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Reconstruction of the Berger measure when the core is of tensor form

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Abstract

Let $\mathfrak{H}_0$ denote the class of commuting pairs of subnormal operators on Hilbert space, and let $\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$, where $c(\mathbf{T})$ is the core of $\mathbf{T}$. We obtain a concrete necessary and sufficient condition for the subnormality of $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$ in terms of $c(\mathbf{T})$, the marginal measures of T_1 and T_2, and the weight α_{01}.

1. Introduction

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. It is well known that the commutativity of the pair is necessary but not sufficient ([1], [18], [19], [20]), and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [11]. Abstract solutions of LPCS were given in [7, Theorem 3.1] and [21, Theorem 2.7], while concrete, necessary conditions, albeit not sufficient, for the lifting were found in [12, Theorem 3.3] and [21, Theorem 2.10]) in the case of 2-variable weighted shifts. In ([11], [12], [13], [7], [8], [21] and [22]) we have shown that many of the nontrivial aspects of LPCS are best detected within the class $\mathfrak{H}_0$ of commuting pairs of subnormal operators; we thus focus our attention on this class. More generally, we will denote the class of subnormal pairs by $\mathfrak{H}_\infty$, and for

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each integer $k \geq 1$ the class of k-hyponormal pairs in $\mathfrak{H}_0$ by $\mathfrak{H}_k$. Clearly, $\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0$; the main results in [11] and [7] show that these inclusions are all proper. The constructions in [11] and [12] have shed light on structural and spectral properties of multivariable weighted shifts, and have brought about some new phenomena in joint spectral theory. More recently, we have made use of the tools and techniques in those papers and in [13] and [7] to approach LPCS from a new angle: to what extent the subnormality of the powers of a 2-variable weighted shift can detect the subnormality of the pair. In [8] we discovered a large class of 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2)$ for which the subnormality of (T_1^2, T_2) and (T_1, T_2^2) does imply the subnormality of $\mathbf{T}$. This is the class $\mathcal{TC}$ (see Definition 1.2 below).

In this paper we study the subnormality of 2-variable weighted shifts $\mathbf{T} \in \mathcal{TC}$. Since a 2-variable weighted shift is subnormal if and only if its weight moments are the moments of a probability measure, known as the Berger-Gellar-Wallen measure (or briefly Berger measure), the search for necessary and sufficient conditions leads to the following concrete problem.

Problem 1.1. *Let $\mathbf{T} \in \mathcal{TC}$ and assume $\mathbf{T}$ is hyponormal. Additionally, assume that $c(\mathbf{T})$ is subnormal, with Berger measure $\xi \times \eta$. Find necessary and sufficient conditions on the rest of the weight data to guarantee the subnormality of $\mathbf{T}$.*

Problem 1.1 is a special instance of the Reconstruction-of-the-Measure Problem, which we now describe. Given $\mathbf{T} \in \mathfrak{H}_0$, the j-th row and the i-th column of the weight diagram have their own Berger measures, ξ_j and η_i, respectively. Solving LPCS in this case amounts to finding a measure μ on $\mathbb{R}_+^2$ which interpolates $\{\xi_j, \eta_i\}_{i,j=0}^\infty$. Without loss of generality we can assume that $\xi_{j+1} \ll \xi_j$ and $\eta_{i+1} \ll \eta_i$ (all $i, j \geq 0$) [12, Theorem 3.3]. Moreover, ξ_j must equal μ_j^X (the marginal measure of μ_j), where $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$; in fact, $d\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j \; d\Phi_s(t)\} \; d\mu^X(s)$, where $d\mu(s,t) \equiv d\Phi_s(t) \; d\mu^X(s)$ is the disintegration of μ by vertical slices [12, Theorem 3.1]; and similarly for η_i. From this perspective, LPCS consists of "compatibly gluing together" the measures ξ_j and η_i on $\mathbb{R}_+$ to produce a measure μ on $\mathbb{R}_+^2$ which satisfies the required properties to be the Berger measure of $\mathbf{T}$. We claim this can be done explicitly for $\mathbf{T} \in \mathcal{TC}$. Note that $\xi_0 = \mu^X$ and $\eta_0 = \mu^Y$.

Our main result is Theorem 2.3, which provides a complete solution to Problem 1.1: $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$ is subnormal if and only if measures ψ and φ given by (2.2) and (2.3), respectively, are positive. As an application, we give a concrete condition for the subnormality of flat 2-variable weighted shifts (Proposition 3.1).

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$. An operator T such that $T^*T \geq TT^*$ is said to be *hyponormal*. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S,T] := ST - TS$. We say that an n-tuple $\mathbf{T} := (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [2], [9]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. The Bram-Halmos criterion for subnormality states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([3], [4]). Using Choleski's algorithm for operator matrices, it is easy to see this is equivalent to the k-tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. For notational convenience, we will write sometimes "$shift(\alpha_0, \alpha_1, \cdots)$" for W_α. In particular, $U_+ := shift(1, 1, \cdots)$ and $S_a := shift(a, 1, 1, \cdots)$. For a weighted shift W_α, *the moments of* α are given as

$$\gamma_k(W_\alpha) \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-variable weighted shift $\mathbf{T} := (T_1, T_2)$ by

$$\begin{cases} T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \\ T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}, \end{cases}$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1T_2 = T_2T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \text{ (all } \mathbf{k} \in \ell^2(\mathbb{Z}_+^2)). \tag{1.1}$$

In an entirely similar way one can define multivariable weighted shifts.

Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, $T_1 \cong I \bigotimes W_\alpha$ and $T_2 \cong W_\beta \bigotimes I$, and $\mathbf{T}$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed. Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is $\gamma_{\mathbf{k}}(\mathbf{T})$

$$\equiv \gamma_{\mathbf{k}}(\alpha,\beta) := \begin{cases} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

(We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1,k_2).) We now recall a well known characterization of subnormality for multivariable weighted shifts [17], due to C. Berger (cf. [4, III.8.16]) and independently established by Gellar and Wallen [16]) in the single variable case: $\mathbf{T} \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a probability measure μ (which we call the Berger measure of $\mathbf{T}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \int_R s^{k_1} t^{k_2} d\mu(s,t)$, for all $\mathbf{k} \in \mathbb{Z}_+^2$. In the single variable case, if W_α is subnormal with Berger measure ξ_α and $h \geq 1$, and if we let $\mathcal{L}_h := \bigvee\{e_n : n \geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{L}_h}$ is $\frac{s^h}{\gamma_h} d\xi(s)$; alternatively, if $S : \ell^\infty(\mathbb{Z}_+) \to \ell^\infty(\mathbb{Z}_+)$ is defined by

$$S(\alpha)(n) := \alpha(n+1) \ (\alpha \in \ell^\infty(\mathbb{Z}_+), n \geq 0), \tag{1.2}$$

then

$$d\xi_{S(\alpha)}(s) = \frac{s}{\alpha_0^2} d\xi(s). \tag{1.3}$$

We now formally define the class $\mathcal{TC}$. First, we need some notation: $\mathcal{M}_1 := \vee\{e_{k_1,k_2} : k_2 \geq 1\}$ and $\mathcal{N}_1 := \vee\{e_{k_1,k_2} : k_1 \geq 1\}$.

Definition 1.2. *(i) The core of a 2-variable weighted shift* $\mathbf{T}$ *is* $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M}_1 \cap \mathcal{N}_1}$*;*
(ii) $\mathbf{T}$ *is said to be of tensor form if* $\mathbf{T} \cong (I \otimes W_\alpha, W_\beta \otimes I)$*. (When* $\mathbf{T}$ *is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product* $\xi \times \eta$*);*
(iii) $\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T})$ *is of tensor form*$\}$*.*

2. Main Results

We now consider 2-variable weighted shifts such as the one given by Figure 1(ii), where $W_x \equiv shift(x_0, x_1, \cdots)$ is subnormal with Berger measure ξ_x and $W_y \equiv shift(y_0, y_1, \cdots)$ is subnormal with Berger measure η_y. Further, let $W_\alpha \equiv shift(\alpha_1, \alpha_2, \cdots)$ (resp. $W_\beta \equiv shift(\beta_1, \beta_2, \cdots)$) be subnormal with Berger measure ξ (resp. η). By (1.3), and without loss of generality, we will always assume that $\frac{1}{s} \in L^1(\xi)$ and $\frac{1}{t} \in L^1(\eta)$. We recall several notions introduced in [11] and [8]: (i) given a probability measure μ on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, with $\frac{1}{t} \in L^1(\mu)$, the *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t)$; and (ii) given a measure μ on $X \times Y$, the *marginal measure* μ^X (resp. μ^Y) is given by $\mu^X := \mu \circ \pi_X^{-1}$ (resp. $\mu^Y := \mu \circ \pi_Y^{-1}$), where $\pi_X : X \times Y \to X$ (resp. $\pi_Y : X \times Y \to Y$) is the canonical projection onto X (resp. Y). Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$ (resp. $\mu^Y(F) = \mu(X \times F)$, $F \subseteq Y$). Observe that if μ is a probability measure, then so are μ^X and μ^Y.

For a measure μ with $\frac{1}{s} \in L^1(\mu)$, we write $d\widetilde{\mu}(s) := \frac{1}{s\left\|\frac{1}{s}\right\|_{L^1(\mu)}} d\mu(s)$. For example,

$$d(\xi \times \eta)_{ext}(s,t) = \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\eta)}} d\xi(s) d\eta(t) = d\xi(s) d\widetilde{\eta}(t) \tag{2.1}$$

and $(\xi \times \eta)^X = \xi$. Finally, for an arbitrary 2-variable weighted shift $\mathbf{T}$, we shall let $\mathcal{R}_{ij}(\mathbf{T})$ denote the restriction of $\mathbf{T}$ to $\mathcal{M}_i \cap \mathcal{N}_j$, where $\mathcal{M}_i$(resp. $\mathcal{N}_j$) is the subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis associated with indices $\mathbf{k} = (k_1, k_2)$, where $k_1 \geq 0$ and $k_2 \geq i$ ($k_1 \geq j$ and $k_2 \geq 0$), respectively. In particular, we simply denote $\mathcal{M} \equiv \mathcal{M}_1$ and $\mathcal{N} \equiv \mathcal{N}_1$. It follows that $\mathcal{R}_{11}(\mathbf{T}) = c(\mathbf{T})$, the core of $\mathbf{T}$. Assume that $c(\mathbf{T})$ is subnormal, with Berger measure $\xi \times \eta$. We let

$$\psi := (\eta_y)_1 - a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \eta \tag{2.2}$$

and

$$\varphi := \xi_x - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \widetilde{\xi}, \tag{2.3}$$

where $(\eta_y)_1$ is the Berger measure of the subnormal shift $shift(y_1, y_2, \cdots)$. Trivially, ψ and φ are measures, but they may or may not be *positive* measures. The following result is a very special case of the Reconstruction-of-the-measure Problem.

Lemma 2.1. *(Subnormal backward extension of a 2-variable weighted shift [11]) Consider the 2-variable weighted shift whose weight diagram is given in Figure 1(i). Assume that $\mathcal{R}_{10}(\mathbf{T}) \equiv \mathbf{T}|_{\mathcal{M}}$ is subnormal, with associated measure $\mu_{\mathcal{M}}$, and that $W_0 \equiv shift(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with associated measure ν. Then $\mathbf{T}$ is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu$.

Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$ then $(\mu_{\mathcal{M}})_{ext}^X = \nu$. In the case when $\mathbf{T}$ is subnormal, the Berger measure μ of $\mathbf{T}$ is given by

$$\begin{aligned} d\mu(s,t) &= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &+ (d\nu(s) - \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)) d\delta_0(t). \end{aligned}$$

Proposition 2.2. *Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ be the 2-variable weighted shift whose weight diagram is given in Figure 1(ii). Then $\mathbf{T}|_{\mathcal{M}} \in \mathfrak{H}_\infty$ if and only if ψ is a positive measure. In this case, the Berger measure of $\mathbf{T}|_{\mathcal{M}}$ is*

$$\mu_{\mathcal{M}} = a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi.$$

Proof. ($\Rightarrow$) If $\mathbf{T}|_{\mathcal{M}} \in \mathfrak{H}_\infty$ then $\mathbf{T}|_{\mathcal{M}\cap\mathcal{N}} \in \mathfrak{H}_\infty$ with Berger measure $\mu_{\mathcal{M}\cap\mathcal{N}} = \xi \times \eta$. Note that $\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} = \left\|\frac{1}{s}\right\|_{L^1(\xi)}$. By Lemma 2.1(iii), if we think of $\mathbf{T}|_{\mathcal{M}}$ as the backward extension of $\mathbf{T}|_{\mathcal{M}\cap\mathcal{N}}$ (in the s direction), we then have

$$\begin{aligned} &(\eta_y)_1 \geq a^2 \left\|\tfrac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} (\mu_{\mathcal{M}\cap\mathcal{N}})_{ext}^Y \\ &\Leftrightarrow (\eta_y)_1 - a^2 \left\|\tfrac{1}{s}\right\|_{L^1(\xi)} \eta \geq 0 \\ &\Leftrightarrow \psi \geq 0. \end{aligned}$$

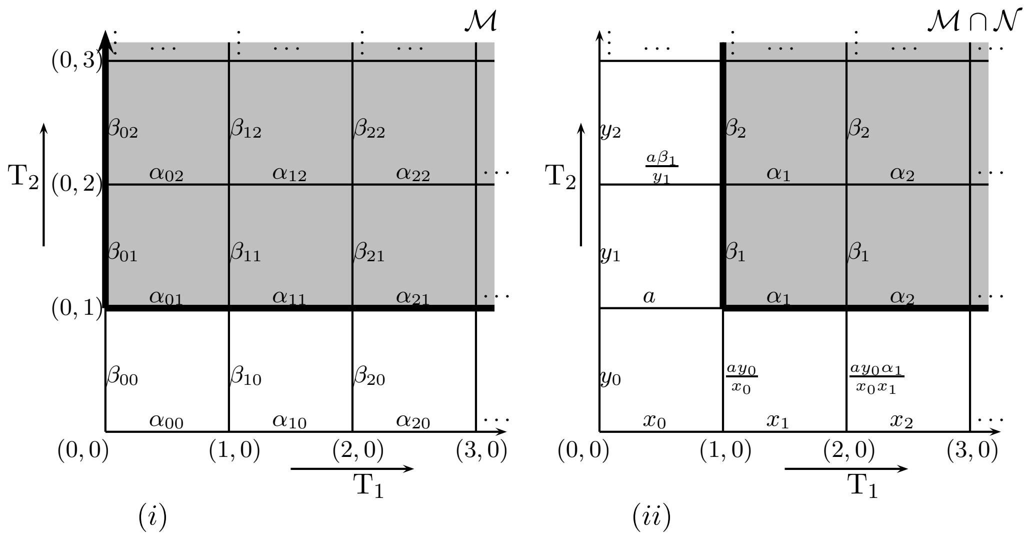

FIGURE 1. Weight diagram of the 2-variable weighted shift in Lemma 2.1, and weight diagram of a 2-variable weighted shift with $\mathcal{R}_{11}(\mathbf{T}) \cong (I \otimes W_\alpha, W_\beta \otimes I)$, respectively.

Thus, ψ is a positive measure.
($\Leftarrow$) If ψ is a positive measure then $\phi := a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi$ is a well defined and positive measure. By a direct calculation, we can see that

$$\left\{ \begin{array}{ll} \iint d\phi(s,t) = 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \iint t^{k_2} d\phi(s,t) = y_1^2 \cdots y_{k_2}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \iint s^{k_1} d\phi(s,t) = a^2\alpha_1^2 \cdots \alpha_{k_1-1}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \iint s^{k_1}t^{k_2} d\phi(s,t) = a^2\alpha_1^2 \cdots \alpha_{k_1-1}^2 \beta_1^2 \cdots \beta_{k_2}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{array} \right\}$$

$= \gamma_{\mathbf{k}}(\mathbf{T}|_{\mathcal{M}})$, where, for notational convenience, we set $\alpha_0 := 1$. Therefore, ϕ interpolates all moments of $\mathbf{T}|_{\mathcal{M}}$, so $\mathbf{T}|_{\mathcal{M}} \in \mathfrak{H}_\infty$ and $\mu_{\mathcal{M}} = \phi \equiv a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi$. □

We now have:

Theorem 2.3. *Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ be the 2-variable weighted shift whose weight diagram is given in Figure 1(ii). Then $\mathbf{T} \in \mathfrak{H}_\infty$ if and only if ψ and φ are positive measures.*

Proof. ($\Leftarrow$) It suffices to find a probability measure μ satisfying

$$\gamma_{\mathbf{k}}(\mathbf{T}) = \int s^{k_1}t^{k_2} d\mu(s,t) \ \ (\text{all } \mathbf{k} \equiv (k_1,k_2) \in \mathbb{Z}_+^2).$$

Let

$$\mu := \varphi \times (\delta_0 - \widetilde{\eta}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \widetilde{\eta} \right) + \xi_x \times \widetilde{\eta}.$$

Clearly, μ is well defined. Observe that

$$\begin{aligned}
\mu &= \varphi \times (\delta_0 - \widetilde{\eta}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \widetilde{\eta} \right) + \xi_x \times \widetilde{\eta} \\
&= (\xi_x - \varphi - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0) \times \widetilde{\eta} + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \widetilde{\psi} + \varphi \times \delta_0 \\
&= a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \widetilde{\xi} \times \widetilde{\eta} + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \widetilde{\psi} + \varphi \times \delta_0. \qquad (2.4)
\end{aligned}$$

Since we are assuming that ψ and φ are positive measures, it follows from 2.4 that μ is also positive. Furthermore, observe that

$$\left\| \frac{1}{t} \right\|_{L^1(\psi)} = \left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \qquad (2.5)$$

and

$$\left\| \frac{1}{t} \right\|_{L^1(\psi)} \widetilde{\psi} = \left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} \widetilde{(\eta_y)}_1 - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \widetilde{\eta}.$$

Thus,

$$\begin{aligned}
\iint d\mu(s,t) &= y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \int d\widetilde{\psi}(t) - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} + 1 \\
&= 1 \ \text{ (since } \widetilde{\psi} \text{ is a probability measure).}
\end{aligned}$$

Therefore, μ is a probability measure.

Next, observe that

$$\varphi([0,+\infty)) = \int d\varphi(s) = 1 - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \qquad (2.6)$$

and, for $k_2 \geq 1$,

$$\left\| \frac{1}{t} \right\|_{L^1(\psi)} \int t^{k_2} d\widetilde{\psi}(t) = \int t^{k_2 - 1} d\,(\eta_y)_1(t) - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \int t^{k_2 - 1} d\eta(t). \qquad (2.7)$$

Thus, if $k_1 = 0$ and $k_2 \geq 1$, we have

$$\begin{aligned}
\iint t^{k_2} d\mu(s,t) &= -\int d\varphi(s) \int t^{k_2} d\widetilde{\eta}(t) \\
&+ y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \int t^{k_2} d\left(\widetilde{\psi} - \widetilde{\eta}\right)(t) + \int t^{k_2} d\widetilde{\eta}(t) \\
&= a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \int t^{k_2} d\widetilde{\eta}(t) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \int t^{k_2} d\widetilde{\psi}(t) \\
&\quad \text{(by (2.6))} \\
&= y_0^2 \int t^{k_2 - 1} d\left(\eta_y\right)_1(t) = y_0^2 \cdots y_{k_2-1}^2 \\
&\quad \text{(by (2.7))}.
\end{aligned}$$

If $k_1 \geq 1$ and $k_2 = 0$, we have

$$\begin{aligned}
\iint s^{k_1} d\mu(s,t) &= \iint s^{k_1} d\varphi(s) d\left(\delta_0 - \widetilde{\eta}\right)(t) + \iint s^{k_1} d\xi_x(s) d\widetilde{\eta}(t) \\
&= \int s^{k_1} d\xi_x(s) \\
&= x_0^2 \cdots x_{k_1-1}^2.
\end{aligned}$$

Finally, if $k_1 \geq 1$ and $k_2 \geq 1$, we have

$$\begin{aligned}
\iint s^{k_1} t^{k_2} d\mu(s,t) &= \iint s^{k_1} t^{k_2} d\varphi(s) d\left(\delta_0 - \widetilde{\eta}\right)(t) + \iint s^{k_1} t^{k_2} d\xi_x(s) d\widetilde{\eta}(t) \\
&= \iint s^{k_1} t^{k_2} (d\xi_x(s) - d\varphi(s)) d\widetilde{\eta}(t) \\
&= a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \iint s^{k_1} t^{k_2} d\widetilde{\xi}(s) d\widetilde{\eta}(t) \\
&= a^2 y_0^2 \alpha_1^2 \cdots \alpha_{k_1-1}^2 \beta_1^2 \cdots \beta_{k_2-1}^2,
\end{aligned}$$

where, for notational convenience, we set $\alpha_0 := 1$ and $\beta_0 := 1$. Thus,

$$\left\{ \begin{array}{ll}
\iint d\mu(s,t) = 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\
\iint t^{k_2} d\mu(s,t) = y_0^2 \cdots y_{k_2-1}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\
\iint s^{k_1} d\mu(s,t) = x_0^2 \cdots x_{k_1-1}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\
\iint s^{k_1} t^{k_2} d\mu(s,t) = a^2 y_0^2 \alpha_1^2 \cdots \alpha_{k_1-1}^2 \beta_1^2 \cdots \beta_{k_2-1}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1
\end{array} \right\}$$

$=\gamma_{\mathbf{k}}(\mathbf{T})$. Therefore, μ interpolates all moments of $\mathbf{T}$, so $\mathbf{T}$ must be subnormal, with Berger measure μ.

($\Rightarrow$) Assume that $\mathbf{T}$ is subnormal with Berger measure μ. Then $\mathbf{T}|_{\mathcal{M}}$ is also subnormal, and by Proposition 2.2 we can see that ψ is a positive measure. We then have

$$\begin{aligned}
\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} &= \int \frac{1}{t} d\mu_{\mathcal{M}}(s,t) \\
&= a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \left\|\frac{1}{t}\right\|_{L^1(\eta)} + \int \frac{1}{t} d\left(\eta_y\right)_1(t) - a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \left\|\frac{1}{t}\right\|_{L^1(\eta)} \\
&= \left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)}.
\end{aligned}$$

Since $\mu_{\mathcal{M}} = a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi$, we get

$$\begin{aligned}
\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)} d(\mu_{\mathcal{M}})_{ext}(s,t) &= \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) \\
&= \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d\left\{a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi\right\}_{ext} (s,t) \\
&= a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} d\widetilde{\xi}(s) \frac{d\eta(t)}{t} + d\delta_0(s) \frac{d\psi(t)}{t} \\
&= a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} d\widetilde{\xi}(s) \frac{d\eta(t)}{t} \\
&+ d\delta_0(s) [\frac{d\left(\eta_y\right)_1(t)}{t} - a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \frac{d\eta(t)}{t}].
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)} (\mu_{\mathcal{M}})^X_{ext} &= \left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)} \int_Y d(\mu_{\mathcal{M}})_{ext}(\cdot,t) \\
&= a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \left\|\frac{1}{t}\right\|_{L^1(\eta)} \widetilde{\xi} \\
&+ \left(\left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)} - a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \left\|\frac{1}{t}\right\|_{L^1(\eta)}\right) \delta_0.
\end{aligned}$$

Now recall that $\beta^2_{00} = y_0^2$, so from Lemma 2.1(iii) we obtain

$$\xi_x \geq a^2 y_0^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \left\|\frac{1}{t}\right\|_{L^1(\eta)} \widetilde{\xi} + y_0^2 \left\|\frac{1}{t}\right\|_{L^1(\psi)} \delta_0 \ \text{(using (2.5))}.$$

Thus, φ is a positive measure, as desired. $\square$

Remark 2.4. The proof of Theorem 2.3 gives a concrete formula for the Berger measure of $\mathbf{T} \equiv (T_1, T_2)$, namely

$$\mu = \varphi \times (\delta_0 - \widetilde{\eta}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \widetilde{\eta} \right) + \xi_x \times \widetilde{\eta}.$$

In Proposition 2.2 and Theorem 2.3 we noted that ψ (resp. φ) is a linear combination of $(\eta_y)_1$ and η (resp. ξ_x, δ_0 and $\widetilde{\xi}$), where $(\eta_y)_1$ and η (resp. ξ_x, δ_0 and $\widetilde{\xi}$) are the Berger measures of the subnormal 1-variable weighted shifts in the vertical (resp. horizontal) slices of $\mathbf{T}$. Thus, the following conjecture for 2-variable weighted shifts seems natural.

Conjecture 2.5. *Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ be the 2-variable weighted shift whose weight diagram is given by Figure 1(i). Then the subnormality of $\mathbf{T}$ is determined by a countable collection of inequalities $\{\psi_k \geq 0\}$, where each measure ψ_k is a linear combination of Berger measures associated to the 1-variable weighted shifts in vertical or horizontal slices of $\mathbf{T}$.*

3. Application to Flat 2-variable Weighted Shifts

We can now give a concrete condition for the subnormality of flat 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2)$. Recall that $\mathbf{T} \equiv (T_1, T_2)$ is called *horizontally flat* if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$, and *vertically flat* if $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$ [13]. If $\mathbf{T}$ is horizontally and vertically flat, then $\mathbf{T}$ is simply called *flat* (see Figure 2). It is straightforward to prove that $\mathbf{T}$ is flat if and only if $\mathbf{T} \in \mathcal{TC}$, with ξ and η 1-atomic. Without loss of generality, we can always assume that $\xi = \delta_1$.

Now recall that for $0 < \alpha < \beta$, $shift(\alpha, \beta, \beta, \cdots)$ is subnormal with Berger measure $(1 - \frac{\alpha^2}{\beta^2})\delta_0 + \frac{\alpha^2}{\beta^2}\delta_{\beta^2}$. Finally, we know from [12, Theorem 3.3] and [13, Section 5] that if $\mathbf{T}$ is flat and subnormal then ξ_x and η_y have the form

$$\begin{aligned} \xi_x &= p\delta_0 + q\delta_1 + [1 - (p+q)]\rho \\ \eta_y &= \ell\delta_0 + m\delta_{b^2} + [1 - (\ell + m)]\sigma, \end{aligned} \tag{3.1}$$

where $0 < p, q, \ell, m < 1$, $p + q \leq 1$, $\ell + m \leq 1$, and ρ and σ are probability measures with $\rho(\{0\} \cup \{1\}) = 0$, $\sigma(\{0\} \cup \{b^2\}) = 0$.

We are now ready to present

Proposition 3.1. *Consider the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ whose weight diagram is given in Figure 2. The following statements are*

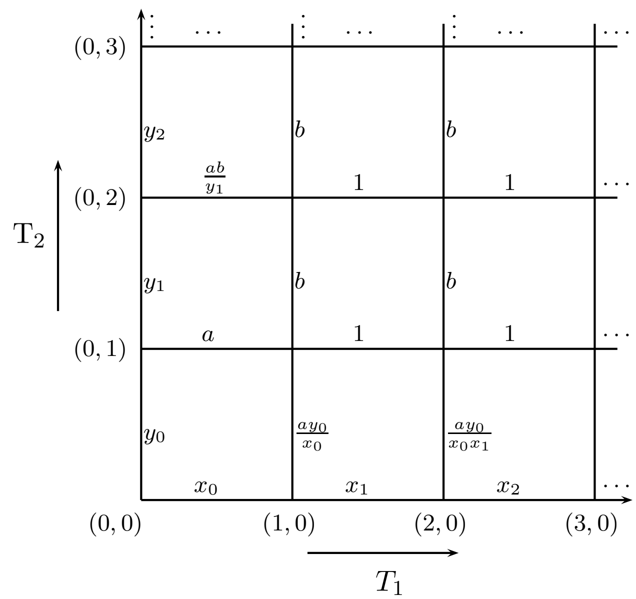

FIGURE 2. Weight diagram of the 2-variable weighted shift in Proposition 3.1.

equivalent.
(i) $\mathbf{T} \in \mathfrak{H}_\infty$*;*
(ii) ψ *and* φ *are positive measures;*
(iii) $\frac{b}{a}\sqrt{m} \geq y_0$ *and*

$$\xi_x \geq y_0^2 \left\{ \left(\left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} - \frac{a^2}{b^2} \right) \delta_0 + \frac{a^2}{b^2} \delta_1 \right\}. \tag{3.2}$$

Moreover, when $\mathbf{T}$ *is subnormal, its Berger measure is given as*

$$\mu = \varphi \times (\delta_0 - \delta_{b^2}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \delta_{b^2} \right) + \delta_1 \times \delta_{b^2} \tag{3.3}$$

Proof. (i) $\Rightarrow$ (ii): This is straightforward from Theorem 2.3.
(ii) $\Rightarrow$ (iii): In (2.2) and (2.3), observe that $\widetilde{\xi} = \delta_1$, $\widetilde{\eta} = \delta_{b^2}$, $\left\|\frac{1}{s}\right\|_{L^1(\xi)} = 1$, $\left\|\frac{1}{t}\right\|_{L^1(\eta)} = \frac{1}{b^2}$ and $\left\|\frac{1}{t}\right\|_{L^1(\psi)} = \left\|\frac{1}{t}\right\|_{L^1((\eta_y)_1)} - \frac{a^2}{b^2}$. Thus, we have

$$\begin{aligned} \psi &= (\eta_y)_1 - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \eta \\ &= \frac{mb^2}{y_0^2} \delta_{b^2} + \frac{[1 - (\ell + m)]}{y_0^2} \sigma_1 - a^2 \delta_{b^2} \end{aligned} \tag{3.4}$$

(where $d\sigma_1(t) := td\sigma(t)$) and

$$\begin{aligned}\varphi &= \xi_x - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \widetilde{\xi} \\ &= \xi_x - y_0^2 \left\{ \left(\left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} - \frac{a^2}{b^2} \right) \delta_0 + \frac{a^2}{b^2} \delta_1 \right\}. \qquad (3.5)\end{aligned}$$

Since we are assuming that ψ and φ are positive measures, it follows from (3.4) and (3.5) that

$$\frac{b}{a}\sqrt{m} \geq y_0 \text{ and } \xi_x \geq y_0^2 \left\{ \left(\left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} - \frac{a^2}{b^2} \right) \delta_0 + \frac{a^2}{b^2} \delta_1 \right\},$$

as desired.

(iii) $\Rightarrow$ (i): Let $\omega := a^2\delta_1 \times \delta_{b^2} + \delta_0 \times \psi$. Then ω is well defined, and by the formula for ψ (given in (3.4)) and the condition $\frac{b}{x}\sqrt{m} \geq y_0$, we see at once that $\omega \geq 0$. Furthermore, ω is the Berger measure of $\mathbf{T}|_{\mathcal{M}}$, so that $\mathbf{T}|_{\mathcal{M}}$ is subnormal. We now wish to apply Lemma 2.1. Note that

$$d\omega_{ext}(s,t) = \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\omega)}} \left\{ a^2 d\delta_1(s) d\delta_{b^2}(t) + d\delta_0(s) d\psi(t) \right\},$$

so that

$$\omega_{ext}^X = \frac{1}{\left\| \frac{1}{t} \right\|_{L^1(\eta_1)}} \left\{ \left(\left\| \frac{1}{t} \right\|_{L^1(\eta_1)} - \frac{a^2}{b^2} \right) \delta_0 + \frac{a^2}{b^2} \delta_1 \right\}.$$

We now see that the conditions (i), (ii) and (iii) in Lemma 2.1 are satisfied, and therefore $\mathbf{T} \in \mathfrak{H}_\infty$.

Finally, since in this case we have $\widetilde{\xi} = \delta_1$ and $\widetilde{\eta} = \delta_{b^2}$, Theorem 2.3 readily implies (3.3). $\square$

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