Power moment problems have received considerable attention over the years, beginning with the pioneering efforts of Stieltjes, Riesz, Hamburger, Hausdorff and Krein, followed by the work of Haviland, Akhiezer, Fuglede, Berg, Atzmon, and many others [AhKr], [Akh], [Atz], [Berg], [BCJ], [BeMa], [Fug], [Hav1], [Hav2], [KrNu], [Put1], [Put2], [PuVa1], [Sch2], [ShTa], [Sto1], [Vas1]. The two articles under review represent significant contributions to our existing knowledge, in terms of providing new criteria for existence and uniqueness of representing measures, and for localization of the support of such measures. They introduce novel ideas, methods and techniques that are bound to have an impact on future developments of the theory. They both appeal to the notion of extendability, in different but compatible directions, and consonant with the main approach to *truncated* moment problems recently developed in [CuFi1], [CuFi3] and [CuFi4], where the starting finite matrix is progressively extended to an infinite positive semi-definite sesquilinear form.

Let $\gamma \equiv \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers, with $\gamma_0 > 0$. The classical Hamburger moment problem on the real line $\mathbb{R}$ entails finding necessary and sufficient conditions on γ that guarantee the existence of a positive Borel measure μ such that $\int t^k d\mu(t) = \gamma_k$ (all $k \geq 0$). It is well known that such a measure μ exists if and only if the Hankel matrix $H \equiv H(\gamma) := (\gamma_{i+j})_{i,j=0}^{\infty}$ is positive semi-definite, as a quadratic form on $\mathbb{R}$ [Akh]. Moreover, $supp \mu \subseteq \mathbb{R}_+ \equiv [\nvdash, \infty)$ (Stieltjes moment problem) if and only if, in addition, $L \equiv L(\gamma) := (\gamma_{i+j+1})_{i,j=0}^{\infty} \geq 0$ [Akh], [ShTa]. Thus, while $H \geq 0$ guarantees the existence of μ, the condition $L \geq 0$ "localizes" the support.

For $n > 1$, the solution of the analogous multivariable Hamburger moment problem for an n-indexed sequence $\gamma \equiv \{\gamma_\alpha\}_{\alpha \in \mathbb{Z}_+^{\ltimes}}$ states that γ admits a representing measure μ if and only if there exists a positive semi-definite $(n+1)$-indexed sequence $\{\delta_{(\alpha,\beta)}\}_{\alpha \in \mathbb{Z}_+^{\ltimes}, \beta \in \mathbb{Z}_+}$ such that

a) $\gamma_\alpha = \delta_{(\alpha,0)}$ (all $\alpha \in \mathbb{Z}_+^{\ltimes}$); and

b) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2\epsilon_1,\beta+1)} + ... + \delta_{(\alpha+2\epsilon_n,\beta+1)}$ (all $\alpha \in \mathbb{Z}_+^{\ltimes}, \beta \in \mathbb{Z}_+$).

Moreover, γ admits a unique representing measure if and only so does δ [Vas2, Theorem 2.3]. Finally, $supp \mu \subseteq \mathbb{R}_+^{\ltimes}$ if and only if (a) and (b) hold, and

c) $(\delta_{(\alpha+\epsilon_j,\beta)})_{\alpha \in \mathbb{Z}_+^{\ltimes}, \beta \in \mathbb{Z}_+} \geq 0$ (all $j = 1, ..., n$) [Vas2, Theorem 2.6].

The key ingredient needed is the idea of building a new moment problem, essentially equivalent to (and extending) the original one, but in a *higher dimensional* setting, where *positivity alone* provides the necessary and sufficient condition, just like in the case $n = 1$. Indeed, the above mentioned results are special cases of a more general result.

Theorem 1. ([PuVa2, Theorem 2.7]) For γ as above and $\mathbf{p} \equiv (p_1, ..., p_m)$ an m-tuple of polynomials in n variables (say $p_k(t) \equiv \sum_\xi a_{k\xi} t^\xi$) , γ admits a representing measure with support in $\bigcap_{k=1}^{m} p_k^{-1}(\mathbb{R}_+)$ if and only if there exists a positive semi-definite sequence $\delta \equiv \{\delta_{(\alpha,\beta)}\}_{\alpha \in \mathbb{Z}_+^{\ltimes}, \beta \in \mathbb{Z}_+}$ such that a $\gamma_\alpha = \delta_{(\alpha,0)}$ (all $\alpha \in \mathbb{Z}_+^\ltimes$);

b' $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \sum_{j=1}^n \delta_{(\alpha+2\epsilon_j,\beta+1)} + \sum_{k=1}^m \sum_{\xi,\eta} a_{k\xi} a_{k\eta} \delta_{(\alpha+\xi+\eta,\beta+1)}$ (all $\alpha \in \mathbb{Z}_+^\ltimes, \beta \in \mathbb{Z}_+$); and

c' the $(n+1)$-sequences $\sum_\xi a_{k\xi}\delta_{(\alpha+\xi,\beta)}$ are positive semi-definite (all $k = 1, ..., n$).

In [PuVa2], the authors pursue the idea of dimensional extension in a concrete way, by attaching to the m-tuple of polynomials $\mathbf{p}$ an embedding of $\mathbb{R}^\ltimes$ into a submanifold of a higher dimensional Euclidean space. In fact, if $\theta_{\mathbf{p}}(t) := (1 + t_1^2 + ... + t_n^2 + p_1(t)^2 + ... + p_n(t)^2)^{-1}$ $(t \in \mathbb{R}^\ltimes)$, if $\mathcal{P}_n$ stands for the space of n-variable polynomials, and if $\mathcal{R}_{\theta_{\mathbf{p}}}$ denotes the $\mathbb{C}$-algebra generated by $\mathcal{P}_n$ and $\theta_{\mathbf{p}}$, then any positive semi-definite map Λ on $\mathcal{R}_{\theta_{\mathbf{p}}}$ that satisfies the localizing condition $\Lambda(p_k\,|r|^2) \geq 0$ $(r \in \mathcal{R}_{\theta_{\mathbf{p}}},\, k = 1, ..., m)$ possesses a uniquely determined representing measure whose support is in the semi-algebraic set $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$; moreover, the algebra $\mathcal{R}_{\theta_{\mathbf{p}}}$ is dense in $L^2(\mu)$ [PuVa2, Theorem 2.5].

Thus, generally speaking, what the authors establish in [PuVa2] is that certain $(n+1)$-dimensional extensions of a moment sequence are naturally characterized by positivity conditions alone; moreover, these extensions parameterize all possible solutions of the moment problem. One motivation behind this is as follows: given a sequence γ with representing measure μ rapidly decaying at infinity (e.g., μ of compact support), the extended sequence $\delta_{\alpha,m} := \int \frac{x^\alpha}{(1+\|x\|^2)^m} d\mu(x)$ $(\alpha \in \mathbb{Z}_+^\ltimes, \gtrdot \in \mathbb{Z}_+)$ can be characterized by a single positivity condition. In addition, the fact that we deal with *power* moment problems allows one to impose natural semi-algebraic restrictions on δ that localize the support of μ in a corresponding semi-algebraic set.

The proof of the main result in [PuVa2] (the above mentioned Theorem 2.5) consists of the following three steps.

Step 1 (Existence) The construction of a sesquilinear form on $\mathcal{R}_{\theta_{\mathbf{p}}}$ via the equation $\langle r_1, r_2\rangle_\Lambda := \Lambda(r_1\bar{r}_2)$, giving rise to a Hilbert space $\mathcal{H}$ (the completion of the quotient of $\mathcal{R}_{\theta_{\mathbf{p}}}$ by the null space of $\langle\cdot,\cdot\rangle_\Lambda$) on which the operators $T_1, ..., T_n$ of multiplication by the coordinates $t_1, ..., t_n$ are symmetric and densely defined. Similarly, the operators $S_1, ..., S_m$ of multiplication by $p_1, ..., p_m$ are also symmetric and densely defined. This in turn allows for the definition of a positive, essentially self-adjoint operator $B := \sum_j T_j^2 + \sum_k S_k^2$ that captures the essence of the embedding $\theta_{\mathbf{p}}$. It then follows that $T_1, ..., T_n, S_1, ..., S_m$ are essentially self-adjoint, and have mutually commuting canonical closures $\bar{T}_1, ..., \bar{T}_n, \bar{S}_1, ..., \bar{S}_m$. Then the joint spectral measure of $\bar{T}_1, ..., \bar{T}_n$ provides the desired solution.

Step 2 (Uniqueness) This is done by exploiting properties of joint spectral measures, following the strategy in [Fug, Theorem 7], and yields as a by-product the density of $\mathcal{R}_{\theta_{\mathbf{p}}}$ in $L^2(\mu)$.

Step 3 (Localization of the support) This is accomplished by observing that $\bar{S}_k = p_k(\bar{T}_1, ..., \bar{T}_n)$ and that the condition $\Lambda(p_k |r|^2) \geq 0$ implies the positivity of S_k.

By contrast, in [StSz2] the authors solve the moment problem using extendability in a different sense. In [StSz2, Theorem 1], they prove that a complex sequence $\{\gamma_{m,n}\}_{m,n=0}^{\infty}$ admits a representing measure if and only if there exists a complex sequence $\{\tilde{\gamma}_{m,n}\}_{m+n\geq 0}$ such that $\tilde{\gamma}_{m,n} = \gamma_{m,n}$ for all $m, n = 0, 1, ...$, and $\sum \tilde{\gamma}_{m+q,n+p}\lambda_{m,n}\bar{\lambda}_{p,q} \geq 0$ for any finite collection $\{\lambda_{m,n}\}_{m+n\geq 0}$ of complex numbers. In other words, if and only if there exists a positive semi-definite extension of γ, not to a higher dimensional setting of the space variables but to a larger collection of indices, that is, the extension of the quarter plane in $\mathbb{Z}^{\nvDash}$ consisting of the half-plane $\mathcal{N}_+$ through the origin that contains the quarter plane and is slanted 45° relative to the canonical coordinate axes.

To capture some of the flavor of [StSz2, Theorem 1], let us see how the above solution applies to the classical Hamburger moment problem. Given a sequence $\{\beta_n\}_{n=0}^{\infty}$ with associated Hankel matrix $H(\beta) \geq 0$, let $\gamma_{m,n} := \beta_{m+n}$ $(m, n \geq 0)$, and define $\tilde{\gamma}_{m,n} := \beta_{m+n}$ (for all (m, n) such that $m+n \geq 0$). A straightforward calculation shows that $\tilde{\gamma}$ is positive semi-definite, so there exists a representing measure ν such that $\beta_{m+n} = \int z^m \bar{z}^n \, d\nu$ $(m + n \geq 0)$. But how do we recover β? Observe that

$$\begin{aligned} \beta_n = \tfrac{1}{2^n} \textstyle\sum_{k=0}^{n} nk\beta_{k+(n-k)} &= \tfrac{1}{2^n} \textstyle\sum_{k=0}^{n} nk \int z^k \bar{z}^{n-k} \, d\nu \\ &= \textstyle\int (\frac{z+\bar{z}}{2})^n \, d\nu = \int (Rez)^n \, d\nu. \end{aligned}$$

It follows that $\beta_n = \int_{\mathbb{R}} x^n \, d\mu(x)$, where x is the real part of z and μ is the canonical measure induced by ν via the map $z \mapsto$Re z.

A similar strategy works for the trigonometric moment problem, and having a solid understanding of it and the above mentioned Hamburger moment problem is essential for the *polar* decomposition approach in [StSz2]. Indeed, [StSz2, Theorem 1] is based on the following result of A. Devinatz (observe that the representing measure ν is supported in $\mathbb{R} \times \mathbb{T}$).

Theorem 2. ([Dev, Theorem 4]) Let $\{\gamma_{m,n}\}_{m\in\mathbb{Z}_+,\ltimes\in\mathbb{Z}}$ be a sequence of complex numbers. There exists a positive Borel measure ν supported in $\mathbb{R} \times \mathbb{T}$ such that $\gamma_{m,n} = \int_{\mathbb{R}\times\mathbb{T}} t^m z^n \, d\nu$ $(m \in \mathbb{Z}_+, \ltimes\in \mathbb{Z})$ if and only if γ is positive semi-definite in the following sense: $\sum_{i,k,j,\ell=0}^{N} \gamma_{i+j,k-\ell}\lambda_{i,k}\bar{\lambda}_{j,\ell}$ (for all complex collections $\{\lambda_{i,k}\}_{i,k=0}^{N}$, where $N \geq 0$).

Detecting the existence of a representing measure by checking the positivity of an extended sequence on the lattice half-plane $\mathcal{N}_+$ may seem coincidental at first sight. If such extensions carry valuable information, why not try the whole of $\mathbb{Z}^{\nvDash}$ instead? Indeed, it is true that if γ admits a positive semi-definite extension to *all* of $\mathbb{Z}^{\nvDash}$ then a representing measure exists, but it is one that carries additional (unwanted) properties. Thus, as the authors remark, "in a sense, the middle brings the solution."

A question immediately arises: under what necessary and sufficient conditions does a sequence γ admit such an extension to $\mathcal{N}_+$? The authors answer this in Section 6, both through Theorem 9 (general *-semigroup case) and Theorem 11 (concrete application to complex moment sequences).

An important consequence of [StSz2, Theorem 1] is the following result, which exemplifies the polar decomposition approach.

Theorem 3. ([StSz2, Theorem 5]) A sequence γ admits a representing measure if and only if there exist a measure space (Ω, Σ, ρ) and Σ-measurable functions $a_n : \Omega \to \mathbb{R}$ $(\ltimes \in \mathbb{Z}_+)$ and $b_n : \Omega \to \mathbb{C}$ $(\ltimes \in \mathbb{Z})$ such that for almost every $\omega \in \Omega$, the sequences $\{a_n(\omega)\}_{n \in \mathbb{Z}_+}$ and $\{b_n(\omega)\}_{n \in \mathbb{Z}}$ are positive semi-definite, and $\gamma_{m,n} = \int_\Omega a_{m+n}(\omega) b_{m-n}(\omega)\, d\rho(\omega)$ $(m, n \geq 0)$.

A related result gives an integral representation for positive semi-definite sequences on $\mathcal{N}_+$.

Theorem 4. γ is positive semi-definite on $\mathcal{N}_+$ if and only if there exist positive Borel measures μ on $\mathbb{C}^* \equiv \mathbb{C} \setminus \{\nvdash\}$ and ν on $\mathbb{T}$ such that $\gamma_{m,n} = \int_{\mathbb{C}^*} z^m \bar{z}^n \; d\mu + \delta_{m+n,0} \int_{\mathbb{T}} z^m \bar{z}^n \; d\nu$, where $\{\delta_{m+n,0}\}_{m+n \geq 0}$ is a *-character of $\mathcal{N}_+$.

It is intriguing to compare this result with the approach to the *truncated* complex moment problem initiated by L. Fialkow and the reviewer in [CuFi1], [CuFi2] and [CuFi3], where the associated moment matrix embodies both the "Hankel type" of a and the "Toeplitz type" of b. The connection between solving *full* moment problems (i.e., those discussed in this review) and truncated ones has been made more even evident quite recently, through [Sto2], in which the author proves that a complete solution of a truncated moment problem can yield, via a weak-* convergence argument, a complete solution to the corresponding full moment problem.

The work in [StSz2] also addresses the issue of localization of the support, along the lines of previous work in [Sto1] and [StSz1]. In fact, the authors provide a description of the complex moment problem on real algebraic curves, and obtain new characterizations of subnormality of unbounded operators having invariant domain.

As is well known, results about moment problems naturally have counterparts in the theory of additive decompositions of nonnegative polynomials into squares of polynomials or rational functions. In [PuVa2] the authors present a new proof of the representation of a positive polynomial as a sum of squares of rational functions, allowing as denominators only powers of $1 + \|t\|^2$. This result, originally proved by B. Reznick [Rez2], sheds new light into the precise structure of positive polynomials, and extends previous work of K. Schmüdgen [Sch1], [Sch2] (see also [Rez1], [PoRe] and [Dem]). The methods in [PuVa2] also permit a description of every homogeneous polynomial that is positive on a semi-algebraic set given by a system of homogeneous polynomial inequalities (Theorem 4.2 and Corollary 4.3). The homogeneity assumption makes possible, after dimensional extension, to reduce the support of a representing measure to an Euclidean sphere, thus triggering an application of G. Cassier's methods [Cas].

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