Polynomially hyponormal operators

Raúl Curto and Mihai Putinar

To the memory of Paul R. Halmos

Abstract. A survey of the theory of k-hyponormal operators starts with the construction of a polynomially hyponormal operator which is not subnormal. This is achieved via a natural dictionary between positive functionals on specific convex cones of polynomials and linear bounded operators acting on a Hilbert space, with a distinguished cyclic vector. The class of unilateral weighted shifts provides an optimal framework for studying k-hyponormality. Non-trivial links with the theory of Toeplitz operators on Hardy space are also exposed in detail. A good selection of intriguing open problems, with precise references to prior works and partial solutions, is offered.

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1. Hyponormal operators

Let $\mathcal{H}$ be a separable complex Hilbert space. A linear operator S acting on $\mathcal{H}$ is called *subnormal* if there exists a linear bounded extension of it to a larger Hilbert space, which is normal. Denoting by $\mathcal{K}$ this larger space and $P = P_{\mathcal{H}}^{\mathcal{K}}$ the orthogonal projection onto $\mathcal{H}$, the above definition can be translated into the identity

$$S = PN|_{\mathcal{H}} = N|_{\mathcal{H}},$$

with N a normal operator acting on $\mathcal{K}$. The spectral theorem asserts that the normality condition

$$[N^*, N] := N^*N - NN^* = 0$$

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implies the existence of a standard functional model for N, specifically described as an orthogonal direct sum of multipliers

$$(Nh)(z) = zh(z), \quad h \in L^2(\mu),$$

where μ is a positive Borel measure, compactly supported in the complex plane $\mathbb{C}$. In turn, the canonical example of a subnormal operator is (a direct sum of) multipliers

$$(Sf)(z) = zf(z), \quad f \in P^2(\mu),$$

where $P^2(\mu)$ stands for the closure of polynomials in the Lebesgue space $L^2(\mu)$.

The self-commutator of a subnormal operator is non-negative:

$$\langle [S^*, S]f, f\rangle = \|Sf\|^2 - \|S^*f\|^2 =$$

$$\|Nf\|^2 - \|PN^*f\|^2 = \|N^*f\|^2 - \|PN^*f\|^2 \geq 0.$$

It was Paul Halmos [36] who isolated the class of *hyponormal* operators, as those linear bounded operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy $[T^*, T] \geq 0$; it was soon discovered that not all hyponormal operators are subnormal. Much later it was revealed that a typical hyponormal operator model departs quite sharply from the above mentioned multipliers. More precisely, such a model is provided by one-dimensional singular integrals, of the form

$$(T\phi)(x) = x\phi(x) + ia(x)\phi(x) - b(x)(\text{p.v.}) \int_{-M}^{M} \frac{b(t)\phi(t)dt}{t-x},$$

where $M > 0$, $a, b \in L^\infty[-M, M]$ are real valued functions and $\phi \in L^2[-M, M; dt]$. The reader will easily verify that

$$(T^*\phi)(x) = x\phi(x) - ia(x)\phi(x) + b(x)(\text{p.v.}) \int_{-M}^{M} \frac{b(t)\phi(t)dt}{t-x},$$

hence

$$[T^*, T]\phi = 2b(x) \int_{-M}^{M} b(t)\phi(t)dt,$$

and consequently

$$\langle [T^*, T]\phi, \phi\rangle = 2\left| \int_{-M}^{M} b(t)\phi(t)dt \right|^2 \geq 0.$$

The question of understanding better the gap between subnormal and hyponormal operators was raised by Halmos; cf. his Hilbert space problem book [37]. In this direction, the following technical problem has naturally appeared: if S is a subnormal operator and p is a polynomial, then it is clear from the definition that $p(S)$ is also subnormal. One can see using simple examples of Toeplitz operators that, in general, T^2 is not hyponormal if T is hyponormal. What happens if $p(T)$ is hyponormal for all polynomials p? Is T in this case subnormal? About 15 years ago we were able to provide a counterexample.

Theorem 1.1. [32] There exists a polynomially hyponormal operator which is not subnormal.

Much more is known today. A whole scale of intermediate classes of operators, a real jungle, was discovered during the last decade. Their intricate structure is discussed in Sections 3 and 4 of this note.

2. Linear operators as positive functionals

The main idea behind the proof of Theorem 1.1 is very basic, and it proved to be, by its numerous applications, more important than the result itself.

Let $A \in \mathcal{L}(\mathcal{H})$ be a bounded self-adjoint operator with cyclic vector ξ; that is, the linear span of the vectors $A^k\xi$ ($k \geq 0$), is dense in $\mathcal{H}$. Denote $M := \|A\|$ and let Σ^2 denote the convex cone of all sums of squares of moduli of complex valued polynomials, in the real variable x. If $p \in \Sigma^2 + (M^2 - x^2)\Sigma^2$, that is

$$p(x) = \sum_i |q_i(x)|^2 + \sum_j (M^2 - x^2)|r_j(x)|^2,$$

with $q_i, r_j \in \mathbb{C}[x]$, then

$$\langle p(A)\xi, \xi\rangle = \sum_i \|q_i(A)\xi\|^2 + \sum_j (M^2\|r_j(A)\xi\|^2 - \|Ar_j(A)\xi\|^2) \geq 0.$$

Since every non-negative polynomial p on the interval $[-M, M]$ belongs to $\Sigma^2 + (M^2 - x^2)\Sigma^2$, the Riesz representation theorem implies the existence of a positive measure σ, supported in $[-M, M]$, so that

$$\langle p(A)\xi, \xi\rangle = \int p(t)d\sigma(t).$$

From here, a routine path leads us to the full spectral theorem; see for details [50]. An intermediate step in the above reasoning is important for our story, namely

Proposition 2.1. *There exists a canonical bijection between contractive self-adjoint operators A with a distinguished cyclic vector ξ and linear functionals $L \in \mathbb{C}[x]'$ which are non-negative on the cone $\Sigma^2 + (1 - x^2)\Sigma^2$. The correspondence is established by the compressed functional calculus map*

$$L(p) = \langle p(A)\xi, \xi\rangle, \quad p \in \mathbb{C}[x].$$

The reader would be tempted to generalize the above proposition to an arbitrary tuple of commuting self-adjoint operators. Although the result is the same, the proof requires a much more subtle Positivestellensatz (that is, a standard decomposition of a positive polynomial, on the polydisk in this case, into a weighted sum of squares). Here is the correspondence.

Proposition 2.2. *There exists a canonical bijection between commuting d-tuples of contractive self-adjoint operators $A_1, \cdots, A_d$ with a distinguished common cyclic vector ξ and linear functionals $L \in \mathbb{C}[x_1, \cdots, x_d]'$ which are non-negative on the* *cone* $\Sigma^2 + (1 - x_1^2)\Sigma^2 + \cdots + (1 - x_d^2)\Sigma^2$*. The correspondence is established by the compressed functional calculus map*

$$L(p) := \langle p(A)\xi, \xi\rangle, \quad p \in \mathbb{C}[x_1, \cdots, x_d].$$

The Positivstellensatz alluded to above (proved by the second named author in 1994 [49]) asserts that a strictly positive polynomial p on the hypercube $[-1,1] \times [-1,1] \times \cdots \times [-1,1] \subset \mathbb{R}^d$ belongs to $\Sigma^2 + (1 - x_1^2)\Sigma^2 + ... + (1 - x_d^2)\Sigma^2$. The survey [40] contains ample remarks on the links between the spectral theorem, Positivestellensätze in real algebra, optimization and applications to control theory.

In order to bring the classes of close-to-normal operators into the picture, we need a non-commutative calculus, applied to an operator and its adjoint. The idea goes back to the quasi-nilpotent equivalence relation introduced by I. Colojoara and C. Foiaş [8], and the hereditary functional calculus cast into a formal definition by J. Agler [2]. Let $T \in \mathcal{L}(\mathcal{H})$ and let z denote the complex variable in $\mathbb{C}$. For every monomial we define the *hereditary functional calculus* by

$$z^m \bar{z}^n(T, T^*) := T^{*n}T^m,$$

that is, we place all powers of T^* to the left of the powers of T. It is clear that some weak positivity of this functional calculus map is persistent for all operators T. More specifically, if $p \in \mathbb{C}[z]$ then

$$p(z)\overline{p(z)}(T, T^*) = p(T)^*p(T) \geq 0.$$

Aiming at a correspondence between positive functionals and operators as in the self-adjoint case, we define Σ_a^2 to be the convex cone generated in the algebra $\mathbb{C}[z, \bar{z}]$ by $|p(z)|^2$, $p \in \mathbb{C}[z]$. On the other hand, we denote as above by Σ^2 the convex cone of all sums of squares of moduli of polynomials, that is, polynomials of the form $|p(z, \bar{z})|^2$.

The main terms of the dictionary are contained in the following result, going back to the works of J. Agler [2], S. McCullough and V. Paulsen [47], and the authors [32].

Theorem 2.3. *a). There exists a bijective correspondence between linear contractive operators* $T \in \mathcal{L}(\mathcal{H})$ *with a distinguished cyclic vector* ξ *and linear functionals* $L \in \mathbb{C}[z, \bar{z}]'$ *which are non-negative on the convex cone* $(1-|z|^2)\Sigma_a^2+\Sigma_a^2$*, established by the hereditary calculus*

$$L(p) := \langle p(T, T^*)\xi, \xi\rangle, \quad p \in \mathbb{C}[z, \bar{z}].$$

b). The operator T *is subnormal if and only if* L *is non-negative on* Σ^2*.*

c). The operator T *is hyponormal if and only if*

$$L(|r + \bar{z}s|^2) \geq 0, \quad r, s \in \mathbb{C}[z].$$

d). The operator T *is polynomially hyponormal if and only if*

$$L(|r + \bar{q}s|^2) \geq 0, \quad q, r, s \in \mathbb{C}[z]. \tag{2.1}$$

The proof of assertion b) is based on the celebrated Bram-Halmos criterion for subnormality [37]. The proofs of a), c) and d) are simple manipulations of the definitions.

The above interpretation of subnormality and polynomial hyponormality invites the study of the filtration given by the condition

$$L(|p|^2) \geq 0 \text{ for all } p(z,\bar{z}) \equiv \sum_{j=0}^{k} \bar{z}^j p_j(z) \ (p_j \in \mathbb{C}[z]), \tag{2.2}$$

which defines the so-called *k-hyponormal* operators. In [32] we proved that polynomial hyponormality does not imply 2-hyponormality, and therefore does not imply subnormality either.

The proof of Theorem 1.1 consists in the construction of a linear functional which separates the convex cones associated to subnormal, respectively polynomially hyponormal operators; see [32] for details. The pioneering work of G. Cassier [5] contains an explicit construction of the same kind. The existence of the separating functional is known in the locally convex space theory community as the Kakutani-Eidelheit Lemma, and it is nowadays popular among the customers of multivariate moment problems (cf. [40]).

3. k-hyponormality for unilateral weighted shifts

Given a bounded sequence $\equiv \{\alpha_n\}_{n=0}^{\infty}$ of positive numbers, the unilateral weighted shift W_α acts on $\ell^2(\mathbb{Z}_+)$ by $W_\alpha e_n := \alpha_n e_{n+1}$ $(n \geq 0)$. Within this class of operators, the condition (2.2) acquires a rather simple form:

$$W_\alpha \text{ is } k\text{-hyponormal} \iff H_n := (\gamma_{n+i+j})_{i,j=0}^{k} \geq 0 \text{ (all } n \geq 0),$$

where $\gamma_0 := 1$ and $\gamma_{p+1} := \alpha_p^2 \gamma_p$ $(p \geq 0)$ [14]. Thus, detecting k-hyponormality amounts to checking the positivity of a sequence of $(k+1) \times (k+1)$ Hankel matrices. With this characterization at hand, it is possible to distinguish between k-hyponormality and $(k+1)$-hyponormality for every $k \geq 1$. Moreover, by combining the main result in [32] with the work in [47], we know that there exists a polynomially hyponormal unilateral weighted shift W_α which is not subnormal; however, it remains an open problem to find a specific weight sequence α with that property.

While k-hyponormality of weighted shifts admits a simple characterization, the same cannot be said of polynomial hyponormality. When one adds the condition deg $q \leq k$ to (2.1), we obtain the notion of weak k-hyponormality. We thus have a staircase leading up from hyponormality to subnormality, passing through 2-hyponormality, 3-hyponormality, and so on. A second staircase starts at hyponormality, goes up to quadratic hyponormality, to cubic hyponormality, and eventually reaches polynomial hyponormality. How these two staircases intertwine is not well understood. A number of papers have been written describing the links for specific families of weighted shifts, e.g., those with recursively generated tails and those obtained by restricting the Bergman shift to suitable invariant subspaces [14], [15], [16], [17], [18], [19], [20], [21], [22], [29], [30], [43], [44]; the overall problem, however, remains largely unsolved.

One first step is to study the precise connections between quadratic hyponormality and 2-hyponormality. While there are several results that establish quantitative differences between these two notions, there are two qualitative results that stand out. The first one has to do with a propagation phenomenon valid for the class of 2-hyponormal weighted shifts.

Theorem 3.1. *(i) If* $\alpha_0 = \alpha_1$ *and* W_α *is* 2*-hyponormal, then* $W_\alpha = \alpha_0 U_+$*, that is,* W_α *is a multiple of the (unweighted) unilateral shift* [14] *(for a related propagation result, see* [6]*);*
(ii) The set $Q := \{(x, y) \in \mathbb{R}^2_+ : W_{1,(1,x,y)^\wedge}$ *is quadratically hyponormal*$\}$ *contains a closed convex set with nonempty interior* [19]*.*
Thus, there exist many nontrivial quadratically hyponormal weighted shifts with two equal weights.

The second result entails completions of weighted shifts. J. Stampfli showed in [51] that given three initial weights $\alpha_0 < \alpha_1 < \alpha_2$, it is always possible to find new weights $\alpha_3, \alpha_4, \cdots$ such that W_α is subnormal; that is, W_α is a subnormal *completion* of the initial segment of weights. In [16] and [17], the first named author and L. Fialkow obtained the so-called Subnormal Completion Criterion, a concrete test that determines when a collection of initial weights $\alpha_0, \cdots, \alpha_m$ admits a subnormal completion W_α. On the other hand, quadratically hyponormal completions require different tools, as discovered in [19].

Theorem 3.2. *Let* $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ *be a given collection of positive weights.*
(i) There always exist weights $\alpha_4, \alpha_5, \cdots$ *such that* W_α *is quadratically hyponormal* [28]*.*
(ii) There exists weights $\alpha_4, \alpha_5, \cdots$ *such that* W_α *is* 2*-hyponormal if and only if*

$$H(2) := \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} \gamma_3 \\ \gamma_4 \end{pmatrix} \in Ran \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{pmatrix} \text{ [16].}$$

In a slightly different direction, attempts have been made to characterize, for specific families of weighted shifts, the weight sequences that give rise to subnormal weighted shifts. We recall a well known characterization of subnormality for weighted shifts due to C. Berger and independently established by R. Gellar and L.J. Wallen: W_α is subnormal if and only if $\gamma_n = \int t^n \, d\mu(t)$, where μ is a probability Borel measure supported in the interval $[0, |W_\alpha|^2]$ [9, III.8.16]. The measure μ is finitely atomic if and only if there exist scalars $\varphi_0, \cdots, \varphi_{r-1}$ such that $\gamma_{n+r} = \varphi_0 \gamma_n + ... + \varphi_{r-1} \gamma_{n+r-1}$ (all $n \geq 0$) [16]; we call such shifts *recursively generated*. The positivity conditions in Theorem 3.2(ii) ensure the existence of a recursively generated subnormal (or, equivalently, 2-hyponormal) completion.

In an effort to unravel how k-hyponormality and weak k-hyponormality are interrelated, researchers have looked at weighted shifts whose first few weights are unrestricted but whose tails are subnormal and recursively generated [4], [7], [34], [35], [41], [42]. A special case involves shifts whose weight sequences are of the form $x, \alpha_0, \alpha_1, \cdots$, with W_α subnormal. Then $W_{x,\alpha}$ is subnormal if and only if $\frac{1}{t} \in L^1(\mu)$ and $x^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu)})^{-1}$ [14]. Thus, the subnormality of a weighted shift can be maintained if one alters the first weight slightly. The following result states that this is the only possible finite rank perturbation that preserves subnormality; quadratic hyponormality, on the other hand, is a lot more stable.

Theorem 3.3. *([29]) (i) Let W_α be subnormal and let F ($\neq cP_{\langle e_0 \rangle}$) be a nonzero finite rank operator. Then $W_\alpha + F$ is not subnormal.*
(ii) Let α be a strictly increasing weight sequence, and assume that W_α is 2-hyponormal. Then $W_{\alpha'}$ remains quadratically hyponormal for all α' such that $\alpha' - \alpha$ is a small nonzero finite rank perturbation.

On a related matter, taking W_α as the restriction of the Bergman shift to the invariant subspace generated by $\{e_2, e_3, \cdots\}$, it is possible to find a range of values for $x > 0$ such that $W_{x,\alpha}$ is quartically hyponormal but not 3-hyponormal [22]. On the other hand, when the weights are given by $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$ ($n \geq 0$), with $a, b, c, d \geq 0$ and $ad - bc > 0$, it was shown in [31] that W_α is always subnormal.

In many respects, 2-hyponormality behaves much like subnormality, particularly within the classes of unilateral weighted shifts and of Toeplitz operators on $H^2(\mathbb{T})$; for instance, a 2-hyponormal operator always leaves the kernel of its self-commutator invariant [26, Lemma 2.2]. The results in [7], [13], [14], [16], [17], [23], [20], [29] and [42] all seem to indicate the existence of a model theory for 2-hyponormal operators, with building blocks given by weighted shifts with recursive subnormal tails and Toeplitz operators with special trigonometric symbols. In [26] the beginnings of such a theory are outlined, including a connection to Agler's abstract model theory [3] - see [26, Section 5]. The proposed model theory involves a new notion, that of *weakly subnormal* operator T, characterized by an extension $\hat{T} \in \mathcal{L}(\mathcal{K})$ such that $\hat{T}^*\hat{T}f = \hat{T}\hat{T}^*f$ (all $f \in \mathcal{H}$); we refer to $\hat{T}$ as a *partially normal extension* of T.

At the level of weighted shifts, it was proved in [26, Theorem 3.1] that if α is strictly increasing then W_α is weakly subnormal precisely when $\limsup u_{n+1}/u_n < \infty$, where $u_n := \alpha_n^2 - \alpha_{n-1}^2$. This characterization allows one to show that every 2-hyponormal weighted shift is automatically weakly subnormal [26, Theorem 1.2] and that the class of weakly subnormal shifts is strictly larger than the class of 2-hyponormal shifts [26, Example 3.7]; however, there exist quadratically hyponormal weighted shifts which are not weakly subnormal [26, Example 5.5]. Moreover, it was shown in [26] that if W_α is 2-hyponormal, then the sequence of quotients u_{n+1}/u_{n+2} is bounded, and bounded away from zero; in particular, the sequence $\{u_n\}$ is eventually decreasing. On the other hand, if T is 2-hyponormal or weakly subnormal, with rank-one self-commutator, then T is subnormal; if, in addition,

T is pure, then T is unitarily equivalent to a linear function of U_+. Weak subnormality can also be used to characterize k-hyponormality, as follows: an operator T is $(k+1)$-hyponormal if and only if T is weakly subnormal and admits a partially normal extension $\hat{T}$ which is k-hyponormal [21].

All of the previous results encourage us to consider the following

Problem 3.4. *Develop a model theory for 2-hyponormality, parallel to subnormal operator theory.*

While it is easy to see that the class of 2-hyponormal contractions forms a *family* (in the sense of J. Agler), it is an open problem whether the same is true of weakly subnormal contractions [26, Question 6.5]. M. Dritschel and S. McCullough found in [33] a sufficient condition for a 2-hyponormal contraction to be extremal.

As we have mentioned before, nontrivial 2-hyponormal weighted shifts are closely related to recursively generated subnormal shifts, i.e., those shifts whose Berger measures are finitely atomic. In [20] a study of extensions of recursively generated weight sequences was done. Given a recursively generated weight sequence $(0 < \alpha_0 < \cdots < \alpha_k)$, and an n-step extension $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$, it was established that

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } ([\frac{k+1}{2}]+1)\text{-hyponormal} & (n=1) \\ W_\alpha \text{ is } ([\frac{k+1}{2}]+2)\text{-hyponormal} & (n>1) \end{cases}.$$

In particular, the subnormality of an extension is *independent* of its length if the length is bigger than 1. As a consequence, if $\alpha(x)$ is a canonical rank-one perturbation of the recursive weight sequence α, then subnormality and k-hyponormality for $W_{\alpha(x)}$ eventually coincide! This means that the subnormality of $W_{\alpha(x)}$ can be detected after finitely many steps. Conversely, if k- and $(k+1)$-hyponormality for $W_{\alpha(x)}$ coincide then $\alpha(x)$ must be recursively generated, i.e., $W_{\alpha(x)}$ is a recursive subnormal.

4. The case of Toeplitz operators

Recall Paul Halmos's Problem 5 (cf. [38], [39]): Is every subnormal Toeplitz operator either normal or analytic? As we know, this was answered in the negative by C. Cowen and J. Long [12]. It is then natural to ask: Which Toeplitz operators are subnormal? We recall the following result.

Theorem 4.1. *([1]) If (i) T_φ is hyponormal; (ii) φ or $\bar{\varphi}$ is of bounded type (i.e., φ or $\bar{\varphi}$ is a quotient of two analytic functions); and (iii)* $\ker[T_\varphi^*, T_\varphi]$ *is invariant for T_φ, then T_φ is normal or analytic.*

(We mention in passing a recent result of S.H. Lee and W.Y. Lee [46]: if $T \in \mathcal{L}(\mathcal{H})$ is a pure hyponormal operator, if $\ker[T^*, T]$ is invariant for T, and if $[T^*, T]$ is rank-two, then T is either a subnormal operator or Putinar's matricial model of rank two.)

Since $\ker[T^*, T]$ is invariant under T for every subnormal operator T, Theorem 4.1 answers Problem 5 affirmatively when φ or $\bar{\varphi}$ is of bounded type. Also, every hyponormal Toeplitz operator which is unitarily equivalent to a weighted shift must be subnormal [52], [10], a fact used in

Theorem 4.2. *([12], [10]). Let $0 < \alpha < 1$ and let ψ be a conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. If $\varphi := (1-\alpha^2)^{-1}(\psi + \alpha\bar{\psi})$, then T_φ is a weighted shift with weight sequence $\alpha_n = (1-\alpha^{2n+2})^{-\frac{1}{2}}$. Therefore, T_φ is subnormal but neither normal nor analytic.*

These results show that subnormality for weighted shifts and for Toeplitz operators are conceptually quite different. One then tries to answer the following

Problem 4.3. *Characterize subnormality of Toeplitz operators in terms of their symbols.*

Since subnormality is equivalent to k-hyponormality for every $k \geq 1$ (this is the Bram-Halmos Criterion), one possible strategy is to first characterize k-hyponormality, and then use it to characterize subnormality. As a first step, we pose the following

Problem 4.4. *Characterize 2-hyponormality for Toeplitz operators.*

As usual, the Toeplitz operator T_φ on $H^2(\mathbb{T})$ with symbol $\varphi \in L^\infty(\mathbb{T})$ is given by $T_\varphi g := P(\varphi g)$, where P denotes the orthogonal projection from $L^2(\mathbf{T})$ to $H^2(\mathbf{T})$. In [25, Chapter 3] the following question was considered:

Problem 4.5. *Is every 2-hyponormal Toeplitz operator T_φ subnormal?*

For the case of trigonometric symbol, one has

Theorem 4.6. *([25]) Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic; in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.*

Theorem 4.6 shows that there is a big gap between hyponormality and quadratic hyponormality for Toeplitz operators. For example, if $\varphi(z) \equiv \sum_{n=-m}^{N} a_n z^n$ $(0 < m < N)$ is such that T_φ is hyponormal, then by Theorem 4.6, T_φ is never quadratically hyponormal, since T_φ is neither analytic nor normal. One can extend Theorem 4.6. First we observe

Proposition 4.7. *([26]) If $T \in \mathcal{L}(\mathcal{H})$ is 2–hyponormal then $T(\ker[T^*, T]) \subseteq \ker[T^*, T]$.*

Corollary 4.8. *If T_φ is 2–hyponormal and if φ or $\bar{\varphi}$ is of bounded type then T_φ is normal or analytic, so that T_φ is subnormal.*

Theorem 4.9. *([27, Theorem 8]) If the symbol φ is almost analytic (i.e., $z^n\varphi$ analytic for some positive n), but not analytic, and if T_φ is 2–hyponormal, then φ must be a trigonometric polynomial.*

In [27, Lemma 9] it was shown that if T_φ is 2–hyponormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product, then T_φ is normal or analytic. Moreover, [27, Theorem 10] states that when $\log|\varphi|$ is not integrable, a 2–hyponormal Toeplitz operator T_φ with nonzero finite rank self–commutator must be analytic. One also has

Theorem 4.10. *(cf. [27]) If T_φ is 2–hyponormal and if φ or $\bar{\varphi}$ is of bounded type (i.e., φ or $\bar{\varphi}$ is a quotient of two analytic functions), then T_φ is normal or analytic, so that T_φ is subnormal.*

The following problem arises naturally:

Problem 4.11. *If T_φ is a 2–hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic? If so, is φ a linear function of a finite Blaschke product?*

A partial positive answer to Problem 4.11 was given in [27, Theorem 10]. In view of Cowen and Long's counterexample [12], it is worth turning attention to hyponormality of Toeplitz operators, which has been studied extensively. An elegant theorem of C. Cowen [11] characterizes the hyponormality of a Toeplitz operator T_φ on $H^2(\mathbf{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbf{T})$. The variant of Cowen's theorem [11] that was first proposed in [48] has been most helpful. We conclude this section with a result that extends the work of Cowen and Long to 2-hyponormality and quadratic hyponormality.

Theorem 4.12. *([23, Theorem 6]) Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi = \psi + \lambda\bar{\psi}$ and let T_φ be the corresponding Toeplitz operator on H^2. Then*
(i) T_φ is hyponormal if and only if λ is in the closed unit disk $|\lambda| \leq 1$.
(ii) T_φ is subnormal if and only if $\lambda = \alpha$ or λ is in the circle $\left|\lambda - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}\right| = \frac{\alpha^k(1-\alpha^2)}{1-\alpha^{2k+2}}$ for $k = 0, 1, 2, \cdots$. (Observe that the case $\lambda = \alpha$ is part of the main result in [12].)
(iii) T_φ is 2–hyponormal if and only if λ is in the unit circle $|\lambda| = 1$ or in the closed disk $\left|\lambda - \frac{\alpha}{1+\alpha^2}\right| \leq \frac{\alpha}{1+\alpha^2}$.
(iv) ([45]) T_φ is 2-hyponormal if and only if T_φ is quadratically hyponormal.

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Raúl Curto
Department of Mathematics, University of Iowa, Iowa City, IA 52246
e-mail: rcurto@math.uiowa.edu

Mihai Putinar
Department of Mathematics, University of California, Santa Barbara, CA 93106
e-mail: mputinar@math.ucsb.edu