Abrahamse’s Theorem and Subnormal Toeplitz Completions

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Reformulation of Halmos’s Problem 5: Which subnormal Toeplitz operators with matrix-valued symbols are either normal or analytic?

M. Abrahamse (1976): Let $\varphi \in L^\infty$ be such that $\varphi$ or $\overline{\varphi}$ is of bounded type. If $T_\varphi$ is subnormal, then $T_\varphi$ is either normal or analytic.

In this talk we will discuss a matrix-valued version of Abrahamse’s Theorem and then apply this result to solve the following subnormal Toeplitz completion problem:
Find the unspecified Toeplitz entries of the partial block Toeplitz matrix

\[ A := \begin{bmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{bmatrix} \quad (\alpha, \beta \in \mathbb{D}) \]

so that \( A \) becomes subnormal, where \( b_\lambda \) is a Blaschke factor of the form

\[ b_\lambda(z) := \frac{z - \lambda}{1 - \lambda z} \quad (\lambda \in \mathbb{D}). \]
Problem. Given two Blaschke products $b_\alpha$ and $b_\beta$ ($\alpha, \beta \in \mathbb{D}$), find necessary and sufficient conditions on $\varphi, \psi$ rational to make

$$G := \begin{bmatrix} T_{b_\alpha} & T_{\varphi} \\ T_{b_\beta} & T_{\psi} \end{bmatrix}$$

subnormal.

Main Idea: Think of $G$ as a block Toeplitz operator
Motivation: A Simple Case

Let $U_+$ be the unilateral shift on $H^2$. Find the unspecified Toeplitz entries $?$ of the partial block Toeplitz matrix

$$A := \begin{bmatrix} U^*_+ & ? \\ ? & U^*_+ \end{bmatrix}$$

so that $A$ becomes subnormal.
A Related Completion Problem

Proposition

\[
\begin{bmatrix}
T_z & ? \\
? & T_{\bar{z}}
\end{bmatrix}
\] is never hyponormal, if ? is Toeplitz.
Recall that the related dilation problem

\[ A := \begin{bmatrix} U^* & ? \\ ? & ? \end{bmatrix} \]

admits the canonical solution

\[ A := \begin{bmatrix} U^*_+ & 0 \\ I - U_+ U^*_+ & U_+ \end{bmatrix}. \]

(But of course the \((1, 2)\)-entry is not Toeplitz.)
One can write down a couple of nontrivial subnormal Toeplitz completions, as follows:

\[ A := \begin{bmatrix} U^*_+ & U_+ \\ U_+ & U^*_+ \end{bmatrix}. \]

and

\[ A := \begin{bmatrix} U^*_+ & \alpha U^*_+ + \sqrt{1 + |\alpha|^2} U_+ \\ \alpha U^*_+ + \sqrt{1 + |\alpha|^2} U_+ & U^*_+ \end{bmatrix}. \]

How general are these solutions?
Notation and Preliminaries

\[ L^\infty \equiv L^\infty(\mathbb{T}); \quad H^\infty \equiv H^\infty(\mathbb{T}); \quad L^2 \equiv L^2(\mathbb{T}); \quad H^2 \equiv H^2(\mathbb{T}), \]

\[ P : L^2 \rightarrow H^2 \] orthogonal projection

\[ T \in \mathcal{L}(\mathcal{H}) : \text{algebra of bounded operators on a Hilbert space } \mathcal{H} \]

- **normal** if \[ T^* T = TT^* \]
- **quasinormal** if \[ T \text{ commutes with } T^* T \]
- **subnormal** if \[ T = N|_{\mathcal{H}}, \text{ where } N \text{ is normal and } N\mathcal{H} \subseteq \mathcal{H} \]
- **hyponormal** if \[ T^* T \geq TT^* \]
- **2-hyponormal** if \( (T, T^2) \) is (jointly) hyponormal \((k \geq 1)\)

\[
\begin{pmatrix}
[T^*, T] & [T^* T^2, T] \\
[T^*, T^2] & [T^* T^2, T^2]
\end{pmatrix} \geq 0
\]
quasinormal $\Rightarrow$ subnormal $\Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal

For $\varphi \in L^\infty$, the Toeplitz operator with symbol $\varphi$ is

$T_\varphi : H^2 \to H^2$, given by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2)$$

$T_\varphi$ is said to be analytic if $\varphi \in H^\infty$

Halmos’s Problem 5 (1970):

Is every subnormal Toeplitz operator either normal or analytic?

C. Cowen and J. Long (1984): No
C. Cowen (1988)
\[ \varphi \in L^\infty, \varphi = \bar{f} + g \ (f, g \in H^2) \]

\[ T_\varphi \text{ is hyponormal} \iff f = c + T_\bar{h} g, \]

for some \( c \in \mathbb{C}, h \in H^\infty, \|h\|_\infty \leq 1. \)

Nakazi-Takahashi (1993)
For \( \varphi \in L^\infty, \) let
\[ \mathcal{E}(\varphi) := \{ k \in H^\infty : \|k\|_\infty \leq 1 \text{ and } \varphi - k\bar{\varphi} \in H^\infty \}. \]

Then
\[ T_\varphi \text{ is hyponormal} \iff \mathcal{E}(\varphi) \neq \emptyset. \]
Natural Questions:
1) When is $T_\varphi$ subnormal?
At present, there’s no known characterization of subnormality in terms of the symbol $\varphi$.
2) Characterize 2-hyponormality for Toeplitz operators

Sample Result:

Theorem

(RC and WY Lee, 2001) Every 2-hyponormal trigonometric Toeplitz operator is subnormal.
Cowen-Long (1984): Let $0 < \alpha < 1$, let $\psi : \mathbb{D} \to E$ be conformal, where $E$ is the interior of the ellipse with vertices $\pm(1 + \alpha)i$ and passing through $\pm(1 - \alpha)$, and let

$$\varphi := \frac{\psi + \alpha \bar{\psi}}{1 - \alpha^2}.$$

Question

Let $\varphi$ be the Cowen & Long symbol. Does it follow that $T_\varphi \sim T_\eta$ for some $\eta \in H^\infty$?

Question

A Reformulation of Halmos’s Problem 5

Let $T_\varphi$ be a non-normal subnormal Toeplitz operator. Does it follow that $T_\varphi \cong T_\eta$ for some $\psi \in H^\infty$?

These two questions remain open.
\( \varphi \in L^\infty \) is of \textit{bounded type} (or in the Nevanlinna class) if

\[ \varphi := \frac{\psi_1}{\psi_2} \quad (\psi_1, \psi_2 \in H^\infty). \]

(Abrahamse, 1976) Assume \( \varphi \) or \( \bar{\varphi} \) is of bounded type. If \( T_\varphi \) is hyponormal and \( \ker[T_\varphi^*, T_\varphi] \) is invariant for \( T_\varphi \), then \( T_\varphi \) is normal or analytic.

Thus, the answer to Halmos’s Problem 5 is \textit{affirmative} if \( \varphi \) is of bounded type.
\( M_n := M_{n \times n} L_{\mathbb{C}^n}^2 = L^2 \otimes \mathbb{C}^n H_{\mathbb{C}^n}^2 = H^2 \otimes \mathbb{C}^n L^\infty_n \equiv L^\infty_n (\mathbb{T}) \)

For \( \Phi \in L^\infty_M \), \( T_\Phi : H_{\mathbb{C}^n}^2 \rightarrow H_{\mathbb{C}^n}^2 \) denotes the \textit{block} Toeplitz operator with symbol \( \Phi \) defined by

\[
T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H_{\mathbb{C}^n}^2,
\]

where \( P_n \) is the orthogonal projection of \( L_{\mathbb{C}^n}^2 \) onto \( H_{\mathbb{C}^n}^2 \).

A \textit{block Hankel} operator with symbol \( \Phi \in L^\infty_M \) is the operator \( H_\Phi : H_{\mathbb{C}^n}^2 \rightarrow H_{\mathbb{C}^n}^2 \) defined by

\[
H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H_{\mathbb{C}^n}^2,
\]

where \( J_n(f)(z) := \overline{z} I_n f(\overline{z}) \) for \( f \in L_{\mathbb{C}^n}^2 \).
We easily see that
\[
T_\Phi = \begin{bmatrix}
T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\
& \vdots & \\
T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}}
\end{bmatrix}
\quad \text{and} \quad
H_\Phi = \begin{bmatrix}
H_{\varphi_{11}} & \cdots & H_{\varphi_{1n}} \\
& \vdots & \\
H_{\varphi_{n1}} & \cdots & H_{\varphi_{nn}}
\end{bmatrix},
\]
where
\[
\Phi = \begin{bmatrix}
\varphi_{11} & \cdots & \varphi_{1n} \\
& \vdots & \\
\varphi_{n1} & \cdots & \varphi_{nn}
\end{bmatrix} \in L_{M_n}^\infty.
\]
For \( \Phi \in L_{M_{n \times m}}^\infty \), write
\[
\tilde{\Phi}(z) := \Phi^*(\overline{z}).
\]
A matrix-valued function \( \Theta \in H_{M_{n \times m}}^\infty ( = H^\infty \otimes M_{n \times m} ) \) is called inner if \( \Theta^* \Theta = I_m \) almost everywhere on \( \mathbb{T} \). Given \( \Phi, \Psi \in L_{M_n}^\infty \),

\[
T^*_\Phi = T_{\Phi^*}
\]

\[
H^*_\Phi = H_{\tilde{\Phi}} \quad \text{(recall that } \tilde{\Phi}(z) := \Phi^*(\bar{z}) \text{)}
\]

\[
T_{\Phi \Psi} - T_{\Phi} T_{\Psi} = H^*_\Phi H_{\Psi}
\]

\[
H_{\Phi} T_{\Psi} = H_{\Phi \Psi}
\]


\[ \Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n} \] is of *bounded type* if each entry \( \varphi_{ij} \) is of bounded type.

\( \Phi \) is *rational* if each entry \( \varphi_{ij} \) is a rational function.
The *shift* operator $S$ on $H^2_{\mathbb{C}^n}$ is defined by

$$S := T_{zI_n}.$$ 

**The Beurling-Lax-Halmos Theorem.** A nonzero subspace $\mathcal{M}$ of $H^2_{\mathbb{C}^n}$ is invariant for $S$ if and only if $\mathcal{M} = \Theta H^2_{\mathbb{C}^m}$, where $\Theta$ is an inner matrix function. Furthermore, $\Theta$ is unique up to a unitary constant right factor.

As a consequence, if $\ker H_\Phi \neq \{0\}$, then

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function $\Theta$. 
(C. Gu, J. Hendricks and D. Rutherford, 2006) For $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \{ K \in H^\infty_{M_n} : \|K\| \leq 1 \text{ and } \Phi - K\Phi^* \in H^\infty_{M_n} \}.$$ 

Then $T_\Phi$ is hyponormal if and only if $\Phi$ is normal (i.e. $\Phi^*\Phi = \Phi\Phi^*$) and $\mathcal{E}(\Phi)$ is nonempty.
Theorem

(Gu, Hendricks and Rutherford, 2006) For $\Phi \in L_{M_n}^\infty$, the following statements are equivalent:

1. $\Phi$ is of bounded type;
2. $\ker H_\Phi = \Theta H_{C^n}^2$ for some square inner matrix function $\Theta$;
3. $\Phi = A\Theta^*$, where $A \in H_{M_n}^\infty$ and $A$ and $\Theta$ are right coprime.

Definition: $\Theta$ and $A$ are right coprime if they do not have a common nontrivial right factor.
Is Abrahamse’s Theorem valid for Toeplitz operators with matrix-valued symbols?

In general, a straightforward matrix-valued version of Abrahamse’s Theorem is doomed to fail: for instance, if

$$\Phi := \begin{bmatrix} z + \overline{z} & 0 \\ 0 & z \end{bmatrix},$$

then both $\Phi$ and $\Phi^*$ are of bounded type and $T_\Phi = \begin{bmatrix} U_+ + U_+^* & 0 \\ 0 & U_+ \end{bmatrix}$ (where $U_+$ is the shift on $H^2$) is subnormal, but neither normal nor analytic.
In 2013, together with Dong-O Kang (CHKL), we were able to obtain a matrix-valued version of Abrahamse’s theorem, in the rational symbol case: If $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function having a matrix pole, i.e., $\exists \alpha \in \mathbb{D}$ for which $\ker H\Phi \subseteq (z - \alpha)H^2_{\mathbb{C}^n}$, and if $T\Phi$ is hyponormal and $\ker [T^*_\Phi, T\Phi]$ is invariant under $T\Phi$, then $T\Phi$ is normal. We will extend this result to the case of bounded type symbols. Thus, we get a full-fledged matrix-valued version of Abrahamse’s Theorem.

**Definition**

A symbol $\Phi$ has a matrix singularity if $\ker H\Phi \subseteq \theta H^2_{\mathbb{C}^n}$ for some nonconstant inner function $\theta$. 
(Abrahamse’s Theorem for matrix-valued symbols) Let $\Phi \in L^\infty_{M_n}$ be such that $\Phi$ and $\Phi^*$ are of bounded type. Assume $\Phi$ has a matrix singularity. If

(i) $T_\Phi$ is hyponormal;

(ii) $\ker [T_\Phi^*, T_\Phi]$ is invariant under $T_\Phi$,

then $T_\Phi$ is normal. Hence, in particular, if $T_\Phi$ is subnormal then $T_\Phi$ is normal.
Corollary (CHKL, 2013)

Let $\Phi \equiv \Phi^- + \Phi^+ \in L^\infty_{M_n}$ be a matrix-valued \textbf{rational} function. We may write

$$\Phi^- = \Theta B^*$$  \text{(right coprime factorization)}.

Assume that $\Theta$ has a nonconstant diagonal-constant inner divisor. Then the following are equivalent:

1. $T_\Phi$ is 2-hyponormal;
2. $T_\Phi$ is subnormal;
3. $T_\Phi$ is normal.
Yakubovich’s Theorem (2006). If $T \in B(\mathcal{H})$ is a pure subnormal operator with finite rank self-commutator and without point masses then it is \textit{unitarily equivalent} to a Toeplitz operator $T_\Phi$ with a matrix-valued analytic rational symbol $\Phi$.

On the other hand, Ito and Wong proved in 1972 that every quasinormal Toeplitz operator is either normal or analytic, i.e., the answer to the Halmos’s Problem 5 is affirmative for quasinormal Toeplitz operators.
However, this is not true for the cases of matrix-valued symbols: indeed, if

$$\Phi \equiv \begin{bmatrix} \bar{z} & \bar{z} + 2z \\ \bar{z} + 2z & \bar{z} \end{bmatrix}. \quad (2)$$

then $T_\Phi$ is quasinormal, but it is neither normal nor analytic. But since

$$T_\Phi = \begin{bmatrix} U^*_+ & U^*_+ + 2U_+ \\ U^*_+ + 2U_+ & U^*_+ \end{bmatrix},$$

one can prove that $T_\Phi$ is unitarily equivalent to

$$2 \begin{bmatrix} U^*_+ + U_+ & 0 \\ 0 & -U_+ \end{bmatrix},$$

that is, the direct sum of a normal operator, $2(U^*_+ + U_+)$, and an analytic Toeplitz operator, $-2U_+$. 

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Theorem (CHKL, 2013)

Every pure quasinormal operator with finite rank self-commutator is unitarily equivalent to a Toeplitz operator with a matrix-valued analytic rational symbol.

Corollary (CHKL, 2013)

Every pure quasinormal Toeplitz operator with a matrix-valued rational symbol is unitarily equivalent to an analytic Toeplitz operator.
Problem. For \( \lambda \in \mathbb{D} \), let \( b_\lambda \) be a Blaschke factor of the form 
\[
b_\lambda(z) := \frac{z-\lambda}{1-\lambda z}.
\]
Complete the unspecified rational Toeplitz operators (i.e., the unknown entries are rational Toeplitz operators) of the partial block Toeplitz matrix
\[
G := \begin{bmatrix}
T_{b_\alpha} & ? \\
? & T_{b_\beta}
\end{bmatrix} \quad (\alpha, \beta \in \mathbb{D})
\] (3)

to make \( G \) subnormal.
Let $\varphi, \psi \in L^\infty$ be rational and consider

$$G := \begin{bmatrix} T_{b_\alpha} & T_{\varphi} \\ T_{\psi} & T_{b_\beta} \end{bmatrix}.$$ 

Then the following statements are equivalent.

1. $G$ is normal.
2. $G$ is subnormal.
3. $G$ is 2-hyponormal.
4. $G$ is hyponormal and $\ker [G^*, G]$ is invariant for $G$. 
Theorem (cont.)

5. \( b_\alpha = b_\beta =: \omega \) and the following condition holds:

\[
\varphi = e^{i\delta_1} \omega + \zeta
\]

and

\[
\psi = e^{i\delta_2} \varphi
\]

with \( \zeta \in \mathbb{C}; \ \delta_1, \delta_2 \in [0, 2\pi) \), except in an exceptional case.
Corollary

Let

\[ A := \begin{bmatrix} U^* & U^* + 2U \\ U^* + 2U & U^* \end{bmatrix}, \]

where \( U \equiv T_z \) is the unilateral shift on \( H^2 \). Then \( A \) is a quasinormal (therefore subnormal) completion of \( \begin{bmatrix} U^* & ? \\ ? & U^* \end{bmatrix} \), and \( A \) is not normal.
Remark

(Example of exceptional case) If

\[ \Phi := \begin{bmatrix} \overline{z}^p & \overline{z}^p + 2z^p \\ \overline{z}^p + 2z^p & \overline{z}^p \end{bmatrix} \quad (p = 1, 2, \ldots) \]

then a straightforward calculation shows that \( T_\Phi \) is quasinormal, but not normal. We note, however, that \( T_\Phi \cong N \oplus T_A \), where \( N \) is normal and \( A \in H^\infty_{M_k} \) (\( \cong \) denotes unitary equivalence).
Remark (cont.)

In fact,

\[ T_\Phi \cong \begin{bmatrix} T_{\overline{z}^p + z^p} & 0 \\ 0 & -T_{z^p} \end{bmatrix}. \]

From this viewpoint, we might expect that this is not a coincidence. Thus we propose:

Conjecture

*Every subnormal rational Toeplitz operator is unitarily equivalent to a direct sum of a normal operator and an analytic Toeplitz operator.*