Truncated Moment Problems:
The Extremal Case
(joint work with L. Fialkow and M. Möller)

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Abstract

- For a degree $2n$ real $d$-dimensional multisequence
  \[ \beta \equiv \beta^{(2n)} = \{ \beta_i \}_{i \in \mathbb{Z}_d^+} : |i| \leq 2n \]
  to have a representing measure $\mu$, it is necessary for the associated moment matrix $M(n)(\beta)$ to be positive semidefinite, and for the algebraic variety associated to $\beta$, $V \equiv V_\beta$, to satisfy $\text{rank} M(n) \leq \text{card} V$ as well as the following consistency condition: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on $V$, then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$.

- In joint work with Lawrence Fialkow and Michael Möller, we prove that for the extremal case ($\text{rank} M(n) = \text{card} V$), positivity of $M(n)$ and consistency are sufficient for the existence of a (unique, $\text{rank} M(n)$-atomic) representing measure.
The new results build on our operator-theoretic approach to truncated moment problems, based on matrix positivity and extension, which, via a “functional calculus” for the columns of the associated moment matrix, allows us to obtain existence theorems in case the columns satisfy one of several natural constraints.

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Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the TCMP entails finding a positive Borel measure $\mu$ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \leq i + j \leq 2n);$$

$\mu$ is called a rep. meas. for $\gamma$.

In earlier joint work with L. Fialkow,

We have introduced an approach based on matrix positivity and extension, combined with a new “functional calculus” for the columns of the associated moment matrix.
We have shown that when the TCMP is of flat data type, a solution always exists; this is compatible with our previous results for:

\[
\begin{align*}
supp \mu & \subseteq \mathbb{R} \quad \text{(Hamburger TMP)} \\
 supp \mu & \subseteq [0, \infty) \quad \text{(Stieltjes TMP)} \\
 supp \mu & \subseteq [a, b] \quad \text{(Hausdorff TMP)} \\
 supp \mu & \subseteq \mathbb{T} \quad \text{(Toeplitz TMP)}
\end{align*}
\]

Along the way we have developed new machinery for analyzing TMP’s in one or several real or complex variables. For simplicity, in this talk we focus on one complex variable or two real variables, although several results have multivariable versions.
Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.

We fully resolve, among others, the cases

\[ \tilde{Z} = \alpha 1 + \beta Z \]

and

\[ Z^k = p_{k-1}(Z, \tilde{Z}) \quad (1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 1; \deg p_{k-1} \leq k - 1) \]

We obtain applications to quadrature problems in numerical analysis.

We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on \( \mathbb{R}^d \).
Very recently, we have begun to use our methods to solve FULL moment problems, by first solving truncated MP’s, and then applying J. Stochel’s limiting argument.

Our matrix extension approach works equally well to **localize the support** of a rep. meas.

In the specific case of \( K := \text{supp } \mu \) a semi-algebraic set determined by a finite collection of complex polynomials \( \mathcal{P} = \{p_i(z, \bar{z})\}_{i=1}^m \), i.e.,

\[
K = K_P := \{z \in \mathbb{C} : p_i(z, \bar{z}) \geq 0, 1 \leq i \leq m\},
\]

we obtain an existence criterion expressed in terms of positivity and extension properties of the moment matrix \( M(n)(\gamma) \) associated to \( \gamma \) and of the localizing matrix \( M_{p_i} \) corresponding to each \( p_i \).
Motivation: extensive literature on the FULL moment problem, in particular the Riesz-Haviland Criterion, which provides an “abstract” solution to the Full Multivariable $K$-Moment Problem.

If $\Lambda_{\gamma}(\bar{z}^rz^s) := \gamma_{rs}$, $\gamma$ admits a rep. meas $\mu$ iff $(p|_{K} \geq 0 \Rightarrow \Lambda_{\gamma}(p) \geq 0)$.

- In general, the cone $C$ of nonnegative poly’s on $K$ cannot easily be characterized, so the Riesz–Haviland criterion is intractable.
- For $K = \mathbb{C}$, no concrete descript. of $C$ exists ($\Rightarrow$ Full CMP unsolved)
- For $K = \overline{D}$, A. Atzmon (1975) found a “concrete” solution to MP
- An alternate solution for $K = \overline{D}$ was later found by M. Putinar
Theorem

(Smul’jan, 1959)

\[
\begin{pmatrix}
A & B \\
B^* & C \\
\end{pmatrix} \geq 0 \iff \begin{cases}
A \geq 0 \\
B = AW \\
C \geq W^*AW
\end{cases}.
\]

Moreover, \( \text{rank} \begin{pmatrix}
A & B \\
B^* & C \\
\end{pmatrix} = \text{rank} \: A \iff C = W^*AW. \)
**Corollary**

Let $A \geq 0$ and assume $\text{rank } \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank } A$. Then

$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$. 
Basic Positivity Condition

\(\mathcal{P}_n\) : polynomials \(p\) in \(z\) and \(\bar{z}\), \(\deg p \leq n\)

Given \(p \in \mathcal{P}_n\), \(p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j\),

\[
0 \leq \int |p(z, \bar{z})|^2 \, d\mu(z, \bar{z})
\]

\[
= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} \, d\mu(z, \bar{z})
\]

\[
= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell,j+k}.
\]

To understand this "matricial" positivity, we introduce the following lexicographic order on the rows and columns of \(M(n)\):

\(1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots\)
Define $M[i, j]$ as in

$$M[3, 2] := \begin{pmatrix}
\gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{14} & \gamma_{23} & \gamma_{32} \\
\gamma_{05} & \gamma_{14} & \gamma_{23}
\end{pmatrix}$$

Then

("matricial" positivity) \[\sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell,j+k} \geq 0\]

\[\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix}
M[0, 0] & M[0, 1] & \ldots & M[0, n] \\
M[1, 0] & M[1, 1] & \ldots & M[1, n] \\
\vdots & \vdots & \ddots & \vdots \\
M[n, 0] & M[n, 1] & \ldots & M[n, n]
\end{pmatrix} \geq 0.\]
For example,

\[ M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}, \]

\[ M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\
\gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}. \]
In general,

\[ M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \]

**Positivity Condition is not sufficient:**

By modifying an example of K. Schmüdgen, we have built a family \( \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{06}, \ldots, \gamma_{60} \) with positive invertible moment matrix \( M(3) \) but no rep. meas. But this can also be done for \( n = 2 \).
Example

(RC - L. Fialkow) For $f > 1$ let

$$M(2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & f & f - 1 & f - 1 \\
1 & 0 & 0 & f - 1 & f & f - 1 \\
0 & 1 & 0 & f - 1 & f - 1 & f
\end{pmatrix}.$$ 

$M(2) \succeq 0$, but $\gamma^{(4)}$ has no rep. measure.
Moment Problems and Nonnegative Polynomials

- $\mathcal{B} := \{ \gamma \equiv \gamma^{(\infty)} : \gamma \text{ admits a rep. meas. } \mu \}$
- $\mathcal{B}_+ := \{ \gamma \equiv \gamma^{(\infty)} : M(\infty)(\gamma) \geq 0 \}$

Clearly, $\mathcal{B} \subseteq \mathcal{B}_+$

- (Berg, Christensen and Ressel) $\gamma \in \mathcal{B}_+$, $\gamma$ bounded $\Rightarrow \gamma \in \mathcal{B}$
- (Berg and Maserick) $\gamma \in \mathcal{B}_+$, $\gamma$ exponentially bounded $\Rightarrow \gamma \in \mathcal{B}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ flat $\Rightarrow \gamma \in \mathcal{B}$

- $\mathcal{P}$ : nonnegative poly’s
- $\Sigma^2$ : sums of squares of poly’s

Clearly, $\Sigma^2 \subseteq \mathcal{P}$
Duality

- (Haviland) $\mathcal{P}^* = \mathcal{B}$
  For, $\mathcal{B} \rightarrow \mathcal{P}^*$, $\gamma \mapsto \Lambda_\gamma(p) := p(\gamma)$ (Riesz functional), and there exists $\mu$ r.m. for $\gamma$ if and only if $\Lambda_\gamma \geq 0$ on $\mathcal{B}$

- $\mathcal{P} = \mathcal{B}^*$ (straightforward once we have a r.m.)

- $\mathcal{B}_+ = (\Sigma^2)^*$ (straightforward)

- (Berg, Christensen and Jensen) $(\mathcal{B}_+)^* = \Sigma^2$

- ($n = 1$) $\mathcal{P} = \Sigma^2 \Rightarrow \mathcal{P}^* = (\Sigma^2)^* \Rightarrow \mathcal{B} = \mathcal{B}_+$ (Hamburger)

- (Hilbert) Description of pairs $(n, d)$ for which every poly of degree $d$ in $n$ indeterminates which is nonnegative on $\mathbb{R}^n$ is a sum of squares (cf. Reznick 2000)
For $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} z^i \bar{z}^j$ define

$$p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j.$$ 

If there exists a rep. meas. $\mu$, then

$$p(Z, \bar{Z}) = 0 \iff \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

\[(\text{RG}) \quad \text{If } p, q, pq \in \mathcal{P}_n, \text{ and } p(Z, \bar{Z}) = 0,\]

\[\text{then } (pq)(Z, \bar{Z}) = 0.\]
• **Observation 1.** If $\mu$ is a rep. meas. and $p(Z, \bar{Z}) = 0$, then $\text{supp } \mu \subseteq Z(p)$.

• **Observation 2.** $p(Z, \bar{Z}) = 0 \Rightarrow \bar{p}(Z, \bar{Z}) = 0$.

**Theorem**

*(Structure Theorem)* Let $M(n) \geq 0$. If $f, g, fg \in P_{n-1}$ and if $f(Z, \bar{Z}) = 0$, then $(fg)(Z, \bar{Z}) = 0$. 
**Theorem**

If $M(n) \geq 0$, RG, and $\{1, Z, \bar{Z}, Z^2\}$ lin. dependent in $C_{M(n)}$, then $\gamma^{(2n)}$ has a rep. meas.

**Remark**

In Example above, $\{1, Z, \bar{Z}, Z^2\}$ lin. indep. ($\Rightarrow M(2)$ RG).
**Theorem**

Let $\gamma$ be a truncated moment sequence. TFAE:

(i) $\gamma$ has a rep. meas. with moments up to order $2n + 1$;

(ii) $\gamma$ has a rep. meas. with moments of all orders;

(iii) $\gamma$ has a compactly supported rep. meas.;

(iv) $\gamma$ has a finitely atomic rep. meas. (with at most $(n + 2)(2n + 3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n + k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n + k + 1)$ (here $k \leq 2n^2 + 6n + 6$).
In special cases,

\[ M(n) \geq 0, \ M(n) \text{ (RG)} \Rightarrow \exists \text{ fin. atom. } \mu \]

(iv) \Rightarrow \text{weak version of flatness:}

\[ \mu \text{ fin. at., with } k \text{ atoms } \Rightarrow M(k)[\mu] \text{ flat.} \]

(J.E. McCarthy, 1995) For \( n \geq 5 \), there exist sequences \( \gamma \equiv \{\gamma_{ij}\}_{0 \leq i+j \leq 2n} \) such that \( M(n) \) admits a rep. meas. but no rank \( M(n) \)-atomic rep. meas.

(L. Fialkow, 1997) McCarthy’s phenomenon for \( n = 3 \).

(RC & LF, 1998) McCarthy’s phenomenon for \( n = 2! \).
Case of Flat Data

Recall: If $\mu$ is a rep. meas. for $M(n)$, then $\text{rank } M(n) \leq \text{card supp } \mu$.

$\gamma$ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

Theorem (RC & LF, 1996) If $\gamma$ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n + 1)$.

Corollary

If $\gamma$ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique positive extension of the form $M(\infty)$, and this is a flat extension of $M(n)$.

To find $\mu$ concretely, let $r := \text{rank } M(n)$ and look for the relation
\[ Z^r = c_01 + c_1Z + \ldots + c_{r-1}Z^{r-1}. \]

We then define

\[ p(z) := z^r - (c_0 + \ldots + c_{r-1}z^{r-1}) \]

and solve the Vandermonde equation

\[
\begin{pmatrix}
1 & \cdots & 1 \\
z_0 & \cdots & z_{r-1} \\
\vdots & \cdots & \vdots \\
z_0^{r-1} & \cdots & z_{r-1}^{r-1}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\vdots \\
\rho_{r-1}
\end{pmatrix}
= 
\begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\vdots \\
\gamma_{0r-1}
\end{pmatrix}.
\]

Then

\[ \mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}. \]
The truncated moment sequence $\gamma$ has a rank $M(n)$-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n + 1)$. 
Consider the full MP

\[ \int \bar{z}^i z^j \, d\mu = \gamma_{ij} \ (i, j \geq 0), \]

where \( \text{supp} \, \mu \subseteq K \), for \( K \) a closed subset of \( \mathbb{C} \).

The **Riesz functional** is given by

\[ \Lambda_{\gamma}(\bar{z}^i z^j) := \gamma_{ij} \ (i, j \geq 0). \]

- **Riesz-Haviland**: There exists \( \mu \) with \( \text{supp} \, \mu \subseteq K \iff \Lambda_{\gamma}(p) \geq 0 \) for all \( p \) such that \( p|_K \geq 0 \).
If $q$ is a polynomial in $z$ and $\bar{z}$, and

$$K \equiv K_q := \{ z \in \mathbb{C} : q(z, \bar{z}) \geq 0 \},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for $\mu$ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p} \, d\mu \geq 0 \quad (\text{all } p).$$

- K. Schmüdgen (1991): If $K_q$ is compact, $\Lambda_\gamma(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all $p$, then there exists $\mu$ with $\text{supp } \mu \subseteq K_q$.
- We shall establish a version of this result for truncated MP’s.
First, recall that \( p(Z, \bar{Z}) = 0 \) implies \( \text{supp} \ \mu \subseteq \mathcal{Z}(p) \). We define the algebraic variety of \( \gamma \) as

\[
\mathcal{V}(\gamma) := \bigcap_{p \in \mathcal{P}_n} \mathcal{Z}(p),
\]

and observe that \( \text{rank } M(n) \leq \text{card supp } \mu \leq \text{card } \mathcal{V}(\gamma) \), from which it follows that

\[
\text{card } \mathcal{V}(\gamma) < \text{rank } M(n) \Rightarrow \text{there is no rep. meas. } \mu.
\]
For $q(z, \bar{z}) \equiv \sum q_{ij} \bar{z}^i z^j$, $M_q(n) := \sum q_{ij} M(n)_{\bar{z}^i z^j}$, where

$$M(n)_{\bar{z}^i z^j} := \begin{pmatrix}
\gamma_{ij} & \gamma_{i+1,j} & \gamma_{i+2,j-1} & \cdots \\
\gamma_{i,j+1} & \gamma_{i+1,j+1} & \gamma_{i+2,j} & \cdots \\
\gamma_{i-1,j+2} & \gamma_{i,j+2} & \gamma_{i+1,j+1} & \cdots \\
\cdots & \cdots & \cdots & \cdots 
\end{pmatrix}.$$

Observe that since we only have a finite supply of $\gamma_{i,j}$'s, the size of $M(n)_q$ is determined by the degree of $q$: the larger the degree the smaller the size of $M(n)_q$.

- $M_q(n)^* = M_{\bar{q}}(n)$
- If $\gamma \equiv \gamma^{(2n)}$ has a rep meas $\mu$, $K = \text{supp } \mu$, $k \leq \frac{n}{2}$, and $q \in \mathcal{P}_{2k}$, then

$$M_q(n) = 0 \Leftrightarrow q|_K \equiv 0.$$
Theorem

Let $M(n) \geq 0$ and suppose $\deg(q) = 2k$ or $2k - 1$ for some $k \leq n$. Then

$\exists \mu$ with rank $M(n)$ atoms and $\text{supp} \ \mu \subseteq K_q$ if and only if $\exists$ a flat extension $M(n+1)$ for which $M_q(n+k) \geq 0$. In this case, $\exists \mu$ with exactly rank $M(n) - \text{rank} \ M_q(n+k)$ atoms in $\mathcal{Z}(q)$.

Remark

M. Laurent has recently found an alternative proof, using ideas from real algebraic geometry.
Let \( r := \text{rank} \mathcal{M}(n) \) and let \( \mathcal{B} \equiv \{ T^{i_k}_k \}_{k=1}^r \) denote a maximal lin. indep. set of columns of \( \mathcal{M}(n) \). For \( \mathcal{V} \equiv \{ v_j \}_{j=1}^r \subseteq \mathbb{R}^N \), let \( W_{\mathcal{B},\mathcal{V}} \) denote the \( r \times r \) matrix whose entry in row \( k \), column \( j \) is \( v_j^{i_k} \) \((1 \leq k, j \leq r)\).

**Theorem**

(Description of \( \mu \) in terms of \( V(\mathcal{M}(n+1)) \)) If \( \mathcal{M}(n) \equiv \mathcal{M}(n)(\beta) \geq 0 \) admits a flat extension \( \mathcal{M}(n+1) \), then \( V := V(\mathcal{M}(n+1)) \) satisfies \( \text{card} \ V = r \) \((\equiv \text{rank}\mathcal{M}(n))\), and \( V \equiv \{ v_j \}_{j=1}^r \) forms the support of the unique rep. meas. \( \mu \) for \( \mathcal{M}(n+1) \). If \( \mathcal{B} \equiv \{ T^{i_k}_k \}_{k=1}^r \) is a maximal lin. indep. subset of columns of \( \mathcal{M}(n) \), then \( W_{\mathcal{B},\mathcal{V}} \) is invertible, and \( \mu = \sum_{i=1}^r \rho_j \delta_{v_j} \), where \( \rho \equiv (\rho_1, \ldots, \rho_r) \) is uniquely determined by \( \rho^t = W_{\mathcal{B},\mathcal{V}}^{-1}(\beta_{i_1}, \ldots, \beta_{i_r})^t \).
Uniqueness in TCMP

**Proposition**

If $\gamma^{(2n)}$ is flat, then there exists a unique rep. meas., which is rank $M(n)$-atomic.

**Proposition**

If $\gamma^{(2n)}$ has a rep. meas. and if $M(n)(\gamma)$ admits an analytic relation

$$Z^k = p_{k-1}(Z, \bar{Z})$$

for some $k \leq n$, then $\gamma$ has a unique rep. meas., which is finitely atomic with at most $k^2$ atoms.

**Key Lemma**

A polynomial of the form $p(z, \bar{z}) \equiv z^k - q(z, \bar{z})$, where $q \in \mathbb{C}_{k-1}[z, \bar{z}]$, has at most $k^2$ roots.
Recall that if $\gamma^{(2n)}$ admits a rep. meas., then

$$M(n) \equiv M(n)(\gamma) \geq 0$$

$M(n)$ is RG

$$\text{card } \mathcal{V}(\gamma) \geq \text{rank } M(n).$$

**Question**

Assume $M(n)$ satisfies (11.1), and $M(n)$ is singular. Does $\gamma$ admit a rep. meas.?
Question

For which \( p \in \mathcal{P}_n \) do (11.1) and \( p(Z, \bar{Z}) = 0 \) imply that \( \gamma \) has a rep. meas.?

(Fialkow) Consider \( p(z, \bar{z}) \equiv z^k - q(z) \), with \( \deg q < k \). If \( k \) is minimal, if \( \gamma \) satisfies (11.1) and if \( p(Z, \bar{Z}) = 0 \), then

\[ B := \{1, Z, Z^2, \ldots, Z^{k-1}\} \]

is lin. indep. Moreover,

\[ k \geq \text{card } \mathcal{V}(\gamma) \geq \text{rank } M(n) \geq k, \]

so \( B \) is indeed a basis for \( \mathcal{C}_{M(n)} \). It follows that \( M(n) \) is flat, and it therefore admits a \( k \)-atomic rep. meas.
(Stochel, 1992) Call $p \in \mathbb{R}[x, y]$ of type $A$ if

$$M(\infty)(\beta(\infty)) \geq 0, \ p(X, Y) = 0 \Rightarrow \beta(\infty) \text{ has a rep. meas.}$$

(i) $\deg p \leq 2 \Rightarrow p$ is of type $A$

(ii) $y(y - x^2)$ is not of type $A$

(Schmüdgen, 2003) considered $y^2 = x^3$ in the context of properties (MP) and (SMP)

We shall see below the analogous notions for TMP.
The Quartic Moment Problem

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A 1$ (Dirac measure)
- $\bar{Z} = A 1 + B Z$ ($\text{supp} \, \mu \subseteq \text{line}$)
- $Z^2 = A 1 + B Z + C \bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A 1 + B Z + C \bar{Z} + D Z^2$

$$D = 0 \Rightarrow \bar{Z}Z = A 1 + B Z + \bar{B} \bar{Z} \text{ and } C = \bar{B}$$

$$\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2$$

$$\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}.$$
**Theorem**

\((D \neq 0)\) TFAE

(i) \(M(2)\) has a f.a.r.m.

(ii) \(M(2)\) admits a 4-atomic (minimal) r.m.

(iii) \(M(2)\) admits a flat extension

(iv) there exists \(\gamma_{23} \in \mathbb{C}\) such that

\[
\bar{\gamma}_{23} - D\gamma_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31}.
\]

When \(|D| \neq 1\), such a \(\gamma_{23}\) exists, and it is unique.
**Example**

\[
M(2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & f & f - 1 & f - 1 \\
1 & 0 & 0 & f - 1 & f & f - 1 \\
0 & 1 & 0 & f - 1 & f - 1 & f
\end{pmatrix}
\]  \quad (f > 1)

\[
\tilde{Z}Z = 1 - \tilde{Z} + Z^2,
\]
so (12.1) is equivalent to \(i l m \gamma_{23} = \frac{f - 1}{2} > 0\), a contradiction. Therefore, \(M(2)\) admits no r.m.
\[ \tilde{Z}^2 = A 1 + B Z + C \tilde{Z} + D Z^2 + E \tilde{Z} Z \]

To each \( \gamma_{23} \in \mathbb{C} \) there corresponds a unique moment matrix block \( B(3) \) satisfying \( \text{Ran}B(3) \subseteq \text{Ran}M(2) \), that is \( M(2) W = B(3) \) for some \( W \). Let \( C(3) := W^* M(2) W \). Then

\[
M(3) \equiv \begin{pmatrix} M(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}
\]

is a flat moment matrix extension of \( M(2) \) if and only if \( C(3) \) is Toeplitz.
Theorem

Suppose $M(2) \geq 0$, $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is independent in $C_{M(2)}$, and $\bar{Z}^2 \in \langle 1, Z, \bar{Z}, Z^2, \bar{Z}Z \rangle$. $M(2)$ admits a flat extension $M(3)$ (and $M(2)$ admits a 5-atomic (minimal) rep. meas.) if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that $C_{21} = C_{32}$.

Example

$M(2) = \begin{pmatrix} 1 & 1+i & 1-i & 1 & 5 & 1 \\ 1-i & 5 & 1 & 9+4i & 9-4i & 9+4i \\ 0 & 0 & 1 & 9-4i & 9+4i & 9-4i \\ 1 & 9-4i & 9+4i & 40 & 15 & 40 \\ 5 & 9+4i & 9-4i & 15 & 40 & 15 \\ 1 & 9-4i & 9+4i & 40 & 15 & 40 \end{pmatrix}$.

Here $\bar{Z}^2 = Z^2$, but $C(3)$-block test fails!
The functional calculus we have constructed is such that $p(Z, \bar{Z}) = 0$ implies $\text{supp } \mu \subseteq \mathcal{Z}(p)$.

When $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_M(2)$, the associated algebraic variety is the zero set of a real quadratic equation in $x := \text{Re}[z]$ and $y := \text{Im}[z]$.

One can then reduce the remaining case to subcases corresponding to the following four real conics:

(a) $\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$  \hspace{1cm} \text{parabola; } y = x^2

(b) $\bar{W}^2 = -4i1 + W^2$  \hspace{1cm} \text{hyperbola; } yx = 1

(c) $\bar{W}^2 = W^2$  \hspace{1cm} \text{pair of intersecting lines; } yx = 0

(d) $\bar{W}W = 1$  \hspace{1cm} \text{unit circle; } x^2 + y^2 = 1.
Theorem

Let $\gamma^{(4)}$ be given, and assume $M(2) \geq 0$ and \{1, $Z$, $\bar{Z}$, $Z^2$, $\bar{Z}Z$\} is a basis for $C_{M(2)}$. Then $\gamma^{(4)}$ admits a rep. meas. $\mu$. Moreover, it is possible to find $\mu$ with $\text{card supp } \mu = \text{rank } M(2)$, except in some cases when $\mathcal{V}(\gamma^{(4)})$ is a pair of intersecting lines, in which cases there exist $\mu$ with $\text{card supp } \mu \leq 6$. 
Theorem

Let $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, \ldots, \beta_{0,2n}, \ldots, \beta_{2n,0}$ be a family of real numbers, $\beta_{00} > 0$, and let $M_{\mathbb{R}}(n)$ be the associated moment matrix. Assume

(a) $M_{\mathbb{R}}(n) \geq 0$

(b) $M_{\mathbb{R}}(n)$ RG and

(c) $Y = X^2$.

TFAE:

(i) $\beta$ admits a rep. meas.

(ii) $\beta$ admits a rank $M_{\mathbb{R}}(n)$-atomic rep. meas.

(iii) $M_{\mathbb{R}}(n)$ admits a flat extension $M_{\mathbb{R}}(n+1)$.

(iv) $M_{\mathbb{R}}(n)$ admits a positive, RG extension $M_{\mathbb{R}}(n+1)$

(v) rank $M_{\mathbb{R}}(n) \leq \text{card } \mathcal{V}(\beta)$. 

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**Corollary**

(Stochel) \( \beta \equiv \beta^{(\infty)} : \beta_{00}, \beta_{01}, \beta_{10}, \ldots, \beta_{0,2n}, \ldots, \beta_{2n,0}, \ldots \) be a full family of real numbers, \( \beta_{00} > 0 \), and let \( M_{\mathbb{R}}(\infty) \) be the associated infinite moment matrix. Assume

(a) \( M_{\mathbb{R}}(n) \geq 0 \) and (b) \( Y = X^2 \). Then \( \beta \) admits a rep. meas. supported on the parabola \( y = x^2 \). *(This says that \( p(x, y) := y - x^2 \) is of type \( \Lambda \).)*

For \( 1 \leq k \leq n \) let

\[
S(k) := \{1, X, Y, YX, Y^2, Y^2X, Y^3, \ldots, Y^{k-1}X, Y^k\}.
\]

**Lemma**

Let \( M_{\mathbb{R}}(n) \) be RG , and assume that \( Y = X^2 \). Then \( S(n) \) spans \( C_{M_{\mathbb{R}}(n)} \).
**Proposition**

Let \( \beta^{(2n)} \) be a family of real numbers, \( \beta_{00} = 1 \), and let \( M_\mathbb{R}(n) \) be the associated moment matrix. Assume that \( M_\mathbb{R}(n) \) is positive, \( RG \), and satisfies \( Y = X^2 \) and \( \text{rank } M_\mathbb{R}(n) \leq \text{card } V(\beta) \). In \( \mathcal{S}(n) \), assume that \( Y^{n-1}X \) is the first dependence relation. Then \( M_\mathbb{R}(n) \) admits a flat extension \( M_\mathbb{R}(n + 1) \).

**Proof.**

Write

\[
Y^{n-1}X = p_{n-1}(Y) + q_{n-2}(Y)X,
\]

and let

\[
\begin{align*}
  r(x, y) &:= y^{n-1}x - \left( p_{n-1}(y) + q_{n-2}(y)x \right) \\
  s(x, y) &:= y - x^2.
\end{align*}
\]
Then

\[ V(\beta) \subseteq Z(r) \cap Z(s). \]

If we substitute \( y = x^2 \) in \( r(x, y) = 0 \), we obtain a polynomial equation in \( x \) of degree at most \( 2n - 1 \). It then follows that

\[ \text{card } V(\beta) \leq 2n - 1, \]

so that

\[ \text{rank } M_R(n) \leq 2n - 1. \]

Then

\[ S(n - 1) \equiv \{1, X, Y, YX, Y^2, Y^2X, Y^3, \ldots, Y^{n-2}X, Y^{n-1}\} \]

is a basis for \( C_{M_R(n)} \), so that \( Y^n \) is then a linear combination of the columns in \( S(n - 1) \). Thus, \( M_R(n) \) is flat, so it admits a flat extension \( M_R(n + 1) \).
(Minimal degree-4 quadrature rules on a parabolic arc) We describe the minimal quadrature rules of degree 4 for arclength measure $\nu$ on the segment of the parabola $y = x^2$ corresponding to $0 \leq x \leq 1$. Let

$$K := \{(x, y) \in \mathbb{R}^2 : y = x^2, \ 0 \leq x \leq 1 \}.$$ 

By a $K$-quadrature rule for $\nu$ of degree 4 we mean a finite collection of points of $K$, $(x_0, y_0), \ldots, (x_d, y_d)$, and corresponding positive weights, $\omega_0, \ldots, \omega_d$, such that for every real polynomial $p(x, y)$ of total degree $\leq 4$,

$$\int_K p(x, y) \, d\nu(x, y) \left( \equiv \int_0^1 p(t, t^2) \sqrt{1 + 4t^2} \, dt \right) = \sum_{i=0}^{d} \omega_i p(x_i, y_i);$$

a minimal quadrature rule is one for which $d$ is as small as possible.
**Example**

First, complexify:

\[
\gamma_{kj} = \int_0^1 (t - it)^k (t + it)^j \sqrt{1 + 4t^2} \, dt, \quad 0 \leq k + j \leq 4.
\]

Each \( \gamma_{kj} \in \mathbb{Q}[i, \sqrt{5}, \ln(2 + \sqrt{5})] \). Since \( M(2)(\gamma) \) has a rep. meas. (namely, \( \nu \)), \( M(2)(\gamma) \geq 0 \). Also, \( \{1, Z, \bar{Z}, Z^2, \bar{Z}Z\} \) is a basis for \( C_{M(2)} \); moreover, \( y = x^2 \) means

\[
Z^2 + 2\bar{Z}Z + \bar{Z}^2 + 2iZ - 2i\bar{Z} = 0
\]

It follows that for each \( \gamma_{23} \equiv r + is \) (\( r, s \in \mathbb{R} \)) there exists a unique moment matrix block \( B(3)[\gamma_{23}] \) satisfying \( \text{Ran}B(3)[\gamma_{23}] \subseteq \text{Ran}M(2) \); moreover, \( \gamma_{23} \) gives rise to a flat extension \( M(3) \) if and only if the relation \( C_{21} = C_{32} \) holds in the \( C(3) \)-block.
We employ localizing matrices. Using nested determinants, we show that

\[ M_x(3) \geq 0 \iff r \geq r_0 \approx 1.04984 \]

\[ M_x(3) \leq M_1(3) \iff r \leq r_1 \approx 1.04986. \]

Thus, precisely for \( r \) satisfying \( r_0 \leq r \leq r_1 \), \( \nu[r] \) is a 5-atomic (minimal) rep. meas. for \( \gamma^{(4)} \) supported in \( K \).
Let \( H := \{(x, y) \in \mathbb{R}^2 : Q(x, y) = 0\} \) be an hyperbola. For \( \beta \equiv \beta^{(2n)} \), assume that \( \mathcal{M}(n) \equiv \mathcal{M}(n)(\beta) \geq 0 \), RG, and satisfies \( Q(X, Y) = 0 \) in \( C_{\mathcal{M}(n)} \). Then rank \( \mathcal{M}(n) \leq 2n + 1 \), and TFAE.

(i) \( \beta \) admits a rep. meas. (necessarily supported in \( H \)).

(ii) \( \beta \) admits a rep. meas. \( \mu \) (necessarily supported in \( H \)) satisfying \( \text{card supp } \mu \leq 1 + \text{rank } \mathcal{M}(n) \). If rank \( \mathcal{M}(n) \leq 2n \), then \( \mu \) can be taken so that \( \text{card supp } \mu = \text{rank } \mathcal{M}(n) \).

(iii) \( \mathcal{M}(n) \) admits a positive, RG extension \( \mathcal{M}(n+1) \), with rank \( \mathcal{M}(n+1) \leq 1 + \text{rank } \mathcal{M}(n) \), and \( \mathcal{M}(n+1) \) admits a flat ext. \( \mathcal{M}(n+2) \). If rank \( \mathcal{M}(n) \leq 2n \), then \( \mathcal{M}(n) \) admits a flat ext. \( \mathcal{M}(n+1) \).

(iv) rank \( \mathcal{M}(n) \leq \text{card } V(\mathcal{M}(n)) \).
Consider the following property for a polynomial $P \in \mathbb{R}_n[x, y]$: 

$$\beta \equiv \beta^{(2n)} \text{ has a rep. meas. supported in } \mathcal{Z}(P) \text{ if and only if } \begin{array}{c} \mathcal{M}(n)(\beta) \text{ is positive semi-definite, } \mathcal{RG}, \\ P(X, Y) = 0 \text{ in } C_{\mathcal{M}(n)}, \text{ and } \text{rank } \mathcal{M}(n) \leq \text{card } \mathcal{V}(\mathcal{M}(n)). \end{array}$$

Polynomials which satisfy $(A'_n)$ form an attractive class, because if $P$ satisfies $(A'_n)$, then the degree-2$n$ moment problem on $P(x, y) = 0$ can be solved by concrete tests involving only elementary linear algebra and the calculation of roots of polynomials.
Theorem

If \( \deg P \leq 2 \), then \( P \) satisfies \((A'_n)\) for every \( n \geq \deg P \).

Despite this theorem, there are differences between the parabolic and elliptic moment problems and the hyperbolic problem. In the former cases, the conditions of \((A'_n)\) always imply the existence of a \( \text{rank } M(n) \)-atomic rep. meas., corresponding to a flat extension of \( M(n) \); for this reason, positive Borel measures supported on these curves always admit Gaussian cubature rules, i.e., \( \text{rank } M(n) \)-atomic cubature rules of degree \( 2n \) (Fialkow and Petrovic, 2003). By contrast, in the hyperbolic case, minimal rep. meas. \( \mu \) sometimes entail \( \text{card supp } \mu > \text{rank } M(n) \) (and Gaussian cubature rules may fail to exist).
The above Theorem is motivated in part by results of J. Stochel, who solved the full moment problem on planar curves of degree at most 2. Paraphrasing Stochel’s work (i.e., translating from the language of moment sequences into the language of moment matrices), we consider the following property of a polynomial $P$:

$$\beta^{(\infty)} \text{ has a rep. meas. supported in } P(x, y) = 0 \quad (A)$$

if and only if $\mathcal{M}(\infty)(\beta) \geq 0$ and $P(X, Y) = 0$ in $C_{\mathcal{M}(\infty)}$.

**Theorem**

*Stochel* If $\deg P \leq 2$, then $P$ satisfies $(A)$. 
Stochel also proved that there exist polynomials of degree 3 that do not satisfy \((A)\).

The link between TMP and FMP is provided by another result of Stochel:

**Theorem**

\[ \beta^{(\infty)} \text{ has a rep. meas. supported in a closed set } K \subseteq \mathbb{R}^2 \text{ if and only if, for each } n, \beta^{(2n)} \text{ has a rep. meas. supported in } K. \]
\[ \beta_i = \int_{\mathbb{R}^d} x^i \, d\mu, \ |i| \leq 2n; \] (15.1)

\( \mathcal{P} \equiv \mathbb{R}^d[x] = \mathbb{R}[x_1, \ldots, x_d] : \) space of real valued \( d \)-variable polynomials.

\( \mathcal{P}_k \equiv \mathbb{R}^d_k[x] : \) the subspace of \( \mathcal{P} \) consisting of polynomials \( p \) with

\( \deg p \leq k \ (k \geq 1). \)

\( \Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R} \) (Riesz functional): if \( p(x) \equiv \sum_{|i| \leq 2n} a_i x^i \), then

\[ \Lambda(p) := \sum_{|i| \leq 2n} a_i \beta_i \]

- In the presence of a representing measure \( \mu \), we have \( \Lambda(p) = \int p \, d\mu. \)

\( \hat{\beta} \) : coefficient vector \( (a_i) \) of \( p \).

\( \mathcal{M}(n) \equiv \mathcal{M}(n)(\beta) : \) moment matrix, with rows and columns \( X^i \) indexed by the monomials of \( \mathcal{P}_n \) in degree-lexicographic order.

\( d = n = 2 : \) the columns of \( \mathcal{M}(2) \) are denoted as \( 1, X_1, X_2, X_1^2, X_2 X_1, X_2^2 \).
• $\mathcal{M}(n)$ is a real symmetric matrix characterized by

$$\langle \mathcal{M}(n)^p, \hat{q} \rangle = \Lambda(pq) \ (p, q \in \mathcal{P}_n). \quad (15.2)$$

• If $\mu$ is a representing measure for $\beta$, then

$$\langle \mathcal{M}(n)^p, \hat{p} \rangle = \Lambda(p^2) = \int p^2 d\mu \geq 0; \text{ it follows that } \mathcal{M}(n) \succeq 0.$$
The algebraic variety of $\beta$ is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$.

- If $\beta$ admits a representing measure $\mu$, then

$$p \in \mathcal{P}_n \text{ satisfies } \hat{p} \in \ker \mathcal{M}(n) \iff \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus $\text{supp } \mu \subseteq \mathcal{V}$, so $r := \text{rank } \mathcal{M}(n)$ and $v := \text{card } \mathcal{V}$ satisfy

$$r \leq \text{card } \text{supp } \mu \leq v.$$

If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$. 
Basic necessary conditions for the existence of a representing measure

(Positivity) \( \mathcal{M}(n) \geq 0 \) \hspace{2cm} (15.3)

(Consistency) \( p \in \mathcal{P}_{2n}, \ p|_{\mathcal{V}} \equiv 0 \iff \Lambda(p) = 0 \) \hspace{2cm} (15.4)

(Variety Condition) \( r \leq \nu \), i.e., \( \text{rank} \ \mathcal{M}(n) \leq \text{card} \ \mathcal{V} \). \hspace{2cm} (15.5)

Consistency implies

(Recursiveness) \( p, q, pq \in \mathcal{P}_n, \ \hat{p} \in \ker \mathcal{M}(n) \implies \hat{p}q \in \ker \mathcal{M}(n) \). \hspace{2cm} (15.6)
Previous results:

- For $d = 1$ (the T Hamburger MP for $\mathbb{R}$), positivity and recursiveness are sufficient
- For $d = 2$, there exists $\mathcal{M}(3) > 0$ for which $\beta$ has no representing measure
- In general (15.3)-(15.5) are not sufficient.

**Question C**

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, $\beta$ is consistent, and $r \leq v$, does $\beta$ admit a representing measure?
More generally, the following question remained unsolved until very recently.

**Question RG**

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, recursively generated, and $r \leq v$, does $\beta$ admit a representing measure?

- **RC-LF**: If $d = 2$ and $\mathcal{M}(n)\hat{\rho} = 0$ for some $p$ with $\deg p \leq 2$, then Question RG has an affirmative answer.
- **RC-LF-MM**: If $d = 2$, $y - x^3 \in \ker \mathcal{M}(n)$ and $r = v \leq 7$, then Question RG has an affirmative answer.
- **RC-LM-MM**: If $d = 2$, $y - x^3 \in \ker \mathcal{M}(n)$ and $r = v = 8$, then Question RG has a negative answer.
The next result gives an affirmative answer to Question C in the extremal case, i.e., \( r = v \).

**Theorem EXT**

*(RC-LF and M. Möller)* For \( \beta \equiv \beta^{(2n)} \) extremal, i.e., \( r = v \), the following are equivalent:

1. \( \beta \) has a representing measure;
2. \( \beta \) has a unique representing measure, which is rank \( M(n) \)-atomic (minimal);
3. \( M(n) \geq 0 \) and \( \beta \) is consistent.
In many cases, the conditions of Theorem EXT provide a concrete solution to the extremal case of TMP. Indeed, only elementary linear algebra is required to verify that $\mathcal{M}(n) \geq 0$, to compute $\text{rank } \mathcal{M}(n)$, and to identify the column relations which define $\mathcal{V}$. 

If the points of $\mathcal{V}$ can be computed exactly, then only elementary linear algebra is required to verify that $\beta$ is consistent.

Question RG is significant because recursiveness is generally a simpler condition to work with than consistency. For example, we often have $\mathcal{M}(n) \geq 0$ and $\mathcal{M}(n-1) > 0$ (positive definite), in which case $\mathcal{M}(n)$ is obviously recursively generated, but we do not know whether $r \leq \nu$ implies that $\mathcal{M}(n)$ is consistent in this case.
The extremal case is inherent in TMP:

C. Bayer and J. Teichmann (extending a classical theorem of V. Tchakaloff and its successive generalizations by I.P. Mysovskikh, M. Putinar and RC-LF) recently proved that if $\beta^{(2n)}$ has a representing measure, then it has a **finitely atomic** representing measure; RC-LF showed that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $M(n)$ admits an extension to a positive moment matrix $M(n+k)$ (for some $k \geq 0$, which in turn admits a rank-preserving (i.e., flat) moment matrix extension $M(n+k+1)$; in many instances, $M(n+k+1)$ is an extremal moment matrix for which there is a computable rank$M(n+k)$-atomic representing measure $\mu$. Clearly, $\mu$ is also a finitely atomic representing measure for $\beta^{(2n)}$, and every finitely atomic representing measure for $\beta^{(2n)}$ arises in this way.
Example LD

(Extremal TCMP of arbitrarily large degree) For $n > 0$, we exhibit an extremal $\gamma \equiv \gamma^{(2n)}$ in one complex variable with

$$\text{rank } M(n)(\gamma) = \text{card } V(\gamma) = 2n.$$ The rows and columns of $M(n)$ are indexed by $1, Z, \bar{Z}, ..., Z^n, \bar{Z}Z^{n-1}, ..., \bar{Z}^{n-1}Z, \bar{Z}^n$. We set

$$\gamma_{ii} = 1 \ (0 \leq i \leq n),$$

and for $0 < a < 1$, we set $\gamma_{0,2n-1} = \gamma_{2n-1,0} := a$ and $\gamma_{0,2n} = \gamma_{2n,0} := 1 - a^2$; the remaining $\gamma_{ij}$ equal 0. For example, with $n = 3$ we have
\[ M(3) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 1 & 0 & 0 & 1 - a^2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 1 - a^2 & 0 & 0 & 1
\end{pmatrix}. \]
Moment theory can sometimes be used to estimate the number and location of the zeros of a prescribed polynomial; indeed, as a by-product of Example LD, we see that the polynomial
\[ p(z) = z^{2n} + az^{2n-1} - az - 1 \quad (0 < a < 1) \] has \(2n\) distinct zeros, all in the unit circle.
If $\beta^{(2n)}$ has a representing measure, then the Riesz functional

$$\Lambda : \mathcal{P}_{2n} \rightarrow \mathbb{R}, \quad \Lambda(x^i) := \beta_i \equiv \int_{\mathbb{R}^d} x^i \, d\mu \ (|i| \leq 2n),$$

is square positive, that is,

$$p \in \mathcal{P}_n \Rightarrow \Lambda(p^2) \geq 0.$$

If we assume that for a representing measure $\mu$ all moments

$$\int_{\mathbb{R}^d} x^i \, d\mu, \quad i \in \mathbb{Z}^d_+$$

are convergent, then we can extend $\Lambda$ to $\mathcal{P}$ by letting

$$\Lambda(x^i) := \int_{\mathbb{R}^d} x^i \, d\mu, \quad i \in \mathbb{Z}^d_+,$$

thus obtaining a square positive functional over $\mathcal{P}$. Under this assumption the set
\[ I := \{ p \in \mathcal{P} : \Lambda(p^2) = 0 \} \]

is a real ideal, i.e., it is an ideal ( \( p_1, p_2 \in I \Rightarrow p_1 + p_2 \in I \) and \( p \in I, \ q \in \mathcal{P} \Rightarrow pq \in I \)) and satisfies one of the two equivalent conditions:

For \( s \in \mathbb{Z}_+, p_1, \ldots, p_s \in \mathcal{P} : \sum_{i=1}^{s} p_i^2 \in I \Rightarrow \{p_1, \ldots, p_s\} \subseteq I \)

There exists \( G \subseteq \mathbb{R}^d \) such that for all \( p \in \mathcal{P} : p|_G \equiv 0 \Rightarrow p \in I. \)

If \( I \) is a real ideal, then one may take for \( G \subseteq \mathbb{R}^d \) the real variety

\[ V_{\mathbb{R}}(I) := \{ w \in \mathbb{R}^d : f(w) = 0 \ (\text{all } f \in I) \}. \]

But one may also take any subset \( G \) of \( V_{\mathbb{R}}(I) \) containing sufficiently many points, such that

\[ p \in \mathcal{P}, \ p|_G \equiv 0 \Rightarrow p|_{V_{\mathbb{R}}(I)} \equiv 0. \]
For instance, if the real variety is a (real) line, one may take for $G$ a subset of infinitely many points on that line. On the other hand, if $V_R(I)$ is a finite set of points, then necessarily $G = V_R(I)$.

If $I$ is an ideal, its subset $I_k := I \cap P_k$ is an $R$-vector subspace of $P_k$. One can then introduce the \textit{Hilbert function} of $I$ by

$$H_I(k) := \dim P_k - \dim I_k, \quad k \in \mathbb{Z}_+.$$
• Both $k \mapsto \dim \mathcal{I}_k$ and $k \mapsto H_{\mathcal{I}}(k)$ are nondecreasing functions.

• For sufficiently large $k$, say $k \geq k_0$, $H_{\mathcal{I}}(k)$ becomes a polynomial in $k$, the so called Hilbert polynomial of $\mathcal{I}$, whose degree equals the dimension of $\mathcal{I}$.

**Example**

Let $G := \{w_1, \ldots, w_m\} \subseteq \mathbb{R}^d$. Then $\mathcal{I} := \{f \in \mathcal{P} : f|_G \equiv 0\}$ is a real ideal with $V_{\mathbb{R}}(\mathcal{I}) = G$. Let $t_1, t_2, t_3, \ldots$ denote the monomials $x^i$ in degree-lexicographic order, such that for each $k \in \mathbb{Z}_+$ the polynomials $t_1, \ldots, t_K$, (with $K := \dim \mathcal{P}_k$) build a basis for the $\mathbb{R}$-vector space $\mathcal{P}_k$.

For $p \in \mathcal{P}_k$ let $\hat{p} := (a_1, \ldots, a_K)$ if $p = \sum_{i=1}^K a_i t_i$. Then obviously

\[
p \in \mathcal{I} \cap \mathcal{P}_k \iff \hat{p} \perp (t_1(w_i), \ldots, t_K(w_i)), \ i = 1, \ldots, m.
\]
Arranging the rows \((t_1(w_i), \ldots, t_K(w_i))\) in a matrix

\[
W_k := (t_j(w_i))_{i=1, \ldots, m, j=1, \ldots, K},
\]

one gets \(\dim \mathcal{I}_k + \text{rank } W_k = \dim \mathcal{P}_k\), or using the Hilbert function,

\[
H_{\mathcal{I}}(k) = \text{rank } W_k, \quad k \in \mathbb{Z}_+.
\]

By construction, \(W_k\) is a submatrix of \(W_{k+1}\). Hence \(\text{rank } W_k \leq \text{rank } W_{k+1}\), reflecting the fact that the Hilbert function increases. If, for a given \(k\), the rank of \(W_k\) is less than \(m\), then one row of \(W_k\), say the last one, depends on the others. This means that every polynomial which is zero in \(w_1, \ldots, w_{m-1}\) also vanishes in \(w_m\). Using Lagrange interpolation polynomials, we can see that for sufficiently big \(k\) this cannot happen. Hence \(\text{rank } W_k = m\) for sufficiently large \(k\). This \(m\) is the constant (degree-0) polynomial in \(k\) which coincides with \(H_{\mathcal{I}}(k)\) for all \(k \geq k_0\). Here \(\mathcal{I}\) is a zero dimensional ideal.
Assume now that $\beta^{(2n)}$ admits a representing measure $\mu$. Then, irrespective of whether the Riesz functional $\Lambda : \mathcal{P}_n \rightarrow \mathbb{R}$ can be extended to a square positive functional $\Lambda : \mathcal{P} \rightarrow \mathbb{R}$, we can define the ideal

$$\mathcal{I}(\mu) := \{ p \in \mathcal{P} : p \big|_{\text{supp } \mu} \equiv 0 \}. \quad (16.1)$$

Since $\text{supp } \mu \subseteq \mathbb{R}^d$, $\mathcal{I}(\mu)$ is a real ideal, which we will call the real ideal of $\beta^{(2n)}$. 
**Lemma**

Assume \( \beta^{(2n)} \) has a representing measure, and let \( \mathcal{I}(\mu) \) be its real ideal. Then

\[
\{ p \in \mathcal{P}_n : \mathcal{M}(n) \hat{p} = 0 \} = \mathcal{I}(\mu) \cap \mathcal{P}_n.
\] (16.2)

If \( t_1, \ldots, t_N \) denote the monomials \( x^i \in \mathcal{P}_n \) in degree-lexicographic order, then the row vectors of \( \mathcal{M}(n) \) and the row vectors of

\[
W_n := \{(t_1(w), \ldots, t_N(w)) : w \in \text{supp } \mu\}
\]

span the same subspace of \( \mathbb{R}^N \); in particular, \( \text{rank } \mathcal{M}(n) = H_{\mathcal{I}(\mu)}(n) \).
We observe that if $H_I(k) = m$ for sufficiently large $k$, then 
\[ \text{card supp } \mu = m, \] and the monotonicity of $H_I(\mu)$ then shows that 
\[ r := \text{rank } M(n) \leq m = \text{card supp } \mu. \] This inequality together with 
\[ \text{card supp } \mu \leq v := \text{card } V \] yields the variety condition (15.5).

**Lemma**

Let $\Lambda : P_{2n} \to \mathbb{R}$ be a linear functional and let $V \equiv \{w_1, \ldots, w_m\} \subseteq \mathbb{R}^d$. The following statements are equivalent.

(a) There exists $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that $\Lambda(p) = \sum_{i=1}^m \alpha_i p(w_i)$ (all $p \in P_{2n}$).

(b) If $p \in P_{2n}$ and $p|_V \equiv 0$, then $\Lambda(p) = 0$. 
Recall that
\[ \mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, p(X) = 0} \mathbb{Z}_p. \]

Let \( \mathcal{P}_n|_\mathcal{V} \) denote the restriction to \( \mathcal{V} \) of the polynomials in \( \mathcal{P}_n \), and consider the mapping \( \varphi_\beta : C_{M(n)} \to \mathcal{P}_n|_\mathcal{V} \) given by \( p(X) \mapsto p|_\mathcal{V} \). The map \( \varphi_\beta \) is well-defined, and \( \beta \) has a representing measure \( \mu \), then \( \varphi_\beta \) is 1-1.

**Proposition**

Let \( \beta, \varphi_\beta \) and \( M(n)(\beta) \) be as before. Then

\[ \beta \text{ consistent } \implies \varphi_\beta \text{ 1-1 } \implies M(n)(\beta) \text{ recursively generated.} \]
**Proposition**

For $d = 2$ (the plane), if $M(n)(\beta)$ is recursively generated and $\mathcal{V}_\beta$ is a proper, infinite irreducible curve, then $\beta$ is consistent.

**Example**

$$M(3) = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 3 & 1 & 1 & 2 \\
0 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 5 \\
1 & 0 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 3 \\
0 & 3 & 1 & 0 & 0 & 0 & 14 & 3 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 3 & 1 & 1 & 2 \\
0 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 2 & 5
\end{pmatrix}.$$
It is straightforward to check that $M(3)$ is positive and recursively generated, with column relations $YX = 1$, $YX^2 = X$, $Y^2X = Y$ in $\mathcal{C}_M(3)$ and $\text{rank } M(3) = 7$. Then $\mathcal{V}_\beta$ is the hyperbola $yx = 1$, and Proposition 41 implies that $\beta$ is consistent. (The existence of a representing measure for $\beta$ is contained in the solution of the T Hyperbolic MP.)
Assume that $\beta \equiv \beta^{(2n)}$ is extremal, i.e., $r := \text{rank } M(n)$ and $v := \text{card } \mathcal{V}$ satisfy $r = v$. Let $\mathcal{V} \equiv \{w_1, \ldots, w_r\}$ denote the distinct points of $\mathcal{V}$. If $\mu$ is a representing measure for $\beta$, then since $\text{supp } \mu \subseteq \mathcal{V}$ and $r \leq \text{card } \text{supp } \mu \leq v$, the extremal hypothesis $r = v$ implies that $\text{supp } \mu = \mathcal{V}$. Thus $\mu$ is necessarily of the form

$$
\mu = \sum_{i=1}^{r} \rho_i \delta_{w_i}.
$$

(18.1)
Let $p_1, ..., p_r$ be polynomials in $\mathcal{P}_n$ such that $\mathcal{B} \equiv \{p_1(X), ..., p_r(X)\}$ is a basis for the column space of $\mathcal{M}(n)$, and set

$$W \equiv W_\mathcal{B} := \begin{pmatrix}
p_1(w_1) & \cdot & \cdot & \cdot & p_1(w_r) \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
p_r(w_1) & \cdot & \cdot & \cdot & p_r(w_r)
\end{pmatrix}.$$  

Also, recall that if $\mathcal{P}_n|\mathcal{V}$ denote the restriction to $\mathcal{V}$ of the polynomials in $\mathcal{P}_n$, the map $\varphi_\beta : \mathcal{C}_{\mathcal{M}(n)} \to \mathcal{P}_n|\mathcal{V}$ is given by $p(X) \mapsto p|\mathcal{V}$. 
**Lemma**

The following are equivalent for $\beta$ extremal:

i) $\varphi_B$ is 1-1, i.e., $p \in P_n$, $p|_V \equiv 0 \implies p(X) = 0$ in $C_M(n)$;

ii) For any basis $B$ of $C_M(n)$, $W_B$ is invertible;

iii) There exists a basis $B$ for $C_M(n)$ such that $W_B$ is invertible.

**Theorem**

For $\beta \equiv \beta^{(2n)}$ extremal, the following are equivalent:

(i) $\beta$ has a representing measure;

(ii) $\beta$ has a unique representing measure, which is a rank $M(n)$-atomic;

(iii) For some (respectively, for every) basis $B$ for $C_M(n)$, $W_B$ is invertible and $\mu_B$ is a representing measure for $\beta$;

(iv) $\beta$ is consistent and $M(n) \geq 0$. 
**Example**

Consider

\[
\mathcal{M}(3) = \begin{pmatrix}
1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\
0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\
1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\
0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\
0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\
0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & 2000 \\
0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & 2000 & 338881
\end{pmatrix}.
\]
We have $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$, $r = 8$,

$$Y = X^3,$$  \hspace{1cm} (19.1)

and

$$Y^3 = q(X, Y).$$  \hspace{1cm} (19.2)

where $q(x, y) := -2285x + 5720y - 34441yx^2 + 578y^2x$. Here $v = 9$.

Thus, $\mathcal{M}(3)$ is positive, recursively generated, $r < v$, and the minimal representing measure for $\beta^{(6)}$ is $v$-atomic.
Conjecture

If \( r < \nu < +\infty \), then \( \mathcal{M}(n) \) has a representing measure, but no \( r \)-atomic representing measure (corresponding to a flat extension \( \mathcal{M}(n + 1) \)); instead, \( \mathcal{M}(n) \) has a minimal representing measure for \( \beta \) which is \( \nu \)-atomic, and corresponds to a rank-\( \nu \) positive extension \( \mathcal{M}(n + k) \) (for some \( k \)), followed by a flat extension \( \mathcal{M}(n + k + 1) \).