques to give a precise description of ξ_j. First, we need to review some basic concepts and general results about disintegration of measures; most of the discussion is taken from Conway's book.

Let X and Z be compact metric spaces and let μ be a positive regular Borel measure on Z. For $\phi : Z \to X$ a Borel mapping, let ν be the Borel measure $\mu \circ \phi^{-1}$ on X; that is,

$$\nu(A) := \mu(\phi^{-1}(A)) \tag{8.1}$$

for all Borel sets A contained in X. Let $\pounds^1(\mu) := \{f : f$ is Borel function on Z such that $\int |f| \, d\mu < \infty\}$, and let $L^1(\mu) := \{[f] : f \in \pounds^1(\mu)\}$, where $[f] := \{g \in \pounds^1(\mu) : \int |f - g| \, d\mu = 0\}$. The map

$$\psi \to \int_Z (\psi \circ \phi) f d\mu \tag{8.2}$$

defines a bounded linear functional on $L^\infty(\nu)$. If attention is restricted to characteristic functions χ_A in $L^\infty(\nu)$,

$$A \to \int_Z (\chi_A \circ \phi) f d\mu = \int_{\phi^{-1}(A)} f d\mu \tag{8.3}$$

is a countably additive measure defined on Borel sets in X, that is absolutely continuous with respect to ν. Hence there is a unique element $E(f)$ in $L^1(\nu)$ such that

$$\int_Z (\chi_A \circ \phi) f d\mu = \int_X \chi_A E(f) d\nu \tag{8.4}$$

for all Borel subsets A of X. By an an approximation argument one can show that

$$\int_Z (\psi \circ \phi) f d\mu = \int_X \psi E(f) d\nu \tag{8.5}$$

for all ψ in $L^\infty(\nu)$. This defines a map

$$E : \pounds^1(\mu) \to L^1(\nu) \tag{8.6}$$

called the expectation operator.

Notation 8.7. The space of all Borel measures on Z will be denoted by $M(Z)$.

Definition 8.8. *A disintegration of the measure μ with respect to ϕ is a function $x \mapsto \Phi_x$ from X to $M(Z)$, such that*
(i) for each x in X, Φ_x is a probability measure;
(ii) if $f \in \pounds^1(\mu)$, $E(f)(x) = \int_Z f d\Phi_x$ a.e. $[\nu]$.

Remark 8.9. In equation (8.5) we can take $f \equiv \chi_A$ for any Borel subset A of Z and $\psi \equiv 1$; the definition of disintegration then implies that

$$\begin{aligned} \int_X \Phi_x(A) d\nu(x) &= \int_X (\int_Z \chi_A d\Phi_x) d\nu(x) \\ &= \int_X E(\chi_A) d\nu = \int_Z (1 \circ \phi) \chi_A d\mu \\ &= \int_{\phi^{-1}(Z)} \chi_A d\mu = \int_X \chi_A d\mu = \mu(A) \end{aligned}$$

Thus disintegration does indeed "disintegrate" the measure into the pieces Φ_x.

Example 8.10. Suppose X is given, ξ is any measure on X, Y is any compact metric space, η is a probability measure on Y, and put

$$Z := X \times Y, \ \phi(x, y) \equiv \pi_X(x, y) := x, \text{ and } \mu := \xi \times \eta.$$

Then $\xi = \mu \circ \phi^{-1}$. Here $E(f)(x) = \int f(x, y) d\eta(y)$ and the disintegration arises by letting

$$\Phi_x(A) := \eta(\pi_Y(A \cap (\{x\} \times Y))).$$

Proposition 8.11. *Let $x \mapsto \Phi_x$ be a disintegration of μ with respect to ϕ. Then supp$\Phi_x = \phi^{-1}(x)$ a.e. $[\nu]$.*

We now list the main theorem on existence and uniqueness of disintegration of measures.

Theorem 8.12. *Given a regular Borel measure μ on a compact metric space Z, and a Borel function ϕ from Z into a compact metric space X, there is a disintegration $x \mapsto \Phi_x$ of μ with respect to ϕ. If $x \mapsto \Phi_x'$ is another disintegration of μ with respect to ϕ, then $\Phi_x = \Phi_x'$ a.e. $[\nu]$.*

9. A new Necessary Condition for the Existence of a Lifting

We are now ready to calculate explicitly the measures ξ_j ($j \geq 0$). Fix $j \geq 0$ and observe that the moments of ξ_j are given by

$$\int_X s^i \, d\xi_j(s) = \alpha_{0j}^2 \cdot ... \alpha_{i-1,j}^2 = \frac{\gamma_{ij}}{\gamma_{0j}} = \frac{1}{\gamma_{0j}} \int \int_R s^i t^j \, d\mu(s, t) \ \text{(all } i \geq 0), \tag{9.1}$$

where $R := X \times Y \equiv [0, a_1] \times [0, a_2]$. Since μ is regular and Borel, we now use Theorem 8.12 to disintegrate μ with respect to $\phi \equiv \pi_X$ and obtain

$$\mu(A) = \int_X \Phi_x(A) \, d\mu^X(x),$$

where as above $\mu^X = \mu \circ \pi_X^{-1}$. Now recall that $supp\Phi_x = \phi^{-1}(x) = \{x\} \times Y$, so with a slight abuse of notation we shall regard Φ_x as a measure on Y and write $d\Phi_x(t)$ for $d\Phi_x(x,t)$. Therefore, for all $i \geq 0$ we have

$$\begin{aligned}
\int\int_R s^i t^j \, d\mu(s,t) &= \int_X (\int\int_R s^i t^j \, d\Phi_x(s,t)) \, d\mu^X(x) \\
&= \int_X (\int\int_{\{x\}\times Y} x^i t^j \, d\Phi_x(x,t)) \, d\mu^X(x) \\
&= \int_X (\int_Y s^i t^j \, d\Phi_s(t)) \, d\mu^X(s). \qquad (9.2)
\end{aligned}$$

Combining (9.1) and (9.2) we obtain

$$\begin{aligned}
\int_X s^i \, d\xi_j(s) &= \frac{1}{\gamma_{0j}} \int\int_R s^i t^j \, d\mu(s,t) \\
&= \frac{1}{\gamma_{0j}} \int_X (\int_Y s^i t^j \, d\Phi_s(t)) \, d\mu^X(s) \\
&= \int_X s^i \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t)\} \, d\mu^X(s) \ \ (\text{all } i \geq 0).
\end{aligned}$$

Thus,

$$d\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t)\} \, d\mu^X(s).$$

We now observe that ξ_j is indeed μ_j^X. First recall that $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$, so the above disintegration of μ with respect to π^X yields $d\mu_j(s,t) = \frac{1}{\gamma_{0j}} t^j d\Phi_s(t) d\mu^X(s)$. For a Borel set $E \subseteq X$, it follows that

$$\begin{aligned}
\mu_j^X(E) &= \mu_j(E \times Y) = \frac{1}{\gamma_{0j}} \int\int_{E\times Y} t^j \, d\Phi_s(t) d\mu^X(s) \\
&= \int_E \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t))\} d\mu^X(s) \\
&= \int_E d\xi_j(s) = \xi_j(E).
\end{aligned}$$

We summarize this in the following result, but first recall

Definition 9.1. *For μ on $X \times Y$, the marginal measure μ^X is given by*

$$\mu^X := \mu \circ \pi_X^{-1},$$

where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus,

$$\mu^X(E) = \mu(E \times Y),$$

for every $E \subseteq X$. If μ is a probability measure, then so is μ^X.

Theorem 9.2. *Let μ be the Berger measure of a subnormal 2-variable weighted shift, and for $j \geq 0$ let ξ_j be the Berger measure of the associated j-th horizontal 1-variable weighted shift $W_{\alpha^{(j)}}$. Then $\xi_j = \mu_j^X$ (cf. Definition 4.6), where $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$; more precisely,*

$$d\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j \; d\Phi_s(t)\} \; d\mu^X(s),$$

where $d\mu(s,t) = d\Phi_s(t) \; d\mu^X(s)$. A similar result holds for the Berger measure η_i of the associated i-th vertical 1-variable weighted shifts $W_{\beta^{(i)}}$ $(i \geq 0)$.

We next need an elementary result.

Lemma 9.3. *Let μ and ν be two regular Borel measures on R, and assume that $\mu \ll \nu$. Then $\mu^X \ll \nu^X$ and $\mu^Y \ll \nu^Y$.*

Theorem 9.4. *Let μ, ξ_j and η_i be as in Theorem 9.2. For every $i, j \geq 0$ we have*

$$\xi_{j+1} \ll \xi_j \tag{9.3}$$

and

$$\eta_{i+1} \ll \eta_i. \tag{9.4}$$

Proof. Straightforward from Theorem 9.2 and Lemma 9.3. □

10. The New Necessary Condition is Not Sufficient

Definition 10.1. *Let μ and ν be two positive measures on R_+. We say that $\mu \leq \nu$ on $X := R_+$, if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq R_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on R_+.*

Definition 10.2. *Let μ be a probability measure on $X \times Y \equiv R_+^2$ and assume that $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t \| \frac{1}{t} \|_{L^1(\mu)}} d\mu(s,t)$.*

Example 10.3. Let B_+ be the Bergman shift on $\ell^2(Z_+)$ and let $M \equiv M_1 := \bigvee\{e_1, e_2, ...\}$. The shift $B_+|_{\mathcal{M}}$ is subnormal, with Berger measure $d\mu(t) := tdt$ on $[0,1]$. Then $d\mu_{ext}(t) = dt$, so the extremal measure μ_{ext} is the Berger measure of B_+.

Lemma 10.4. *Let μ and ω be two measures on $X \times Y$, and assume that $\mu \leq \omega$. Then $\mu^X \leq \omega^X$.*

Proposition 10.5. *(Subnormal backward extension of a 2-variable weighted shift) For a 2-variable weighted shift $\mathbf{T}$, consider $\mathbf{T}|_{\mathcal{M}}$, where $\mathcal{M}$ is the subspace associated to indices $\mathbf{k}$ with $k_2 \geq 1$. Assume that $T_{\mathcal{M}}$ is subnormal with associated measure $\mu_{\mathcal{M}}$ and that $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with associated measure ξ_0. Then $\mathbf{T}$ is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\beta_{00}^2 \leq (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \xi_0$.
Moreover, if $\beta_{00}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} = 1$*, then* $(\mu_{\mathcal{M}})_{ext}^X = \xi_0$*. In the case when* $\mathbf{T}$ *is subnormal, the Berger measure* μ *of* $\mathbf{T}$ *is given by*

$$\begin{aligned} d\mu(s,t) &= \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &\quad + (d\xi_0(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)) d\delta_0(t). \end{aligned}$$

Corollary 10.6. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *and* $\mathcal{M}$ *be as in Proposition 4.11, and assume that* $\mathbf{T}_{\mathcal{M}}$ *is subnormal with Berger measure* $\delta_1 \times \eta$*. Assume further that* T_1 *and* T_2 *are contractions, that* $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with associated measure* ξ_0*, and that* T_2 *is subnormal. Then* $\mathbf{T}$ *is subnormal.*

One might wonder what happens if one insists on having all measures ξ_j and η_i 1-atomic. The next result shows that the lifting is not guaranteed in that case either; as a matter of fact, it is even possible for the pair to fail to be hyponormal.

Proposition 10.7. *Consider the 2-variable weighted shift given by the weight diagram in Figure 5, where* $\alpha < 1$*. Then* $\mathbf{T} \equiv (T_1, T_2)$ *is commuting, each of* T_1 *and* T_2 *is subnormal, and all horizontal and vertical marginal measures* ξ_j *and* η_i $(i, j \geq 0)$ *are 1-atomic, but* $\mathbf{T}$ *is not hyponormal.*

Proposition 10.8. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be the 2-variable weighted shift in Figure 6. Then*
(i) $\mathbf{T}$ *is hyponormal* $\Leftrightarrow 1 - 2x^2 + y^2 \geq 0$
(ii) $\mathbf{T}$ *is subnormal* $\Leftrightarrow 1 - 2x^2 + x^2y^2 \geq 0$.
As a consequence, for $(x, y) \in \mathbb{R}_+^2$ *such that* $1 - 2x^2 + x^2y^2 < 0 \leq 1 - 2x^2 + y^2$, $\mathbf{T}$ *is hyponormal but not subnormal (cf. Figure 7 below).*

We now allow the 0-th horizontal Berger measure ξ_0 to be 3-atomic. The following example shows that it is still possible for T_1 and T_2 to be commuting subnormals, for the pair $T \equiv (T_1, T_2)$ to be hyponormal, for the marginal measures to satisfy the conditions

$$\cdots \ll \mu_{\alpha_2} \ll \mu_{\alpha_1} \ll \mu_{\alpha_0}$$

and

$$\cdots \ll \mu_{\beta_2} \ll \mu_{\beta_1} \ll \mu_{\beta_0},$$

and for $\mathbf{T}$ not to be subnormal. We first need some preliminary facts.

Proposition 10.9. *Let*

$$\alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n + \frac{1}{2}}{2^n + 1}}, & \text{if } n \geq 1 \end{cases}. \tag{10.1}$$

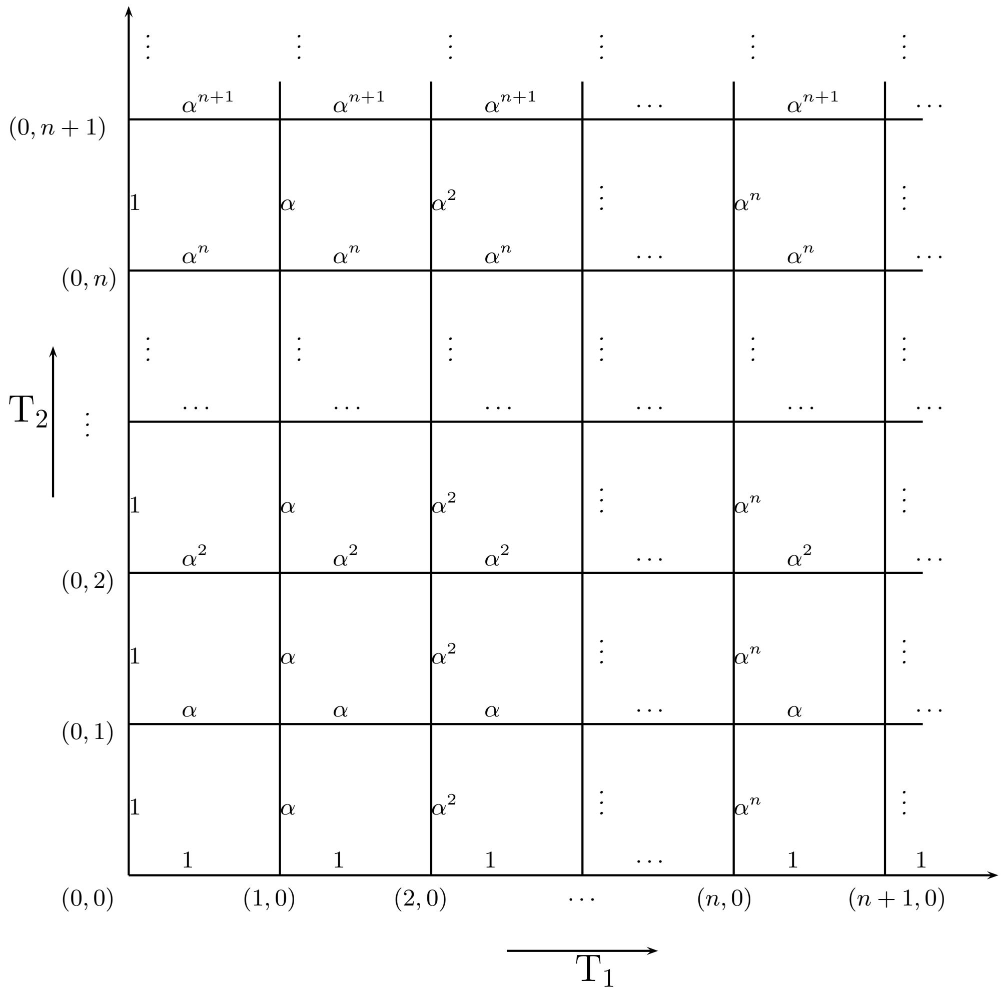


FIGURE 5. Weight diagram the of 2-variable weighted shift in Proposition 10.7

Then W_α is subnormal.

Lemma 10.10. *Let*

$$\widehat{\alpha_n} := \begin{cases} \sqrt{2}, & \text{if } n = 0 \\ \sqrt{\frac{2^n+1}{2^n+\frac{1}{2}}}, & \text{if } n \geq 1 \end{cases},$$

then $\prod_{n=0}^{\infty} \widehat{\alpha_n} = \sqrt{3}$. (Observe that $\widehat{\alpha_n} = \frac{1}{\alpha_n}$, for α_n given by (10.1).)

Proposition 10.11. *Consider the weighted shift $\mathbf{T} \equiv (T_1, T_2)$ in Figure 8,*

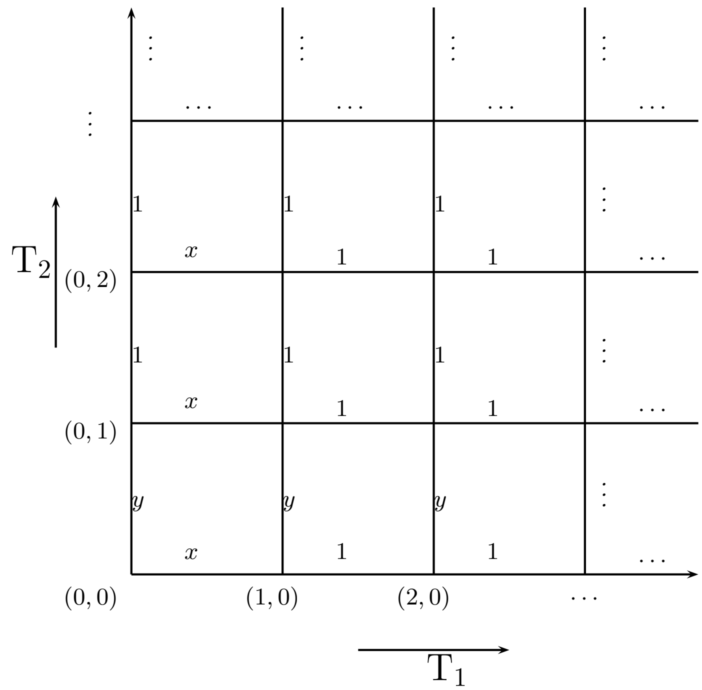


FIGURE 6. Weight diagram of 2-variable weighted shift in Proposition 10.8

where $y < 1$ and $ay \le \frac{1}{\sqrt{3}}$. Let $W_\alpha := shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$, with α_n as in (10.1), i.e.,

$$\alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n + \frac{1}{2}}{2^n + 1}}, & \text{if } n \ge 1 \end{cases}.$$

Then T is subnormal if and only if $y \le \min\{\frac{1}{\sqrt{3(1-a^2)}}, \frac{1}{\sqrt{3}a}\}$.

For the next result, we will need the following lemma.

Lemma 10.12. *Let $\mathbf{T}$ be a commuting 2-variable weighted shift such that*
(i) $\mathbf{T}|_{\vee\{e_{(i,j)}: i \ge 0, j \ge 1\}} \cong (U_+ \bigotimes I, I \bigotimes U_+)$; and
(ii) $\mathbf{T}|_{\vee\{e_{(i,0)}: i \ge 0\}}$ is subnormal.
Then $\mathbf{T}$ is subnormal $\Leftrightarrow$ $\mathbf{T}$ is hyponormal $\Leftrightarrow$ T_2 is subnormal.

Proposition 10.13. *The 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ defined in Proposition 10.11 is hyponormal if and only if $y \le \min\{\sqrt{\frac{2}{5-12a^2+12a^4}}, \frac{1}{\sqrt{3}a}\}$.*

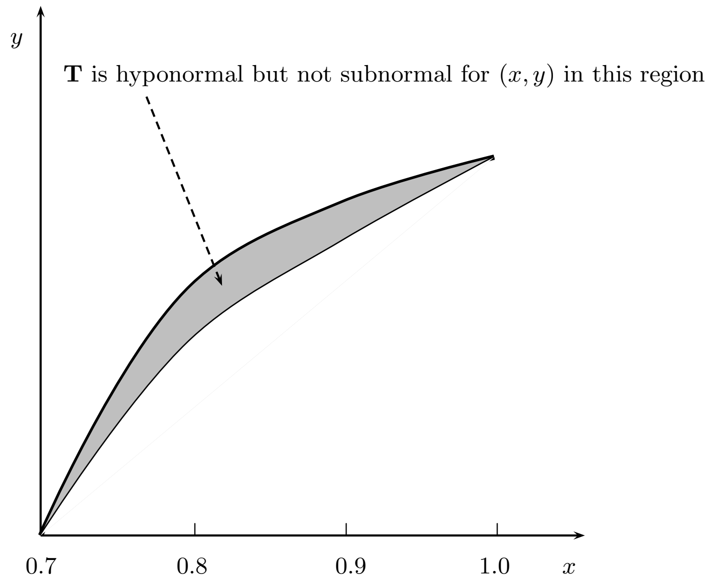


FIGURE 7. Graph of the regions of hyponormality and subnormality in Proposition 10.8

Theorem 10.14. *The 2-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ *defined in Proposition 10.11 is hyponormal and not subnormal if and only if*

$$m_{sub}(a) \quad : \quad = \min\{\frac{1}{\sqrt{3(1-a^2)}}, \frac{1}{\sqrt{3}a}\} < y \qquad (10.2)$$
$$\leq \quad \min\{\sqrt{\frac{2}{5-12a^2+12a^4}}, \frac{1}{\sqrt{3}a}\} =: m_{hyp}(a).$$

Remark 10.15. Observe that for $0 < a < 1$, the left-hand side of (10.2) in Theorem 10.14 is strictly less than the right-hand side, so there is indeed a nonempty range of values for y that guarantees that T is hyponormal but not subnormal. For, if we let $f(a) := \frac{1}{\sqrt{3(1-a^2)}}$, $g(a) := \sqrt{\frac{2}{5-12a^2+12a^4}}$ and $h(a) := \frac{1}{\sqrt{3}a}$, we can analyze the relative sizes of f, g and h to detect the set of points

$$\{a \;\in\; (0,1] : m_{sub}(a) \equiv \min\{f(a), h(a)\}$$
$$< \; \min\{g(a), h(a)\} \equiv m_{hyp}(a)\}$$

(cf. Figure 9 below).

Here $a_1 = \frac{\sqrt{3-\sqrt{\frac{7}{3}}}}{2} \cong 0.607$, $a_2 = \frac{\sqrt{2}}{2} \cong 0.707$ and $a_3 = \frac{\sqrt{1+\sqrt{\frac{7}{3}}}}{2} \cong 0.795$ are the a-coordinates of the points of intersection between the graphs of g and h, f and h, and f and g, respectively. Thus,

$$m_{sub}(a) = \begin{cases} f(a) & \text{if } 0 < a \leq a_2 \\ h(a) & \text{if } a_2 < a < 1 \end{cases}$$

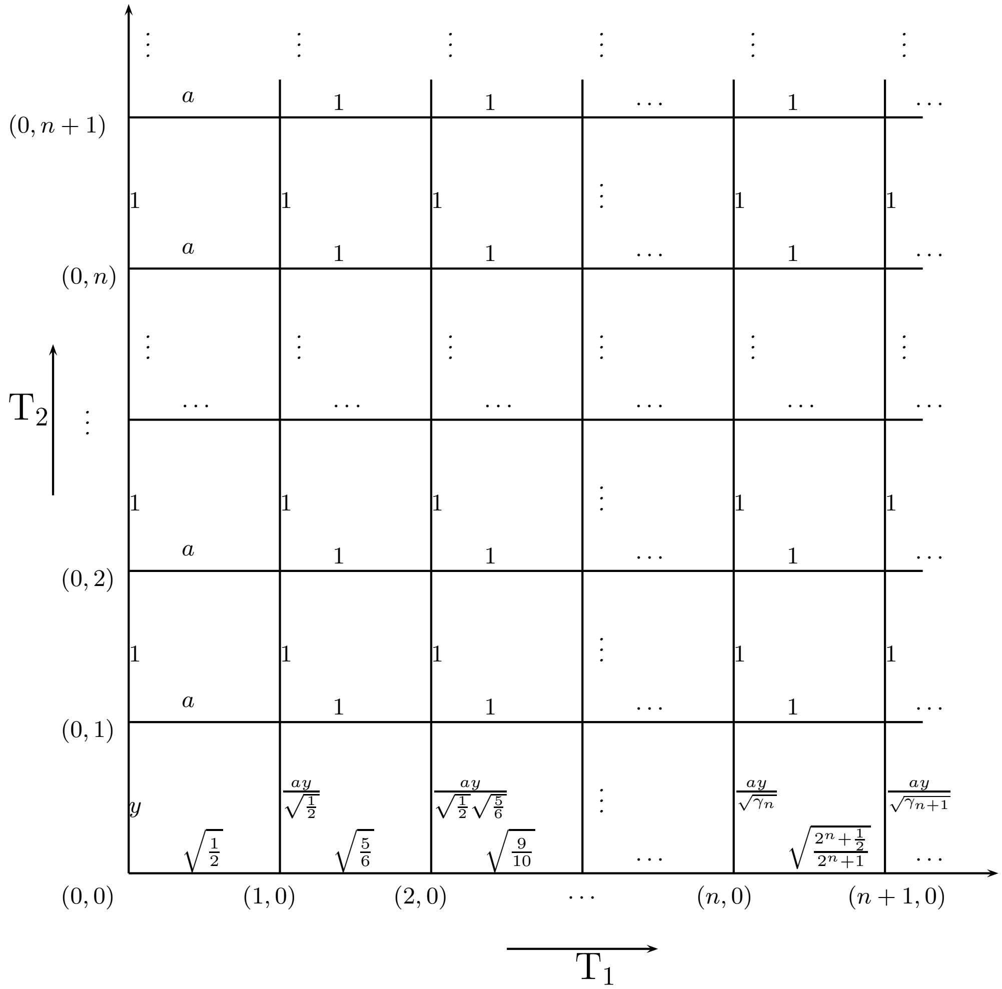


FIGURE 8. Weight diagram of 2-variable weighted shift in Proposition 10.11

and

$$m_{hyp}(a) = \begin{cases} g(a) & \text{if } 0 < a \leq a_1 \\ h(a) & \text{if } a_1 < a < 1 \end{cases} .$$

It easily follows that $m_{sub}(a) < m_{hyp}(a)$ precisely when $0 < a < a_2$. For this a-interval one can always build a 2-hyponormal weighted shift $T \equiv T(a, y)$ with the required properties.

In the following result, we strengthen considerably the necessary conditions (9.3) and (9.4), that is, we require that the Berger measures of all horizontal and vertical slices be mutually absolutely continuous, but that still does not suffice to yield (joint) subnormality.

Proposition 10.16. *Consider the following 2-variable weighted shift (see Figure 10).*

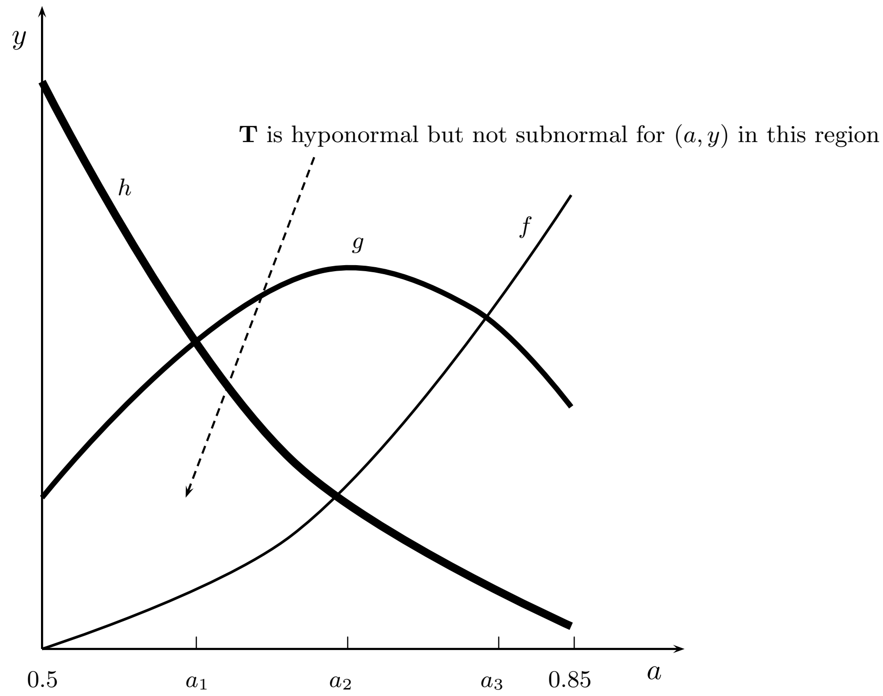


FIGURE 9. Graphs of f, g and h on the interval $[0.5, 0.85]$, showing $a_1, a_2 = \frac{\sqrt{2}}{2}$ and a_3.

Then $\mathbf{T} \equiv (T_1, T_2)$ *is commuting, hyponormal, with each of* T_1 *and* T_2 *subnormal, and* $\xi_{j+1} \approx \xi_j$ $(j \geq 0)$ *and* $supp \eta_i = \{0, 1\}$ $(i \geq 0)$*, but* $\mathbf{T}$ *is not subnormal.*

Definition 10.17. *Let* $B_+^{(\ell)} := \{shift(\sqrt{\ell - \frac{1}{n+2}}) : \ell \geq 1, n \geq 0\}$*; in particular,* $B_+^{(1)} \equiv B_+ := shift(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots)$*. This is the class of Bergman-like weighted shifts.*

Theorem 10.18. *(RC, Y.T. Poon and J. Yoon, 2004) All Bergman-like weighted shifts* $B_+^{(\ell)}$ $(\ell \geq 1)$ *are subnormal.*

Proof of Proposition 10.16. The Berger measure of $shift(\alpha_{00}, \alpha_{10}, ...)$ is $d\xi_0(s) = \frac{ds}{\pi\sqrt{2s-s^2}}$ on $[0, 2]$, and of course $d\xi_j(s) = ds$ for all $j \geq 1$. Thus, $\xi_{j+1} \approx \xi_j$ for all $j \geq 0$. Moreover, $\eta_i = \delta_1$ for all $i \geq 0$. Thus, T_1 and T_2 are subnormal. We will show later that the spectral picture of $\mathbf{T}$ fails to fulfill the description in (RC-K. Yan, JFA, 1995) of the spectral picture corresponding to a subnormal 2-variable weighted shift. It then follows that $\mathbf{T}$ cannot be subnormal.

We now use the Six-point Test to show that $\mathbf{T}$ is hyponormal. In this case

$$H(\mathbf{0}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{1}{2}} \\ -\frac{1}{2}\sqrt{\frac{1}{2}} & \frac{1}{2} \end{pmatrix} \geq 0,$$

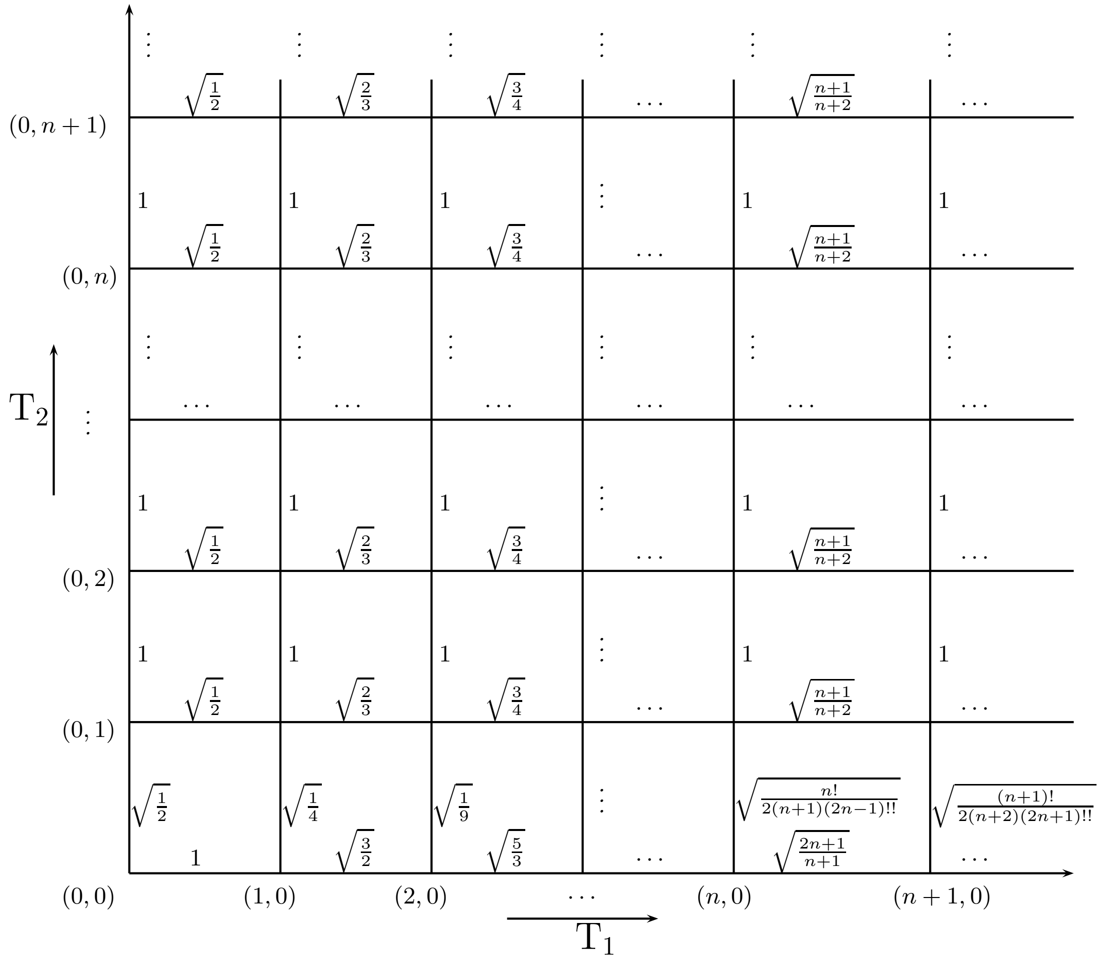


FIGURE 10. Weight diagram of the 2-variable weighted shift in Proposition 10.16

and for $n \geq 1$, $H((n,0)) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where

$$\begin{aligned} a &:= \frac{2n+3}{n+2} - \frac{2n+1}{n+1} \\ b &:= \sqrt{\frac{n+1}{n+2}}\sqrt{\frac{(n+1)!}{2(n+2)(2n+1)!!}} \\ &\quad -\sqrt{\frac{2n+1}{n+1}}\sqrt{\frac{n!}{2(n+1)(2n-1)!!}} \\ c &:= 1 - \frac{n!}{2(n+1)(2n-1)!!}. \end{aligned}$$

After simplification, we have

$$\det H((n,0)) = \frac{2(2+5n+2n^2)(2n-1)!!-(3+8n+5n^2+n^3)n!}{2(n+1)(n+2)^2(2n+1)!!} \geq 0 \text{ (all } n \geq 1).$$

It follow that **T** is hyponormal. □

11. The Spectral Picture of Subnormal 2-variable Weighted Shifts

The 1-variable case. Recall that the spectrum of a unilateral weighted shift W_α is always a closed disk centered at the origin, of radius $r(W_\alpha)$. When W_α is hyponormal, the essential spectrum is the circle centered at the origin, of radius $r(W_\alpha) \equiv \|W_\alpha\|$.

The 2-variable case. For subnormal 2-variable weighted shifts, RC-K. Yan gave in 1995 a complete description of the spectral picture, by exploiting the groupoid machinery in Muhly-Renault and RC-Muhly, and the presence of a Berger measure, which they use to analyze the asymptotic behavior of sequences of weights.

Notation 11.1. μ : compactly supported finite positive Borel measure on $\mathbb{C}^n$ $(n \geqslant 1)$
$P^2(\mu)$: norm closure in $L^2(\mu)$ of $\mathbb{C}[z_1, \cdots, z_n]$
$M_z \equiv M_z^{(\mu)} := (M_{z_1}^{(\mu)}, \cdots, M_{z_n}^{(\mu)})$: multiplication operators acting on $P^2(\mu)$
M_z on $P^2(\mu)$ is the universal model for cyclic subnormal n-tuples of operators

Definition 11.2. *(i)* $E \subseteq \mathbb{C}^n$ *is Reinhardt if for every* $z \in E$ *and every* $\theta \in \mathbb{R}^n$, $e^{i\theta}z := (e^{i\theta_1}z_1, \cdots, e^{i\theta_n}z_n) \in E$.
(ii) μ *is Reinhardt if* $\mu(e^{i\theta}E) = \mu(E)$ *for every Borel subset* $E \subseteq \mathbb{C}^n$ *and every* $\theta \in \mathbb{R}^n$
(suppμ *is always a Reinhardt set)*
(iii) For μ *Reinhardt,*

$$b.p.e(\mu) = \{\lambda \in \mathbb{C}^n : p \to p(\lambda),\ p \in \mathbb{C}[z], \text{ extends to a bounded point evaluation from } P^2(\mu) \text{ to } \mathbb{C}.\}$$

(iv) The kernel function associated with μ *is*

$$K(z,w) \equiv K^{(\mu)}(z,w) := \sum_\alpha \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|^2},$$
$$\alpha \equiv (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{Z}_+^n,\ z \text{ and } z \in \mathbb{C}^n.$$

(v) The set of convergence of K *is*

$$\mathcal{C}(K) := \{\lambda \in \mathbb{C}^n : K(\lambda,\lambda) < \infty\}$$

(vi) The (joint) point spectrum of $M_z^{(\mu)*}$ *is*

$$\sigma_p(M_z^{(\mu)*}) = \sigma_p(M_z^*) := \{\lambda \in \mathbb{C}^n : M_z^* - \overline{\lambda} \text{ has a joint eigenvector}\}.$$

Proposition 11.3. *(RC-K. Yan)*$\sigma_p(M_z^{(\mu)*})^* = b.p.e(\mu) = C(K)$.

Corollary 11.4. *(RC-K. Yan) Let μ be a Reinhardt measure. Then $\overline{\sigma_p(M_z^{(\mu)*})}$ is polynomially convex.*

Theorem 11.5. *(RC-K. Yan) Let μ be a Reinhardt measure on C^n and let $K := supp\mu$. Then $int.\widehat{K} \subseteq b.p.e.(\mu)$.*

Lemma 11.6.

Theorem 11.7. *(RC-K. Yan, 1995) Let μ be a Reinhardt measure on $\mathbb{C}^2$, and let $C := \log\left|\widehat{K}\right|$. Then*
(i) $\partial\widehat{K} \supseteq \sigma_\ell(M_z, P^2(\mu)) = \sigma_{\ell_e}(M_z, P^2(\mu)) \supseteq (\exp(\partial^0 C \times \mathbb{T}^2))^-$
(i') If, in addition, μ is well-behaved, then $\sigma_{\ell_e}(M_z,, P^2(\mu)) = (\exp(\partial^0 C \times \mathbb{T}^2))^-$.
(ii) $\sigma_T(M_z, P^2(\mu)) = \sigma_r(M_z, P^2(\mu)) = \widehat{K}$
(iii) $\sigma_{Te}(M_z, P^2(\mu)) = \sigma_{r_e}(M_z, P^2(\mu)) = \partial\widehat{K}$
(iv) $int.\widehat{K} \subseteq \sigma_p(M_z, P^2(\mu)) = b.p.e.(\mu) \subseteq \widehat{K}$
(v) $index(M_z - \lambda) = \begin{cases} 1 & if\ \lambda \in int.(\widehat{K}) \\ 0 & if\ \lambda \notin int.(\widehat{K}) \end{cases}$
(vi) $kerD^1_{M_z-\lambda} = ranD^0_{M_z-\lambda}$ *for all* $\lambda \in int.(\widehat{K})$.

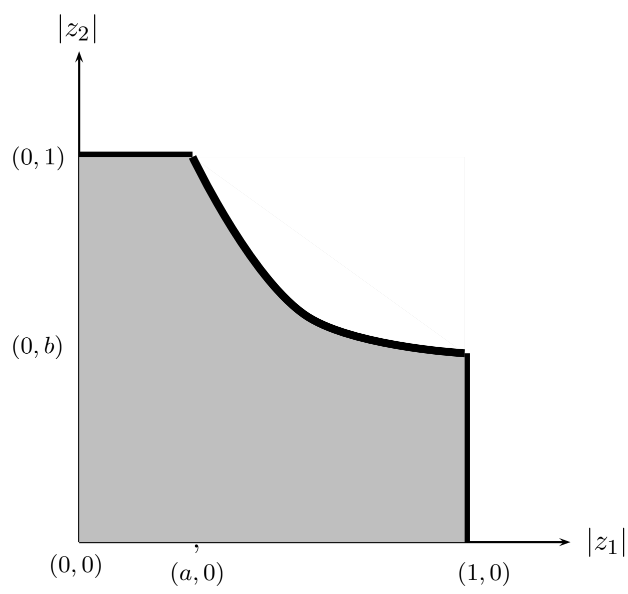


FIGURE 11. Graph of the Taylor spectrum of a typical subnormal 2-variable weighted shift

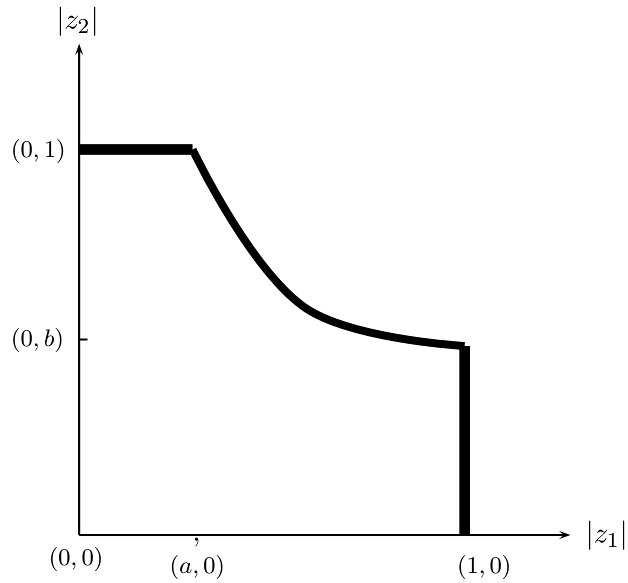


FIGURE 12. Graph of the Taylor essential spectrum of a typical subnormal 2-variable weighted shift

12. The Spectral Picture of the Shift in Proposition 10.16

Theorem 12.1. *Consider the following* 2*-variable weighted shift (see Figure 10). Then*

$$\overline{\mathcal{C}(K)} = (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \{0\}),$$

where $\overline{\mathcal{C}(K)}$ *means closure of* $C(K)$.

Corollary 12.2. $\overline{\mathcal{C}(K)} \subsetneq \sigma_r(T_1, T_2)$.

Theorem 12.3. $\sigma_T(T_1, T_2) = \sigma_r(T_1, T_2) = \sigma_H(T_1, T_2) = (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \cup (\sqrt{2} \cdot \overline{\mathbb{D}} \times \{0\}) \neq \overline{\mathcal{C}(K)}$

Lemma 12.4. $(0, y) \nsubseteq \sigma_\ell(T_1, T_2)$, *for* $0 < y \leq 1$.

Theorem 12.5.

$$\sigma_{\ell_e}(T_1, T_2) = (\{1\} \times \mathbb{D}) \cup (\sqrt{2} \cdot \mathbb{T} \times \{0\})$$

and

$$\sigma_{r_e}(T_1, T_2) = (\mathbb{D} \times \{1\}) \cup (\sqrt{2} \cdot \mathbb{T} \times \{0\}).$$

Corollary 12.6. $\sigma_\ell(T_1, T_2) = \sigma_{\ell_e}(T_1, T_2)$.

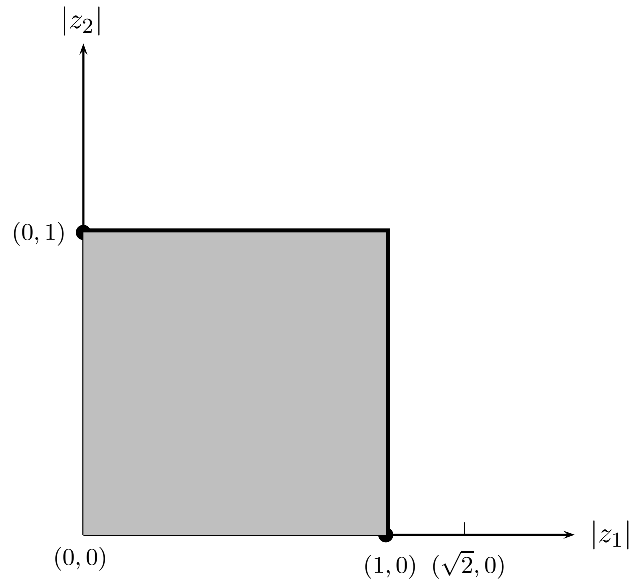


FIGURE 13. Graph of $\mathcal{C}(K)$ for the 2-variable weighted shift of Example 10.16

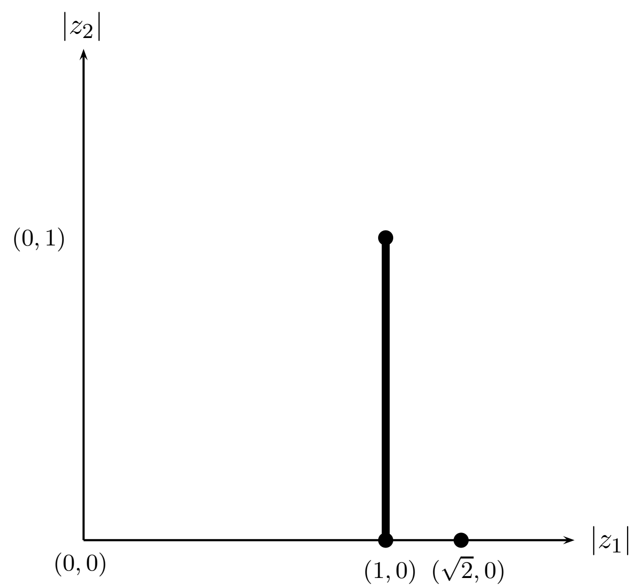


FIGURE 14. Graph of the left essential spectrum of the 2-variable weighted shift of Example 10.16

Theorem 12.7. $\sigma_{Te}(T_1,T_2) = \sigma_{He}(T_1,T_2) = (\sqrt{2}\mathbb{T},0) \cup ((\mathbb{T}\times\overline{\mathbb{D}}) \cup (\overline{\mathbb{D}}\times\mathbb{T}))$.

Theorem 12.8. *Let* $K := (\frac{1}{\sqrt{2}}\overline{\mathbb{D}}\times\overline{\mathbb{D}}) \cup (\overline{\mathbb{D}}\times\{0\})$. *Then* $\widehat{K} = K$, *where* $\widehat{K}$ *means polynomially convex hull of set* K.

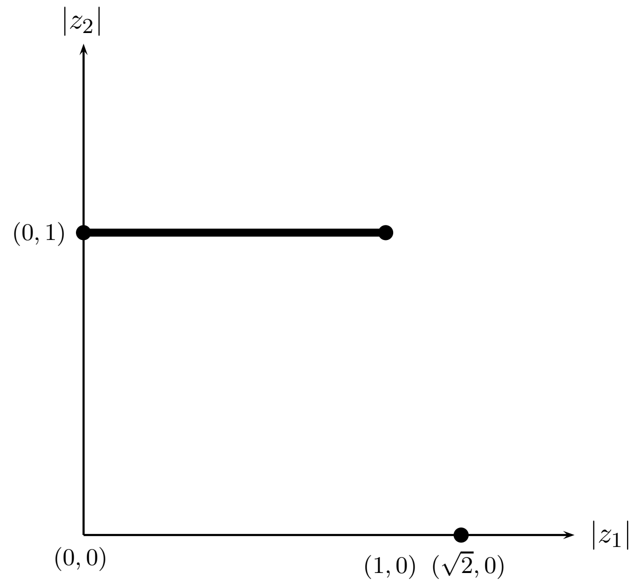


FIGURE 15. Graph of the right essential spectrum of the 2-variable weighted shift of Example 10.16

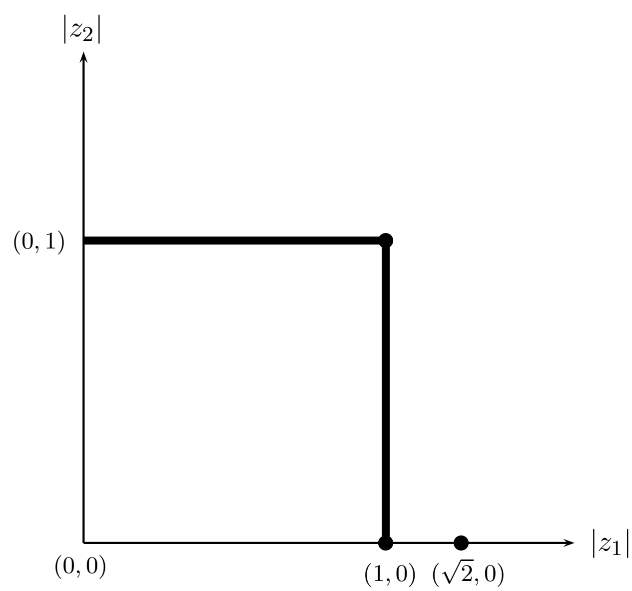


FIGURE 16. Graph of the Taylor essential spectrum of the 2-variable weighted shift of Example 10.16

Corollary 12.9. $(\exp(\partial^0 \log \left|\widehat{K}\right| \times T^2)) = \frac{1}{\sqrt{2}} T \times T$

Corollary 12.10. $(\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \widehat{\cup (\sqrt{2} \cdot \overline{\mathbb{D}}} \times \{0\}) = (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \cup (\sqrt{2} \cdot \overline{\mathbb{D}} \times \{0\})$.

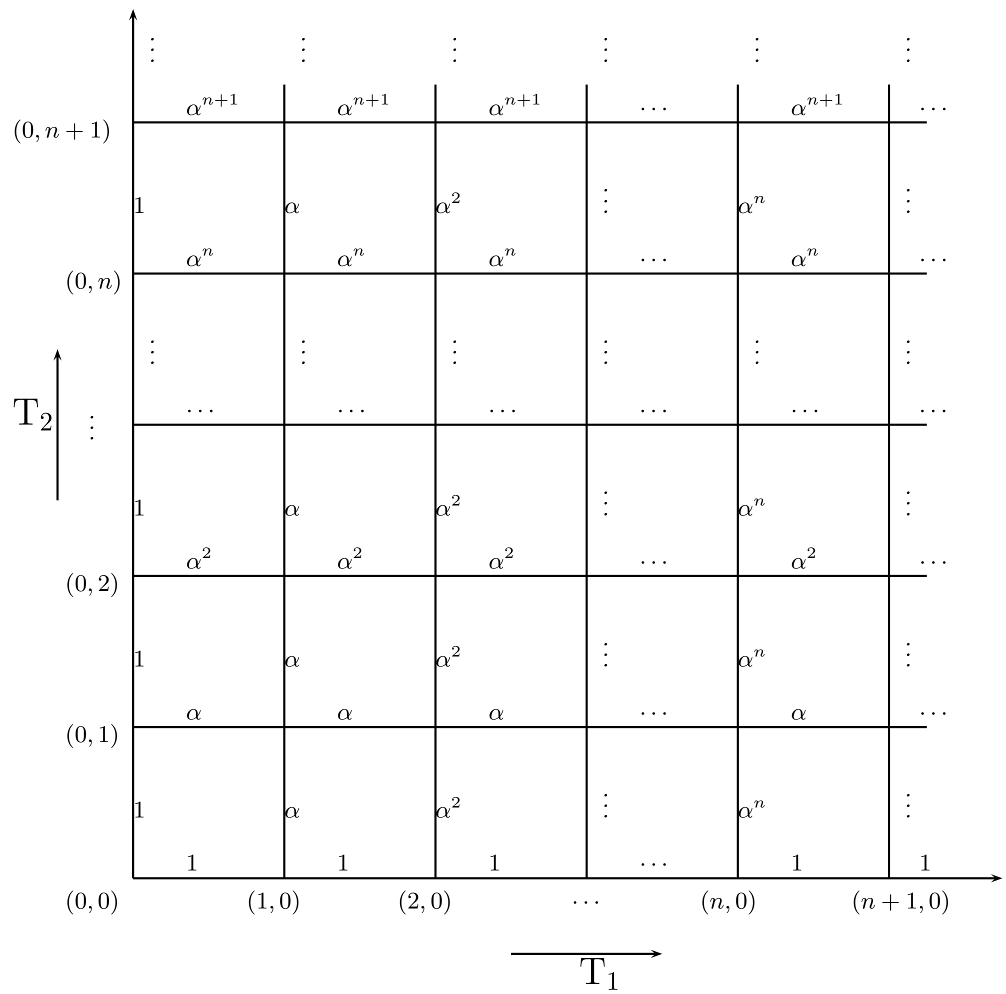


FIGURE 17. Weight diagram the of 2-variable weighted shift in Proposition 10.7

13. The Spectral Picture of the Shift in Proposition 10.7

Proposition 13.1. *Consider the* 2*-variable weighted shift given by the weight diagram in Figure 17, where* $\alpha < 1$. *Then* $\mathbf{T} \equiv (T_1, T_2)$ *is commuting, each of* T_1 *and* T_2 *is subnormal, and all horizontal and vertical marginal measures* ξ_j *and* η_i $(i, j \geq 0)$ *are* 1*-atomic, but* $\mathbf{T}$ *is not hyponormal.*

Theorem 13.2.

$$\sigma_T(\mathbf{T}) = \sigma_r(T_1, T_2) = \overline{\mathcal{C}(K)} = [(|z_1| \leq 1) \times \{0\}] \bigcup [\{0\} \times (|z_2| \leq \beta_0)]$$

and

$$\begin{aligned}\sigma_{Te}(\mathbf{T}) &= \sigma_\ell(T_1,T_2)=\sigma_{r_e}(T_1,T_2)=\sigma_{\ell_e}(T_1,T_2)\\ &= \{(0,0)\}\bigcup\ \{[\bigcup_{k=0}^{\infty}(|z_1|=\alpha^k)]\times\{0\}\}\bigcup\ \{\{0\}\times[\bigcup_{\ell=0}^{\infty}(|z_2|=\beta_0\alpha^\ell)]\}.\end{aligned}$$

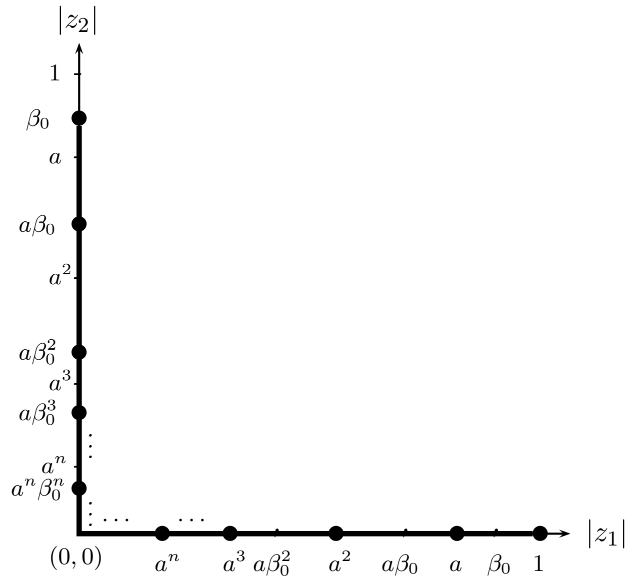


FIGURE 18. Graph of the Taylor spectrum of the 2-variable weighted shift in Theorem 13.2

14. JOINT k-HYPONORMALITY

Problem 14.1. *(Lifting Problem for Commuting Subnormals) Find necessary and sufficient conditions on T_1 and T_2 to guarantee the subnormality of $\mathbf{T}\equiv(T_1,T_2)$.*

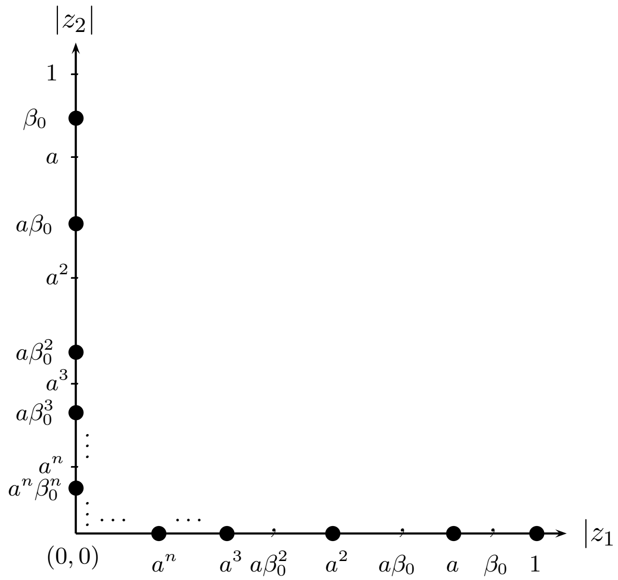
We recall that a pair $\mathbf{S}=(S_1,S_2)$ of commuting subnormal operators is called *polynomially subnormal* if $p(\mathbf{S})$ is subnormal for all 2-variable polynomials $p\in\mathbb{C}[z_1,z_2]$.

- (Franks, 1994) If $p(\mathbf{S})$ is subnormal for all p with $\deg p\le 5$, then $\mathbf{S}$ is subnormal.

Using Franks' result, we can give an abstract answer to Problem 14.1.

Definition 14.2. *A commuting pair $\mathbf{T}\equiv(T_1,T_2)$ is called k-hyponormal if $\mathbf{T}(k):=(T_1,T_2,T_1^2,T_2T_1,T_2^2,\cdots,T_1^k,T_2T_1^{k-1},\cdots,T_2^k)$ is hyponormal, or equivalently*

$$([(T_2^qT_1^p)^*,T_2^nT_1^m])_{\substack{1\le m+n\le k\\1\le p+q\le k}}\ge 0.$$



Proposition 13.3.

FIGURE 19. Graph of the Taylor essential spectrum of the 2-variable weighted shift in Theorem 13.2

Clearly, subnormal $\Rightarrow$ $(k+1)$-hyponormal $\Rightarrow$ k-hyponormal for every $k \geq 1$.

We now present our multivariable version of the Bram-Halmos criterion for subnormality. When combined with Theorem 14.5 below, Theorem 14.3 provides an abstract answer to Problem 14.1, by showing that no matter how k-hyponormal the pair $\mathbf{T}$ might be, it may still fail to be subnormal.

Theorem 14.3. *(Multivariable Version of the Bram-Halmos Criterion) Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting pair of operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent.*
(i) $\mathbf{T}$ is subnormal.
(ii) $\mathbf{T}(k)$ is subnormal for all $k \in \mathbb{Z}_+$.
(iii) $\mathbf{T}$ is k-hyponormal for all $k \in \mathbb{Z}_+$.

In the single variable case, there are useful criteria for k-hyponormality, and for 2-variable weighted shifts, we have the Six-point Test for joint hyponormality. We now present a characterization of k-hyponormality for 2-variable weighted shifts.

Theorem 14.4. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift with weight sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}$. The following statements are equivalent.*
(a) $\mathbf{T}$ is k-hyponormal.
(b) $((T_2^n T_1^m)^* [(T_2^q T_1^p)^*, T_2^n T_1^m](T_2^q T_1^p))_{\substack{1 \leq m+n \leq k \\ 1 \leq p+q \leq k}} \geq 0$.
(c) $(\langle [(T_2^q T_1^p)^*, T_2^n T_1^m] e_{\mathbf{u}+(m,n)}, e_{\mathbf{u}+(p,q)} \rangle)_{\substack{1 \leq m+n \leq k \\ 1 \leq p+q \leq k}} \geq 0$ for all $\mathbf{u} \in \mathbb{Z}_+^2$.
(d) $(\gamma_{\mathbf{u}} \gamma_{\mathbf{u}+(m,n)+(p,q)} - \gamma_{\mathbf{u}+(m,n)} \gamma_{\mathbf{u}+(p,q)})_{\substack{1 \leq m+n \leq k \\ 1 \leq p+q \leq k}} \geq 0$ for all $\mathbf{u} \in \mathbb{Z}_+^2$.

(e) $M_{\mathbf{u}}(k) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{\substack{0\le m+n\le k\\ 0\le p+q\le k}} \geq 0$ *for all* $\mathbf{u} \in \mathbb{Z}_+^2$. *(For a subnormal pair* $\mathbf{T}$*, the matrix* $M_{\mathbf{u}}(k)$ *is the truncation of the moment matrix associated to the Berger measure of* $\mathbf{T}$*.)*

As an application of Theorem 14.4, we build a two-parameter family of 2-variable weighted shifts (see Figure 20 below), and we identify the precise parameter ranges that separate hyponormality from 2-hyponormality, 2-hyponormality from 3-hyponormality, etc., and k-hyponormality from subnormality. We believe these are the first examples in the literature of commuting pairs of subnormal operators which are k-hyponormal but not $(k+1)$-hyponormal.

For $0 < y \leq 1$, let $x \equiv \{x_n\}_{n=0}^{\infty}$ where

$$x_n := \begin{cases} y\sqrt{\frac{3}{4}}, & \text{if } n = 0 \\ \frac{\sqrt{(n+1)(n+3)}}{(n+2)}, & \text{if } n \geq 1. \end{cases}$$

It can be shown that $W_x \equiv shift(x_0, x_1, \cdots)$ is subnormal.

Theorem 14.5. *For* $0 < a \leq \frac{1}{\sqrt{2}}$*, the* 2*-variable weighted shift* $\mathbf{T}$ *given by Figure 20 is*
(i) hyponormal $\Leftrightarrow 0 < y \leq \frac{\sqrt{32-48a^4}}{\sqrt{59-72a^2}}$;
(ii) k-hyponormal $\Leftrightarrow 0 < y \leq \sqrt{\frac{\frac{(k+1)^2}{2k(k+2)}-a^2}{a^4-\frac{5}{2}a^2+\frac{(k+1)^2}{2k(k+2)}+\frac{2k^2+4k+3}{4(k+1)^2}}}$ $(k \geq 2)$;
(iii) subnormal $\Leftrightarrow 0 < y \leq \sqrt{\frac{1}{2-a^2}}$.
In particular, $\mathbf{T}$ *is hyponormal and not subnormal if and only if* $\sqrt{\frac{1}{2-a^2}} < y \leq \frac{\sqrt{32-48a^4}}{\sqrt{59-72a^2}}$.

Remark 14.6. (i) Even for 1-variable weighted shifts, it is generally difficult to provide concrete parameterizations that separate k-hyponormality from $(k+1)$-hyponormality. That we can accomplish the same separation for 2-variable weighted shifts is an indication that the condition in Theorem 14.4(e) is sharp.
(ii) Theorem 14.5 gives a new family of examples, with explicit parameter values to distinguish between k-hyponormality and $(k+1)$-hyponormality, and a fortiori between hyponormality and subnormality.

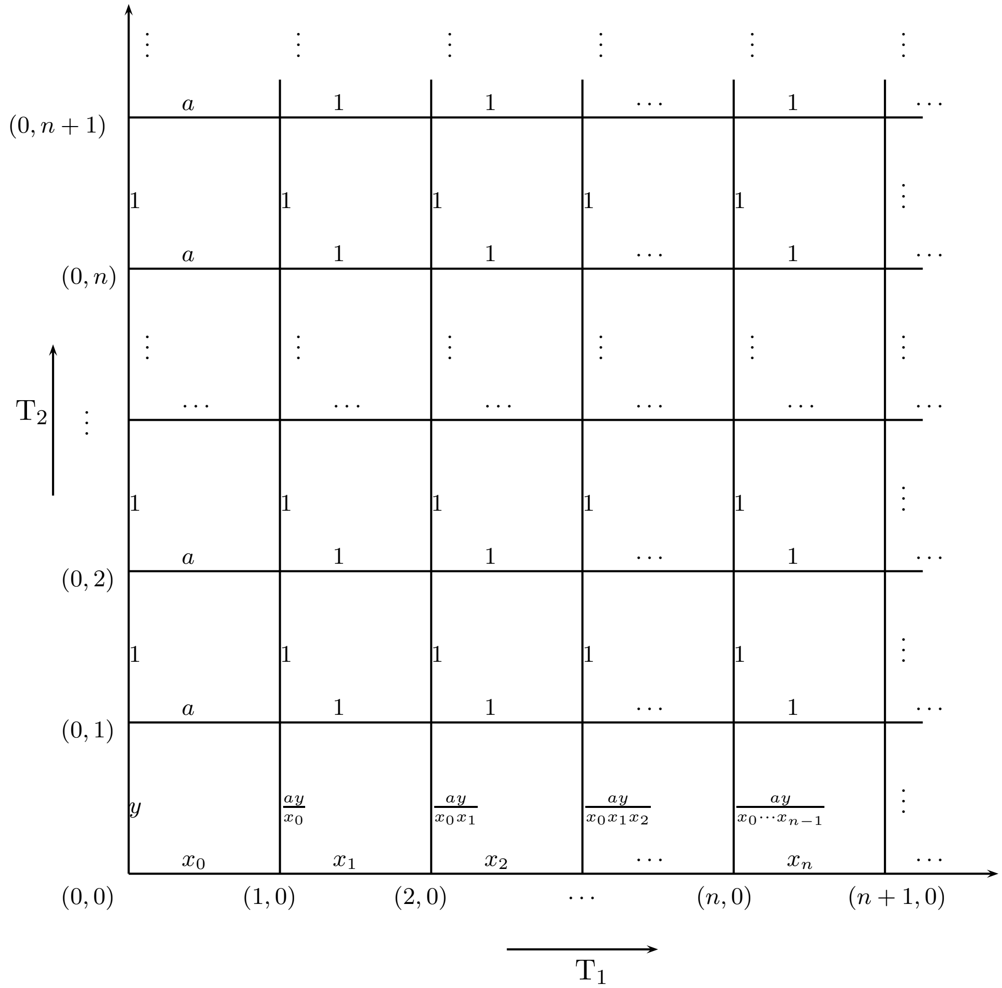


FIGURE 20. Weight diagram of the 2-variable weighted shift in Theorem 14.5

- (15-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then

$$\mathbf{T} \text{ is } 2-\text{hyponormal}$$

$$\Leftrightarrow M_{\mathbf{k}}(2) := (\gamma_{\mathbf{k}+(n,m)+(p,q)})_{\substack{0\le n+m\le 2\\ 0\le p+q\le 2}} \geq 0 \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2)$$

$$\Leftrightarrow \begin{pmatrix} \gamma_{k_1,k_2} & \gamma_{k_1+1,k_2} & \gamma_{k_1,k_2+1} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} \\ \gamma_{k_1+1,k_2} & \gamma_{k_1+2,k_2} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} \\ \gamma_{k_1,k_2+1} & \gamma_{k_1+1,k_2+1} & \gamma_{k_1,k_2+2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3} \\ \gamma_{k_1+2,k_2} & \gamma_{k_1+3,k_2} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+4,k_2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2} \\ \gamma_{k_1+1,k_2+1} & \gamma_{k_1+2,k_2+1} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1+3,k_2+1} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3} \\ \gamma_{k_1,k_2+2} & \gamma_{k_1+1,k_2+2} & \gamma_{k_1,k_2+3} & \gamma_{k_1+2,k_2+2} & \gamma_{k_1+1,k_2+3} & \gamma_{k_1,k_2+4} \end{pmatrix} \geq 0 \text{ (all } \mathbf{k} \in \mathbf{Z}_+^2).$$

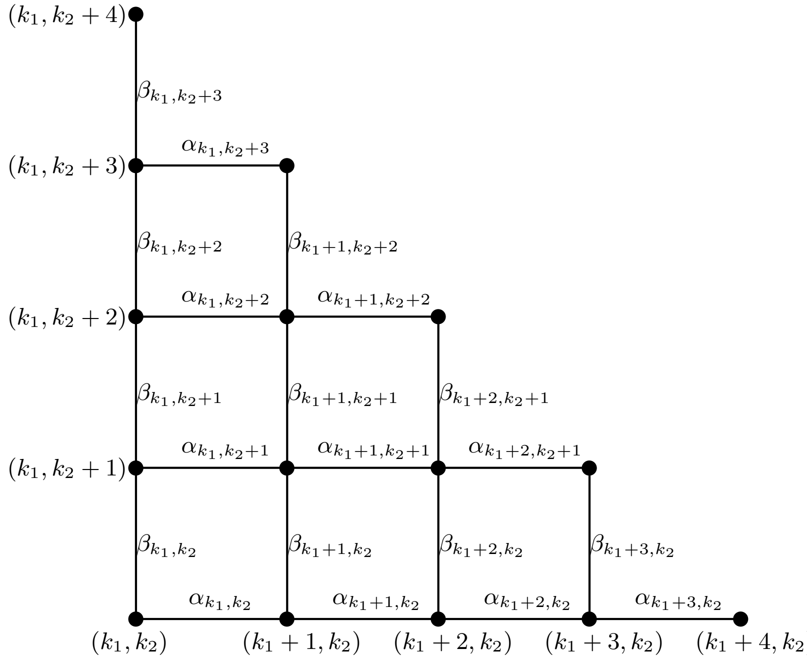


FIGURE 21. Weight diagram used in the 15-point Test

15. PROPAGATION IN THE 2-VARIABLE HYPONORMAL CASE

We show that if a commuting, hyponormal pair $\mathbf{T} \equiv (T_1, T_2)$ with T_1 quadratically hyponormal satisfies $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1, T_2(I \otimes U_+^{k_2-1}))$ is horizontally flat. We also prove that this result is optimal in the following sense: the propagation does not extend either to the left (0-th column) or down (below k_2-th level).

Definition 15.1. *A 2-variable weighted shift $\mathbf{T}$ is horizontally flat (resp. vertically flat) if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$). We say that $\mathbf{T}$ is flat if $\mathbf{T}$ is horizontally and vertically flat (cf. Figure 22), and we say that $\mathbf{T}$ is symmetrically flat if $\mathbf{T}$ is flat and $\alpha_{11} = \beta_{11}$.*

Theorem 15.2. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting, hyponormal 2-variable weighted shift.*
(i) If T_1 is quadratically hyponormal and $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1, T_2(I \otimes U_+^{k_2-1}))$ is horizontally flat.
(ii) If, instead, T_2 is quadratically hyponormal and $\beta_{(k_1,k_2)+\varepsilon_1} = \beta_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1(U_+^{k_1-1} \otimes I), T_2)$ is vertically flat.

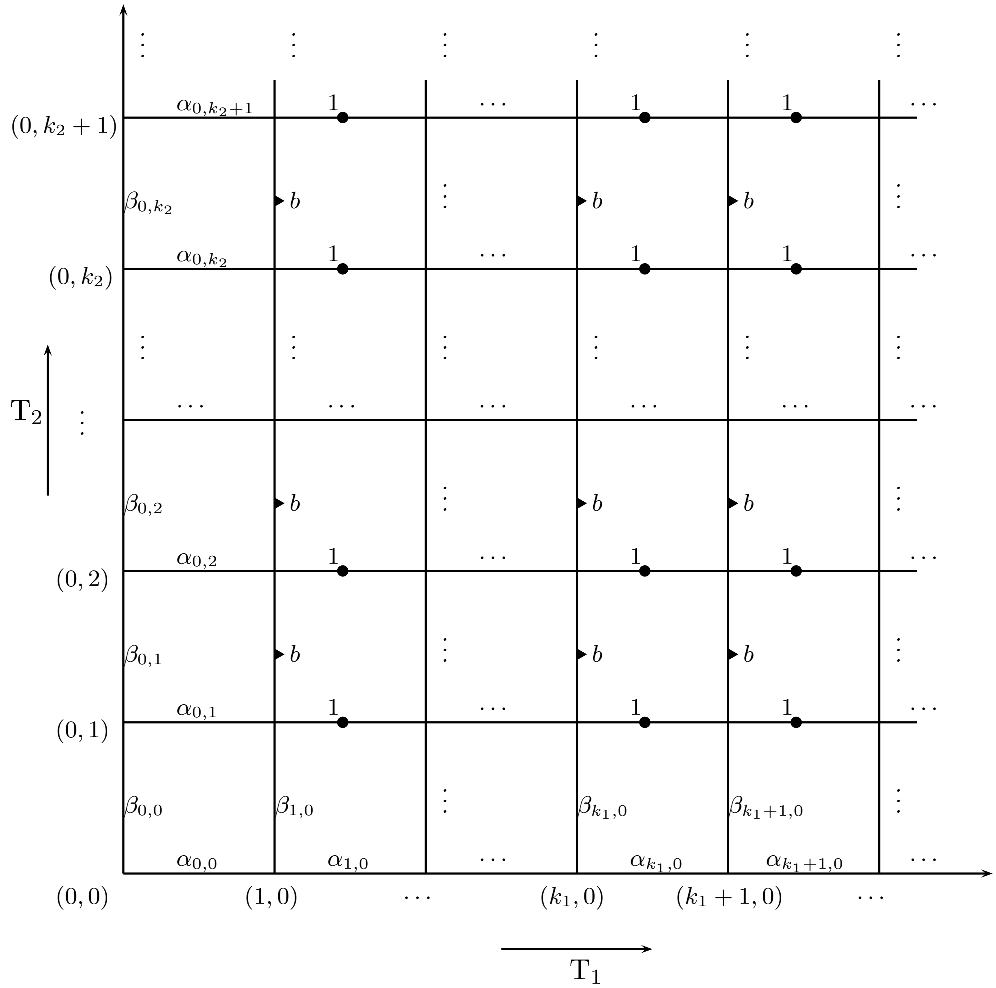


FIGURE 22. Weight diagram of a flat 2-variable weighted shift (with round dots for horizontal flatness, triangular dots for vertical flatness)

Remark 15.3. The proof of Theorem 15.2 shows that for $\mathbf{T} \equiv (T_1, T_2)$ commuting and hyponormal, and for $k_1, k_2 \geq 0$,

$$\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} \Rightarrow \beta_{(k_1,k_2)} = \beta_{(k_1,k_2)+\varepsilon_1} \tag{15.1}$$

Moreover, if $k_2 \geq 1$,

$$\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} \text{ and } \alpha_{(k_1,k_2)+\varepsilon_1-\varepsilon_2} = \alpha_{(k_1,k_2)-\varepsilon_2} \Rightarrow \alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)+\varepsilon_1-\varepsilon_2}.$$

To demonstrate optimality, we recall the class of Bergman-like weighted shifts.

Definition 15.4. *For $\ell \geq 1$, the Bergman-like weighted shift acting on $\ell^2(Z_+)$ is* $B_+^{(\ell)} := shift(\{\sqrt{\ell - \frac{1}{k+2}}) : k \geq 0\})$*; that is,*

$$B_+^{(\ell)} e_k := \sqrt{\ell - \frac{1}{k+2}} e_{k+1} \ \ (k \geq 0).$$

In particular, $B_+^{(1)} \equiv B_+ := shift(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots)$ *is the Bergman shift.*

Remark 15.5. (i) B_+ is subnormal with Berger measure $d\xi(s) := ds$.
(ii) (RC, Y.T. Poon and J. Yoon) $B_+^{(2)}$ is subnormal with Berger measure $d\xi(s) := \frac{sds}{\pi\sqrt{2s-s^2}}$.

Theorem 15.6. *For every $k_2 \geq 1$ there exist*
(i) a family $\{B_+^{(\ell_i)}\}_{i=0}^{k_2-1}$ of Bergman-like weighted shifts,
(ii) a hyponormal weighted shift $W_\beta := shift(\beta_0, \beta_1, \beta_2, \cdots)$ (with $\beta_n < \beta_{n+1}$ for all $n \geq 0$), and
(iii) a constant $\alpha_0 < 1$,
such that the commuting 2-variable weighted shift with a weight diagram whose first k_2 rows are $B_+^{(\ell_0)}, \cdots, B_+^{(\ell_{k_2-1})}$, whose remaining rows are S_{α_0}, and whose 0th column is given by W_β is hyponormal (see Figure 23 for the case $k_2 = 2$).

16. References

- J. Agler, Hypocontractions and subnormality, J Operator Th. 13(1985), 203-217.
- N.I. Akhiezer, The Classical Moment Problem, Hafner Publ. Co., New York, 1965.
- A. Athavale, On joint hyponormality of operators, Proc. Amer. Math. Soc. 103(1988), 417-423.
- K. Atkinson, Introduction to Numerical Analysis, Wiley and Sons, 2nd. Ed. 1989.
- J. Bae, G. Exner and I.B. Jung, Criteria for positively quadratically hyponormal weighted shifts, Proc. Amer. Math. Soc. 130(2002), 3287-3294.
- J. Bak and D.J. Neuman, Complex Analysis, Springer-Verlag, New York, 1982.
- J. Bram, Subnormal operators, Duke Math. J. 22(1955), 75-94.
- J. Bunce, The joint spectrum of commuting nonnormal operators, Proc. Amer. Math. Soc. 29(1971), 499-505.
- Y. Choi, A Propagation of the quadratically hyponormal weighted shifts, Bull. Korean Math. Soc. 37(2000), 347-352.
- J. Conway, The Theory of Subnormal Operators, Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.
- J.B. Conway and W. Szymanski, Linear combination of hyponormal operators, Rocky Mountain J. 18 (1988), 695-705.
- RC, Application of several complex variables to multiparameter spectral theory, Surveys of Some Recent Results in Operator Theory, Vol. II, J.B. Conway and B.B. Morrel, editors, Longman Publishing Co., London (1988), 25-90.

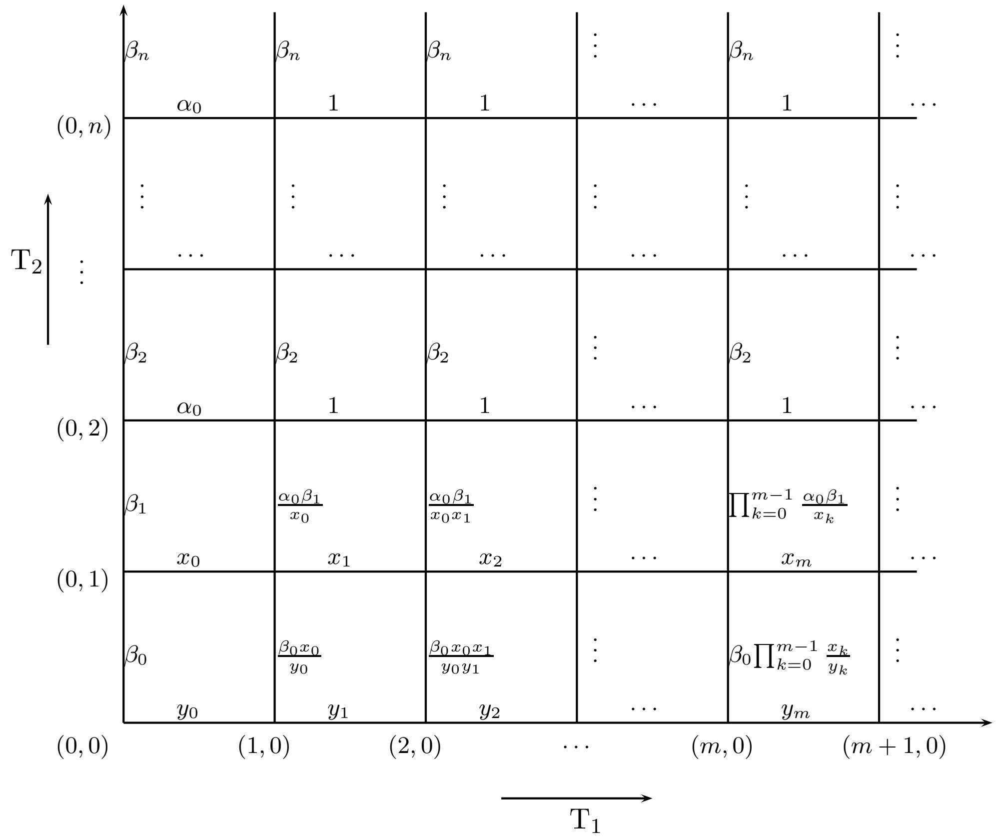


FIGURE 23. Weight diagram of the 2-variable weighted shift in Theorem 15.6 ($k_2 = 2$)

- RC, Joint hyponormality: A bridge between hyponormality and subnormality, Proc. Symposia Pure Math. 51(1990), 69-91.
- RC, Quadratically hyponormal weighted shifts, Integral Equations Operator Theory 13(1990), 49-66.
- RC, An operator-theoretic approach to truncated moment problems, in Linear Operators, Banach Center Publ., vol. 38, 1997, pp. 75-104.
- RC and L. Fialkow, Recursiveness, positivity, and truncated moment problems, Houston J. Math. 17(1991), 603-635.
- RC and L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, Integral Equations Operator Theory 17(1993), 202-246.
- RC and L. Fialkow, Recursively generated weighted shifts and the subnormal completion problem, II, Integral Equations Operator Theory, 18(1994), 369-426.
- RC and L. Fialkow, Solution of the truncated complex moment problem with flat data, Memoirs Amer. Math. Soc. no. 568, Amer. Math. Soc., Providence, 1996.

- RC and L. Fialkow, Solution of the singular quartic moment problem, J. Operator Theory 48(2002), 315-354.
- RC and I.B. Jung, Quadratically hyponormal weighted shifts with two equal weights, Integral Equations Operator Theory 37(2000), 208-231.
- RC, S.H. Lee and W.Y. Lee, A new criterion for k-hyponormality via weak subnormality, PAMS, 2005.
- RC, S.H. Lee and J. Yoon, k-hyponormality of multivariable weighted shifts, JFA, to appear.
- RC and W.Y. Lee, Towards a model theory for 2-hyponormal operators, Integral Equations Operator Theory 44(2002), 290-315.
- RC and P. S. Muhly, C*-algebras of multiplication operators on Bergman spaces, J. Funct. Anal. 64(1985), 315-329.
- RC, P. Muhly and J. Xia, Hyponormal pairs of commuting operators, Operator Theory: Adv. Appl. 35(1988), 1-22.
- RC, Y.T. Poon and J. Yoon, Subnormality of Bergman-like weighted shifts, JMAA, to appear.
- RC and M. Putinar, Existence of non-subnormal polynomially hyponormal operators, Bull. Amer. Math. Soc. 25(1991), 373-378.
- RC & M. Putinar, Nearly subnormal operators and moment problems, JFA 115(1993).
- RC and K. Yan, The spectral picture of Reinhardt measures, J. Funct. Anal. 131(1995), 279-301.
- RC and J. Yoon, Jointly hyponormal pairs of subnormal operators need not be jointly subnormal, TAMS, to appear.
- RC and J. Yoon, Disintegration of measure techniques for commuting multivariable weighted shifts, preprint 2004.
- RC and J. Yoon, Flatness and propagation for hyponormal 2-variable weighted shifts, preprint 2005.
- RC and J. Yoon, The spectral picture of 2-variable weighted shifts, in preparation.
- R.G. Douglas, On majorization factorization and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc. 17 (1966), 413-415.
- M. Dritschel and S. McCullough, private communication.
- P. Fan, A note on hyponormal weighted shifts, Proc. Amer. Math. Soc. 92(1984), 271-272.
- P. Fan, Note on subnormal weighted shifts, Proc. Amer. Math. Soc. 103(1988), 801-802.
- E. Franks, Polynomially subnormal operator tuples, *J. Operator Theory* 31(1994), 219-228.
- P.R. Halmos, Ten problems in Hilbert space, Bull. Amer. Math. Soc. 76(1970), 887-933.
- P.R. Halmos, A Hilbert Space Problem Book, 2nd ed., Springer, New York 1982.
- N.P. Jewell and A.R. Lubin, Commuting weighted shifts and analytic function theory in several variables, J. Operator Theory 1(1979), 207-223.

- S. McCullough and V. Paulsen, A note on joint hyponormality, Proc. Amer. Math. Soc., 107(1989), 187-195.
- V. Paulsen, Completely bounded maps and dilations, Pitman Research Notes in Mathematics Series, vol. 146, Longman Sci. Tech., New York, 1986.
- A. Shields, Weighted shift operators and analytic function theory, Math. Surveys, 13(1974), 49-128.
- J.A. Shohat and J.D. Tamarkin, The Problem of Moments, Math. Surveys I, American Math. Soc., Providence, 1943.
- Ju. L. Smul'jan, An operator Hellinger integral, Mat. Sb. (N.S.) 49 (1959), 381–430 (in Russian).
- J. Stampfli, Hyponormal operators, Pacific J. Math. , 12(1962), 1453-1458.
- J. Stampfli, Which weighted shifts are subnormal?, Pacific J. Math. 17(1966), 367-379.
- Wolfram Research, Inc. Mathematica, Version 4.2, Wolfram Research Inc., Champaign, IL, 2002.
- D. Xia, Spectral Theory of Hyponormal Operators, Operator Theory: Advances and Applications, vol. 10, Birkhäuser Verlag, Basel-Boston, 1983.
- D. Xia, On the semi-hyponormal n-tuple of operators, Integral Equations Operator Theory 6(1983), 879-898.