Truncated Moment Problems: An Introductory Survey
(based on joint work with L.A. Fialkow, H.M. Möller and S. Yoo)

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Real Algebraic Geometry With a View Toward Moment Problems and Optimization
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add work of Kuhlmann, Kuhlmann-Infusino, etc

A) Low-order polynomial approx. on subintervals of decreasing size

**Commonly used Newton-Cotes formulas**

\[ \int_a^b f(x) \, dx = \]

\[
\begin{align*}
\text{T} & \quad n = 1 & \quad \frac{h}{2} [f(a) + f(b)] - \frac{h^3}{12} f''(\xi) \\
\text{S} & \quad n = 2 & \quad \frac{h}{3} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] - \frac{h^5}{90} f^{(4)}(\xi) \\
& \quad n = 3 & \quad \left\{ \begin{array}{l}
\frac{3h}{8} [f(a) + 3f(a+h) + 3f(b-h) + f(b)] \\
-\frac{3h^5}{80} f^{(4)}(\xi)
\end{array} \right. \\
& \quad n = 4 & \quad \left\{ \begin{array}{l}
\frac{2h}{45} [7f(a) + 32f(a+h) + 12f\left(\frac{a+b}{2}\right)] \\
+ 32f(b-h) + 7f(b) - \frac{8h^7}{945} f^{(6)}(\xi)
\end{array} \right.
\end{align*}
\]
B) Polynomial approximation of increasing degree, using fewer, strategically-placed nodes

**Definition**

A quadrature (or cubature) rule of size $p$ and precision $m$ is a numerical integration formula which uses $p$ nodes, is exact for all polynomials of degree at most $m$, and fails to recover the integral of some polynomial of degree $m + 1$.

**Gaussian Quadrature (size $n$, precision $2n - 1$)**

$$
\int_{-1}^{1} f(t) \, dt = \sum_{j=0}^{n-1} \rho_j f(t_j^{(n)}) \text{ for every polynomial } f \in \mathbb{R}_{2n-1}[t]
$$

(Gaussian means minimum number of nodes possible)
Interpolating Equations:

\[
\sum_{j=0}^{n-1} \rho_j t_j^k = \int_{-1}^{1} t^k \, dt = \begin{cases} 
0 & k = 1, 3, \ldots, 2n - 1 \\
\frac{2}{k+1} & k = 0, 2, \ldots, 2n - 2
\end{cases}
\]
Example: \( n = 2 \)

\[
\begin{align*}
\rho_0 + \rho_1 &= 2 \\
\rho_0 t_0 + \rho_1 t_1 &= 0 \\
\rho_0 t_0^2 + \rho_1 t_1^2 &= \frac{2}{3} \\
\rho_0 t_0^3 + \rho_1 t_1^3 &= 0 
\end{align*}
\]

\( \rho_0 = \rho_1 = 1; \ t_0 = -\frac{\sqrt{3}}{3}, \ t_1 = \frac{\sqrt{3}}{3}. \)

\[
\int_{-1}^{1} \sum_{k=0}^{3} a_k t^k = \sum_{j=0}^{1} \rho_j \sum_{k=0}^{3} a_k t_j^k
\]

NA textbooks prove this by using orthogonal Legendre polynomials

(\( t_0 < \ldots < t_{n-1} \) are the zeros of the \( n \)th Legendre polynomial)
Can do this as follows:

\[ \gamma_0 := 2, \ \gamma_1 := 0, \ \gamma_2 := \frac{2}{3}, \ \gamma_3 := 0, \ \gamma_4 := \frac{2}{5}, \text{ etc.} \]

Assume \( n \) even, and form the Hankel matrix

\[
H(n) := \begin{pmatrix}
2 & 0 & \frac{2}{3} & \cdots & 0 & \vdots & \frac{2}{n+1} \\
0 & \frac{2}{3} & 0 & \cdots & \frac{2}{n+1} & \vdots & 0 \\
\frac{2}{3} & 0 & \frac{2}{5} & \cdots & 0 & \vdots & \frac{2}{n+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \frac{2}{n+1} & 0 & \cdots & \frac{2}{2n-1} & \vdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{2}{n+1} & 0 & \frac{2}{n+3} & \cdots & 0 & \vdots & \text{NEWMOMENT}
\end{pmatrix},
\]

label the columns \( 1, T, T^2, \ldots, \)

require that \( T^n = \varphi_0 1 + \ldots + \varphi_{n-1} T^{n-1}, \)

build the polynomial
find its zeros \((t_0 < \ldots < t_{n-1})\),

and

compute the densities using the Vandermonde system

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
 t_0 & t_1 & \cdots & t_{n-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 t_0^{n-1} & t_1^{n-1} & \cdots & t_{n-1}^{n-1}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\vdots \\
\rho_{n-1}
\end{pmatrix}
= \begin{pmatrix}
\gamma_0 \\
\gamma_1 \\
\vdots \\
\gamma_{n-1}
\end{pmatrix}.
\]
To solve the Gaussian quadrature problem, RC and Fialkow’s basic idea was to augment the original Hankel matrix by one row and one column at a time, preserving the rank (which a fortiori preserves positivity):

\[ H(n) \prec H(n + 1) \prec \ldots H(\infty) \]

Then define

\[ \langle p, q \rangle_{H(\infty)} := (H(\infty) \hat{p}, \hat{q})_{\ell^2}, \]

and show that

\[ \langle p, q \rangle_{H(\infty)} = \int p \bar{q} \, d\mu \]

for some finitely atomic rep. meas., with \( \text{supp} \, \mu = \mathcal{Z}(g) \).
The Truncated Real Moment Problem

Given a family of real numbers $\beta: \beta_0, \beta_1, \ldots, \beta_{2n}$ with $\beta_0 > 0$, the **TMP** entails finding a positive Borel measure $\mu$ supported in the real line $\mathbb{R}$ such that

$$\beta_i = \int t^i \, d\mu \quad (0 \leq i \leq 2n);$$

$\mu$ is called a **representing measure** for $\beta$.

**Theorem**

**FULL MP** *(Hamburger, 1920)*

$$\exists \mu \iff A(n) := (\beta_{i+j})_{i,j=0}^n \equiv \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \cdots \\ \beta_2 & \beta_3 & \cdots & \cdot & \cdots \\ \beta_3 & \cdots & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succeq 0 \ \forall \ n \geq 0.$$
**Theorem**

**FULL MP (Stieltjes, 1894)**

\[ \exists \mu \text{ with } \text{supp } \mu \subseteq [0, +\infty) \]

\[ \iff (\beta_{i+j})_{i,j=0}^n \geq 0 \text{ and } (\beta_{i+j+1})_{i,j=0}^n \geq 0 \ \forall \ n \geq 0. \]

\[
\begin{pmatrix}
\beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \cdots \\
\beta_2 & \beta_3 & \cdots & \cdots & \cdots \\
\beta_3 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix} \geq 0 \text{ and } \quad \begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\
\beta_2 & \beta_3 & \beta_4 & \cdots & \cdots \\
\beta_3 & \beta_4 & \cdots & \cdots & \cdots \\
\beta_4 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix} \geq 0
\]

(localizing matrix)
The Truncated Complex Moment Problem

- Given \( \gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0} \), with \( \gamma_{00} > 0 \) and \( \gamma_{ji} = \bar{\gamma}_{ij} \), the TCMP entails finding a positive Borel measure \( \mu \) supported in the complex plane \( \mathbb{C} \) such that

\[
\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \leq i + j \leq 2n);
\]

\( \mu \) is called a rep. meas. for \( \gamma \).

- In earlier joint work with L. Fialkow,

- We have introduced an approach based on matrix positivity and extension, combined with a new “functional calculus” for the columns of the associated moment matrix.
We have shown that when the TCMP is of flat data type, a solution always exists; this is compatible with our previous results for

\[
\begin{align*}
\text{supp } \mu &\subseteq \mathbb{R} \quad \text{(Hamburger TMP)} \\
\text{supp } \mu &\subseteq [0, \infty) \quad \text{(Stieltjes TMP)} \\
\text{supp } \mu &\subseteq [a, b] \quad \text{(Hausdorff TMP)} \\
\text{supp } \mu &\subseteq \mathbb{T} \quad \text{(Toeplitz TMP)}
\end{align*}
\]

Along the way we have developed new machinery for analyzing TMP’s in one or several real or complex variables. For simplicity, in this talk we focus on one complex variable or two real variables, although several results have multivariable versions.
Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.

We have fully resolved, among others, the cases

\[ \tilde{Z} = \alpha 1 + \beta Z \]

and

\[ Z^k = p_{k-1}(Z, \tilde{Z}) \quad (1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1; \text{deg} \ p_{k-1} \leq k - 1). \]

We obtain applications to quadrature problems in numerical analysis.

We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on \( \mathbb{R}^d \).
Applications

- Subnormal Operator Theory (unilateral weighted shifts)

For $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$, the weighted shift $W_\alpha$ is subnormal if and only if the moment problem $\alpha_0^2 \alpha_1^2 \cdots \alpha_{k-1}^2 = \int s^k d\mu(s)$ is soluble.

- Physics (determination of contours)

- Computer Science (image recognition and reconstruction)

- Geography (location of proposed distribution centers)

- Probability (reconstruction of p.d.f.’s)
Environmental Science (oil spills, via quadrature domains)

Engineering (tomography)

Optimization (finding the global minimum of a real polynomial in several real variables - J. Lasserre)

Function Theory (a dilation-type structure theorem in Fejér-Riesz factorization theory - S. McCullough)

Geophysics (inverse problems, cross sections)

**Typical Problem:** Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.
**Basic Positivity Condition**

\( \mathcal{P}_n \): polynomials \( p \) in \( z \) and \( \bar{z} \), \( \deg p \leq n \)

Given \( p \in \mathcal{P}_n \), \( p(z, \bar{z}) = \sum_{0 \leq i + j \leq n} a_{ij} \bar{z}^i z^j \),

\[
0 \leq \int |p(z, \bar{z})|^2 \ d\mu(z, \bar{z}) = \sum_{i,j,k,l} a_{ij} \bar{a}_{kl} \int \bar{z}^i \bar{z}^j z^k d\mu(z, \bar{z}) = \sum_{i,j,k,l} a_{ij} \bar{a}_{kl} \gamma_{i+j+k+l}.
\]

To understand this "matricial" positivity, we introduce the following lexicographic order on the rows and columns of \( M(n) \):

\[1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots\]
Define $M[i, j]$ as in

$$M[3, 2] := \begin{pmatrix}
\gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{14} & \gamma_{23} & \gamma_{32} \\
\gamma_{05} & \gamma_{14} & \gamma_{23}
\end{pmatrix}$$

Then

$$(\text{"matricial" positivity}) \quad \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell,j+k} \geq 0$$

$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix}
M[0, 0] & M[0, 1] & \ldots & M[0, n] \\
M[1, 0] & M[1, 1] & \ldots & M[1, n] \\
\vdots & \vdots & \ddots & \vdots \\
M[n, 0] & M[n, 1] & \ldots & M[n, n]
\end{pmatrix} \geq 0.$$
For example,

\[ M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}, \]

\[ M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}. \]
In general,

\[ M(n + 1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \]

Similarly, one can build \( M(\infty) \).

**Positivity Condition is not sufficient:**

By modifying an example of K. Schmüdgen, we have built a family \( \gamma_0, \gamma_1, \gamma_{10}, \ldots, \gamma_{06}, \ldots, \gamma_{60} \) with positive invertible moment matrix \( M(3) \) but **no** rep. meas. But this can also be done for \( n = 2 \).
For the Real TMP, one defines

\[ \mathcal{M}(n)_{ij} := \gamma_{i+j}, \quad i, j \in \mathbb{Z}_+. \]

The TCMP and TRMP are structurally equivalent, meaning that there is a bijection linking TCMP in \( d \) variables with TRMP in \( 2d \) variables, via the map \( z \equiv x + iy \). Moreover, it is possible to modify a TRMP and obtain an equivalent TRMP using degree-one transformations of the form

\[ \varphi(x, y) := (ax + by + e, cx + dy + f), \]

where \( ad - bc \neq 0 \).
For moment problems in \( \mathbb{C} \),

\[
M(3) = \begin{pmatrix}
1 & Z & \bar{Z} & Z^2 & \bar{Z} Z & \bar{Z}^2 & \cdots & Z^3 & \bar{Z} Z^2 & \bar{Z}^2 Z & \bar{Z}^3 \\
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \cdots & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \cdots & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \cdots & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \cdots & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \cdots & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\
\gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \cdots & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \cdots & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\
\gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \cdots & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\
\gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \cdots & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\
\gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \cdots & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33}
\end{pmatrix}.
\]
For moment problems in $\mathbb{R}^2$,

$$M(3) = \begin{pmatrix}
1 & X & Y & X^2 & XY & Y^2 & : & X^3 & X^2Y & XY^2 & Y^3 \\
\beta_{00} & \beta_{01} & \beta_{10} & \beta_{02} & \beta_{11} & \beta_{20} & : & \beta_{03} & \beta_{12} & \beta_{21} & \beta_{30} \\
\beta_{01} & \beta_{02} & \beta_{11} & \beta_{03} & \beta_{12} & \beta_{21} & : & \beta_{04} & \beta_{13} & \beta_{22} & \beta_{31} \\
\beta_{10} & \beta_{11} & \beta_{20} & \beta_{12} & \beta_{21} & \beta_{30} & : & \beta_{13} & \beta_{22} & \beta_{31} & \beta_{40} \\
\beta_{02} & \beta_{03} & \beta_{12} & \beta_{04} & \beta_{13} & \beta_{22} & : & \beta_{05} & \beta_{14} & \beta_{23} & \beta_{32} \\
\beta_{11} & \beta_{12} & \beta_{21} & \beta_{13} & \beta_{22} & \beta_{31} & : & \beta_{14} & \beta_{23} & \beta_{32} & \beta_{41} \\
\beta_{20} & \beta_{21} & \beta_{30} & \beta_{22} & \beta_{31} & \beta_{40} & : & \beta_{23} & \beta_{32} & \beta_{41} & \beta_{50} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{03} & \beta_{04} & \beta_{13} & \beta_{05} & \beta_{14} & \beta_{23} & : & \beta_{06} & \beta_{15} & \beta_{24} & \beta_{33} \\
\beta_{12} & \beta_{13} & \beta_{22} & \beta_{14} & \beta_{23} & \beta_{32} & : & \beta_{15} & \beta_{24} & \beta_{33} & \beta_{42} \\
\beta_{21} & \beta_{22} & \beta_{31} & \beta_{23} & \beta_{32} & \beta_{41} & : & \beta_{24} & \beta_{33} & \beta_{42} & \beta_{51} \\
\beta_{30} & \beta_{31} & \beta_{40} & \beta_{32} & \beta_{41} & \beta_{50} & : & \beta_{33} & \beta_{42} & \beta_{51} & \beta_{60}
\end{pmatrix}$$
Moment Problems and Nonnegative Polynomials (FULL MP Case)

- $\mathcal{M} := \{\gamma \equiv \gamma^{(\infty)} : \gamma \text{ admits a rep. meas. } \mu\}$
- $\mathcal{B}_+ := \{\gamma \equiv \gamma^{(\infty)} : M(\infty)(\gamma) \geq 0\}$
  
  Clearly, $\mathcal{M} \subseteq \mathcal{B}_+$

- (Berg, Christensen and Ressel) $\gamma \in \mathcal{B}_+$, $\gamma$ bounded $\Rightarrow \gamma \in \mathcal{M}$
- (Berg and Maserick) $\gamma \in \mathcal{B}_+$, $\gamma$ exponentially bounded $\Rightarrow \gamma \in \mathcal{M}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ finite rank $\Rightarrow \gamma \in \mathcal{M}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ flat $\Rightarrow \gamma \in \mathcal{M}$
\( \mathcal{P}_+ \): nonnegative poly’s

\( \Sigma^2 \): sums of squares of poly’s

Clearly, \( \Sigma^2 \subseteq \mathcal{P}_+ \)

Duality

For \( C \) a cone in \( \mathbb{R}^{\mathbb{Z}^2}_+ \), we let

\[
C^* := \{ \xi \in \mathbb{R}^{\mathbb{Z}^2}_+ : \text{supp}(\xi) \text{ is finite and } \langle p, \xi \rangle \geq 0 \text{ for all } p \in C \}.
\]

(Riesz-Haviland) \( \mathcal{P}_+^* = \mathcal{M} \)

For, consider the Riesz functional \( \Lambda_\gamma(p) := p(\gamma) \equiv \langle p, \gamma \rangle \), which induces a map \( \mathcal{M} \to \mathcal{P}_+^* \ (\gamma \mapsto \Lambda_\gamma) \); Haviland’s Theorem says that this maps is onto, that is, there exists \( \mu \) r.m. for \( \gamma \) if and only if \( \Lambda_\gamma \geq 0 \) on \( \mathcal{P}_+ \).

There exists a version of this result for TMP, as we will see shortly.
Can one localize the support of a representing measure?

J. Stochel solved the full moment problem on planar curves of degree at most 2. Paraphrasing Stochel’s work (i.e., translating from the language of moment sequences into the language of moment matrices), we consider the following property of a polynomial $P$:

$$\beta^{(\infty)} \text{ has a rep. meas. supported in } P(x, y) = 0 \quad (A)$$

if and only if $\mathcal{M}(\infty)(\beta) \geq 0$ and $P(X, Y) = 0$ in $C_{\mathcal{M}(\infty)}$.

**Theorem**

*(Stochel, 1992)* If $\deg P \leq 2$, then $P$ satisfies (A).
Stochel also proved that there exist polynomials of degree 3 that do not satisfy (A).
The link between TMP and FMP is provided by another result of Stochel (2001):

**Theorem**

\[ \beta(\infty) \text{ has a rep. meas. supported in a closed set } K \subseteq \mathbb{R}^2 \text{ if and only if, for each } n, \beta(2n) \text{ has a rep. meas. supported in } K. \]
Localizing Matrices

Consider the **full, complex** MP

\[
\int \bar{z}^i z^j \, d\mu = \gamma_{ij} \quad (i, j \geq 0),
\]

where \(\text{supp} \, \mu \subseteq K\), for \(K\) a closed subset of \(\mathbb{C}\).

- The **Riesz functional** is given by

\[
\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij} \quad (i, j \geq 0).
\]

- **Riesz-Haviland:**

There exists \(\mu\) with \(\text{supp} \, \mu \subseteq K \iff \Lambda_\gamma(p) \geq 0\) for all \(p\) such that \(p|_K \geq 0\).
If \( q \) is a polynomial in \( z \) and \( \bar{z} \), and

\[
K \equiv K_q := \{ z \in \mathbb{C} : q(z, \bar{z}) \geq 0 \},
\]

then \( L_q(p) := L(qp) \) must satisfy \( L_q(p \bar{p}) \geq 0 \) for \( \mu \) to exist. For,

\[
L_q(p \bar{p}) = \int_{K_q} qp \bar{p} \, d\mu \geq 0 \quad \text{(all } p)\]

- K. Schmüdgen (1991): If \( K_q \) is compact, \( \Lambda_\gamma(p \bar{p}) \geq 0 \) and \( L_q(p \bar{p}) \geq 0 \) for all \( p \), then there exists \( \mu \) with \( \text{supp } \mu \subseteq K_q \).
Given a moment sequence $\beta$, the Riesz functional is

$$L_\beta(p) := p(\beta) \ (p \in \mathbb{C}[z, \bar{z}]).$$

Recall the Riesz-Haviland Theorem:

$$\exists \mu \text{ rep. meas. for } \beta \iff L \equiv L_\beta \geq 0 \text{ on } \mathcal{P}_+.$$  

- For TMP, the natural analogue won't work.
- We say that the Riesz functional $L$ is $K$-positive if

$$p \in \mathcal{P} \text{ and } p|K \geq 0 \Rightarrow L(p) \geq 0.$$
Consider the case

\( d = 1, \; K = \mathbb{R}, \) and

\[
M(2) := \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 2
\end{pmatrix} \geq 0.
\]

In this case,

\[
L(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) := a_0 + a_1 + a_2 + a_3 + 2a_4
\]

One proves that \( L \) is \( K \)-positive, but \( \beta \) has no representing measure.
In TMP, $K$-positivity is a necessary (but not sufficient) condition for a $K$-representing measure $\mu$.

**Theorem (TMP Version of Riesz-Haviland)**

(RC-LF, 2007) $\beta \equiv \beta^{(2n)}$ admits a $K$-representing measure if and only if $L_\beta$ admits a $K$-positive linear extension $L : \mathcal{P}_{2n+2} \rightarrow \mathbb{R}$.

This Theorem implies the classical Riesz-Haviland, via Stochel’s Theorem.
In general it is quite difficult to directly verify that an extension \( \tilde{L} : \mathcal{P}_{2n+2} \to \mathbb{R} \) is \( K \)-positive. One approach to establishing \( K \)-positivity or the existence of representing measures is through extensions of moment matrices.
(Smul’jan, 1959)

\[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix} \succeq 0 \iff \begin{cases}
A \geq 0 \\
B = AW \\
C \geq W^* AW
\end{cases}.
\]

Moreover, \( \text{rank} \left( \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \right) = \text{rank} A \iff C = W^* AW. \)
Corollary

Assume \( \text{rank } \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank } A \). Then

\[ A \geq 0 \iff \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0. \]

We say that

\[ \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \]

is a flat extension of \( A \). Observe that

\[ \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix}. \]
Corollary

Assume that

\[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix} \geq 0.
\]

Then

\[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix} = \begin{pmatrix}
A & AW \\
W^*A & W^*AW
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & C - W^*AW
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\sqrt{A} & \sqrt{AW} \\
\sqrt{A} & \sqrt{AW}
\end{pmatrix}^* \begin{pmatrix}
\sqrt{A} & \sqrt{AW} \\
\sqrt{A} & \sqrt{AW}
\end{pmatrix} + \begin{pmatrix}
0 & \sqrt{C - W^*AW} \\
0 & \sqrt{C - W^*AW}
\end{pmatrix}^* \begin{pmatrix}
0 & \sqrt{C - W^*AW} \\
0 & \sqrt{C - W^*AW}
\end{pmatrix}
\]

(sum-of-squares representation).
For $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$, let $\hat{p}$ denote the vector of coefficients and define

$$p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j \equiv M(n)\hat{p}.$$ 

If there exists a rep. meas. $\mu$, then

$$p(Z, \bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq Z(p).$$

The following is our analogue of recursiveness for the TCMP

(Recursiveness) If $p, q, pq \in \mathcal{P}_n$, and $p(Z, \bar{Z}) = 0$,

then $(pq)(Z, \bar{Z}) = 0$. 
Singular TMP; Real Case

- Given a finite family of moments, build moment matrix.
- Label the columns, $1, X, Y, X^2, XY, Y^2, \cdots$.
- Identify column relations, as $p(X, Y) = 0$.
- Observe that $p(X, Y) = 0$ is equivalent to $M(n)\hat{p} = 0$.
- Build algebraic variety

$$V := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker M(n)} \mathbb{Z}_p.$$ 

- Always true: in the presence of a measure,

$$\text{supp } \mu \subseteq V.$$
Therefore,

\[ r := \text{rank} \mathcal{M}(n) \leq \text{card supp } \mu \leq \nu := \text{card } \mathcal{V}. \]

It follows that if \( r > \nu \) then \( \mathcal{M}(n) \) has no representing measure.

If the variety is finite there’s a natural candidate for \( \text{supp } \mu \), i.e., \( \text{supp } \mu = \mathcal{V} \)

(\text{It is possible for the inclusion } \text{supp } \mu \subseteq \mathcal{V} \text{ to be proper.})
### Truncated Moment Problems in Two Real Variables

<table>
<thead>
<tr>
<th></th>
<th>Complex</th>
<th>Real</th>
</tr>
</thead>
<tbody>
<tr>
<td>moments</td>
<td>$\gamma_{ij} \in \mathbb{C}$, $\gamma_{ji} = \bar{\gamma}_{ij}$</td>
<td>$\beta_{ij} \in \mathbb{R}$</td>
</tr>
<tr>
<td>moment matrix</td>
<td>$M(n)$</td>
<td>$M(n)$</td>
</tr>
<tr>
<td>functional calculus</td>
<td>$p(Z, \bar{Z})$</td>
<td>$p(X, Y)$</td>
</tr>
<tr>
<td>algebraic variety</td>
<td>$\mathcal{V}(\gamma) := \bigcap_{p(Z, \bar{Z})=0} \mathbb{Z}_p$</td>
<td>$\mathcal{V}(\beta) := \bigcap_{p(X, Y)=0} \mathbb{Z}_p$</td>
</tr>
</tbody>
</table>

A new notion, of **core variety**, will be presented in L. Fialkow’s talk.
## General Strategy for Solving Truncated Moment Problems

<table>
<thead>
<tr>
<th>$n$</th>
<th><strong>Invertible</strong> $M(n)$</th>
<th><strong>Singular</strong> $M(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$r = 3$; there exists a flat extension $M(2)$.</td>
<td>$r \leq 2$; there exists a flat extension $M(2)$.</td>
</tr>
<tr>
<td>2</td>
<td>$r = 6$; there exists a flat extension $M(3)$.</td>
<td>$r \leq 5$; for $r \leq 4$, there exists a flat extension $M(3)$; for $r = 5$, there exists a measure $\mu$ with $\text{card supp } \mu \leq 6$.</td>
</tr>
<tr>
<td>Invertible $M(n)$</td>
<td>Singular $M(n)$</td>
<td></td>
</tr>
<tr>
<td>------------------</td>
<td>----------------</td>
<td></td>
</tr>
<tr>
<td>$n = 3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r = 10$; there exists $M(3)$ with no representing measure.</td>
<td>$r \leq 9$; we need to distinguish between finite and infinite algebraic varieties.</td>
<td></td>
</tr>
<tr>
<td>$n = 4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r = 15$; little else is known</td>
<td>very little is known</td>
<td></td>
</tr>
<tr>
<td>$n = 5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r = 21$; there exists $M(5)$ with 22-atomic representing measure, but no 21-atomic representing measure. This was proved by J.E. McCarthy via a topological dimension argument that uses the Open Mapping Theorem.</td>
<td>nothing is known</td>
<td></td>
</tr>
</tbody>
</table>
Theorem

(RC-L. Fialkow, 1998) Let $\gamma$ be a truncated moment sequence. TFAE:

(i) $\gamma$ has a rep. meas.;

(ii) $\gamma$ has a rep. meas. with moments of all orders;

(iii) $\gamma$ has a compactly supported rep. meas.;

(iv) $\gamma$ has a finitely atomic rep. meas. (with at most $(n + 2)(2n + 3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n + k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n + k + 1)$ (here $k \leq 2n^2 + 6n + 6$).
Case of Flat Data

Recall: If $\mu$ is a rep. meas. for $M(n)$, then $\text{rank } M(n) \leq \text{card supp } \mu$.

$\gamma$ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

**Theorem**

(RC-L. Fialkow, 1996) If $\gamma$ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

**Theorem**

(RC-L. Fialkow, 1996) The truncated moment sequence $\gamma$ has a rank $M(n)$-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.

To find $\mu$ concretely, let $r := \text{rank } M(n)$ and look for the relation...
\[
Z^r = c_0 1 + c_1 Z + \ldots + c_{r-1} Z^{r-1}.
\]

We then define
\[
p(z) := z^r - (c_0 + \ldots + c_{r-1} z^{r-1})
\]
and solve the **Vandermonde** equation

\[
\begin{pmatrix}
1 & \ldots & 1 \\
z_0 & \ldots & z_{r-1} \\
\vdots & \ddots & \vdots \\
z_0^{r-1} & \ldots & z_{r-1}^{r-1}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\vdots \\
\rho_{r-1}
\end{pmatrix}
= \begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\vdots \\
\gamma_{0r-1}
\end{pmatrix}.
\]

Then
\[
\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.
\]
Consider the problem

\[ p^* := \inf p(x) \text{ subject to } h_1 \geq 0, \cdots, h_m \geq 0; \]

that is, we try to minimize the values of the polynomial \( p \) over the semialgebraic set \( F \) determined by the polynomials \( h_1, \cdots, h_m \).

Let \( d_0 := [(deg p)/2] \) and \( d_i := [(deg h_i)/2] \). For \( t \geq \max\{d_0, d_1, \cdots, d_m\} \), consider the associated optimization problem
An Application to Optimization, cont.

\[ p_t^* := \inf p^T \beta \]

subject to

\[ \beta_0 = 1, \ M(t)[\beta] \geq 0 \quad \text{and} \quad M_h(t - d_j)[\beta] \geq 0 \quad (j = 1, \cdots, m). \]

This is a semidefinite program. One proves that

\[ p_t^* \leq p_{t+1}^* \leq p^*. \]

That is, the sequence \((p_t^*)_t\) approximates the absolute minimum \(p^*\) from below.
J. Lasserre was able to use the Flat Extension Theorem to prove that the sequence converges to $p^*$ when the semialgebraic set $F$ is compact. Hence, the above mentioned semidefinite program can be used to approximate the minimum value of $p$ over $F$. Moreover, in the 0/1 and grid cases, Lasserre was able to prove finite convergence. The significant outcome of this is that for certain optimization problems, the Flat Extension Theorem allows one to establish finite stopping times for suitable algorithms.
Consider the **full** MP

$$\int \bar{z}^i z^j \, d\mu = \gamma_{ij} \ (i, j \geq 0),$$

where supp $\mu \subseteq K$, for $K$ a closed subset of $\mathbb{C}$.

- **The Riesz functional** is given by

$$\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij} \ (i, j \geq 0).$$

- **Riesz-Haviland:**

  There exists $\mu$ with supp $\mu \subseteq K \iff \Lambda_\gamma(p) \geq 0$ for all $p$ such that $p|_K \geq 0$. 
If $q$ is a polynomial in $z$ and $\bar{z}$, and

$$K \equiv K_q := \{ z \in \mathbb{C} : q(z, \bar{z}) \geq 0 \},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for $\mu$ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p} \, d\mu \geq 0 \quad \text{(all } p).$$

- K. Schmüdgen (1991): If $K_q$ is compact, $\Lambda_\gamma(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all $p$, then there exists $\mu$ with $\text{supp } \mu \subseteq K_q$.

- We shall establish a version of this result for truncated MP’s.
First, recall that \( p(Z, \bar{Z}) = 0 \) implies \( \text{supp } \mu \subseteq \mathcal{Z}(p) \). We define the algebraic variety of \( \gamma \) as

\[
\mathcal{V}(\gamma) := \bigcap_{p \in \mathcal{P}_n} \mathcal{Z}(p),
\]

and observe that \( \text{rank } M(n) \leq \text{card } \text{supp } \mu \leq \text{card } \mathcal{V}(\gamma) \), from which it follows that

\[
\text{card } \mathcal{V}(\gamma) < \text{rank } M(n) \Rightarrow \text{there is no rep. meas. } \mu.
\]
Localization of Support: Main Theorem

**Theorem**

(RC-LF, 2000) Let $M(n) \geq 0$ and suppose $\deg(q) = 2k$ or $2k - 1$ for some $k \leq n$. Then there exists $\mu$ with rank $M(n)$ atoms and $\text{supp} \, \mu \subseteq K_q$ if and only if there exists a flat extension $M(n+1)$ for which $M_q(n+k) \geq 0$. In this case, there exists $\mu$ with exactly $\text{rank} \, M(n) - \text{rank} \, M_q(n+k)$ atoms in $\mathbb{Z}(q)$.

**Remark**

M. Laurent (2005) has found an alternative proof, using ideas from real algebraic geometry.
M. Laurent has been able to use techniques from algebraic geometry to obtain an alternative proof of the Flat Extension Theorem.

Once the matrix $M(n)$ has been extended to $M(\infty)$, one observes that $\ker M(\infty)$ is a polynomial ideal.

For an ideal $\mathcal{I}$, let

$$V(\mathcal{I}) := \{ z \in \mathbb{C}^n : p(z) = 0 \text{ for all } f \in \mathcal{I} \};$$

$V(\mathcal{I})$ is the complex variety associated to $\mathcal{I}$.

Both $V(\mathcal{I})$ and

$$\sqrt{\mathcal{I}} := \{ p : p^k \in \mathcal{I} \text{ for some integer } k \geq 1 \}$$

are again ideals, and they both contain $\mathcal{I}$. 
$\mathcal{I}$ is called **radical** if $\mathcal{I} = \sqrt{\mathcal{I}}$.

**Hilbert Nullstellensatz.** $\sqrt{\mathcal{I}} = \mathcal{I}(V(\mathcal{I}))$.

**Corollary.** If $\mathcal{I}$ is radical, then every polynomial that vanishes in $V(\mathcal{I})$ belongs to $\mathcal{I}$.

(Laurent, 2005) $\ker M(\infty)$ is a radical ideal.

**Corollary.** If $M(\infty) \geq 0$ and rank $M(\infty) < \infty$, then the cardinality of $V(\ker M(\infty))$ is rank $M(\infty)$. 


Quadratures (Eswaran and Fialkow)
Subnormal Completion Problem (RC, S.H. Lee and J. Yoon) add picture
Work of D. Kimsey, H. Woerdeman, J.B. Lasserre
D. Henrion, M. Laurent, S. Kuhlmann, M. Infusino, J. Nie, G.
Blekherman, B. Reznick, W. Helton, M. Schweighofer, C. Scheiderer, K.
Schmüdgen mention the quintic paper with S. Yoo
A second approach to positivity for an extension $\tilde{L} : \mathcal{P}_{2n+2} \to \mathbb{R}$ concerns the **structure of positive polynomials**. Recall that the main difficulty associated with the Riesz-Haviland Theorem is that for a general closed set $K \subseteq \mathbb{R}^d$ there is no concrete representation theorem for polynomials that are nonnegative on $K$. 
In this sense, the Riesz-Haviland Theorem is an “abstract” solution to the moment problem, and, similarly, our analogue is an abstract solution to the truncated moment problem.

When $K$ is a compact semialgebraic set, Schmüdgen’s Theorem provides a concrete test for the $K$-positivity of $L_{\beta(\infty)}$, which we have generalized to the case of TMP, as we described above.

We will now look at the second approach, based on representations of positive polynomials.
Let $Q = \{q_0, q_1, \ldots, q_m\} \subseteq \mathcal{P}$ (with $q_0 \equiv 1$) and consider the semialgebraic set

$$K_Q = \{x \in \mathbb{R}^d : q_i(x) \geq 0 \ (1 \leq i \leq m)\}.$$ 

Moreover, let $Q^\pi$ denote the set of products of distinct polynomials in $Q$, that is,

$$Q^\pi := \{q_{i_1} \cdots q_{i_s} : q_{i_j} \in Q, 0 \leq i_1 < \cdots < i_s \leq m, 1 \leq s \leq m + 1\};$$

Observe that $Q \subseteq Q^\pi$ and that $K_Q = K_{Q^\pi}$. 
Theorem (FMP Case)

(K. Schmüdgen) Suppose $K_Q$ is compact. The sequence $\beta \equiv \beta^{(\infty)}$ has a representing measure supported in $K_Q$ if and only if $M_r \geq 0$ for each polynomial $r \in Q^\pi$. 
What about TMP?

Let $K_Q$ be as above and choose $n$ so that $2n \geq \deg q_i$ for $i = 1, \cdots, m$. For $k \geq 0$, consider the following properties for $K_Q$:

$$\left( S_{n,k} \right) \begin{cases} \beta^{(2n)} \text{ has a } K_Q\text{-representing measure if and only if} \\ M(n) \text{ admits a positive extension } M(n + k) \text{ such that} \\ M_{q_i}(n + k) \geq 0 \text{ for } i = 1, \cdots, m \end{cases}$$

and

$$\left( R_{n,k} \right) \begin{cases} \beta^{(2n)} \text{ has a } K_Q\text{-representing measure if and only if} \\ M(n) \text{ admits a positive, recursively generated extension} \\ M(n + k) \text{ such that } M_{q_i}(n + k) \geq 0 \text{ for } i = 1, \cdots, m. \end{cases}$$

Clearly, if $K_Q$ satisfies $(S_{n,k})$, then it also satisfies $(R_{n,k})$. 
Whereas Schmüdgen works with the cone $\Sigma_{Q^n} \cap \mathcal{P}_{2n}$, we focus on the sub-cone $\Sigma_{Q,n}$, defined by:

$$\Sigma_{Q,n} := \{ p \in \mathcal{P}_{2n} : p = \sum_j f^2_{0j} + q_1 \sum_j f^2_{1j} + \ldots + q_m \sum_j f^2_{mj}, \; q_i f^2_{ij} \in \mathcal{P}_{2n} \}.$$ 

Here’s our TMP analog of one direction of Schmüdgen’s Thm.

**Theorem**

(i) Assume that $K_Q$ satisfies $(S_{n,k})$ for some $n$ and $k$. Then every polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_Q$ belongs to $\Sigma_{Q,n+k}$.

(ii) Assume that $K_Q$ satisfies $(R_{n,k})$ for some $n$ and $k$. Then each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_Q$ belongs to $\Sigma_{Q,n+k+1}$. 

Raúl Curto (Oberwolfach, 03.07.2017)
The previous theorem can be extended to nonnegative polynomials in those cases where the cone $\Sigma_{Q,n+k}$ is closed in $\mathcal{P}_{2(n+k)}$.

What about converses?

**Theorem**

(i) If $k \geq 1$ and each polynomial in $\mathcal{P}_{2n+2}$ that is strictly positive on $K_Q$ belongs to $\Sigma_{Q,n+k}$, then $K_Q$ satisfies $(S_{n,k})$.

(ii) If $k = 0$, $K_Q$ is compact, and each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_Q$ belongs to $\Sigma_{Q,n}$, then $K_Q$ satisfies $(S_{n,0})$. 
Each polynomial $p \in \mathcal{P}_2$ satisfying $p|_{\overline{D}} \geq 0$ admits a representation
$$p = \sum_{i=1}^{5} f_i^2 + \alpha(1 - x^2 - y^2),$$
where $\deg f_i \leq 1$ ($1 \leq i \leq 6$) and $\alpha \geq 0$.

We know that $\overline{D}$ fails to satisfy $(S_{3,k})$ for all $k$, and it appears to be open whether the disk satisfies $(S_{2,k})$ for some $k$. 
The Quartic Moment Problem

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \tilde{Z}, Z^2, \tilde{Z}Z, \tilde{Z}^2$$

- $Z = A 1$ (Dirac measure)
- $\tilde{Z} = A 1 + B Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A 1 + B Z + C \tilde{Z}$ (flat extensions always exist)
- $\tilde{Z}Z = A 1 + B Z + C \tilde{Z} + D Z^2$

$$D = 0 \implies \tilde{Z}Z = A 1 + B Z + \tilde{B} \tilde{Z} \text{ and } C = \tilde{B}$$

$$\Rightarrow (\tilde{Z} - B)(Z - \tilde{B}) = A + |B|^2$$

$$\Rightarrow \tilde{W}W = 1 \text{ (circle), for } W := \frac{Z - \tilde{B}}{\sqrt{A + |B|^2}}.$$
Case $r = 5$

With $x := \text{Re}[z]$ and $y := \text{Im}[z]$, and using the flat data result, one can reduce the study to cases corresponding to the following five real conics:

(a) $\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$ \hspace{1cm} \text{parabola; } y = x^2

(b) $\bar{W}^2 = -4i1 + W^2$ \hspace{1cm} \text{hyperbola; } yx = 1

(c) $\bar{W}^2 = W^2$ \hspace{1cm} \text{pair of intersect. lines; } yx = 0

(d) $\bar{W}W = 1$ \hspace{1cm} \text{unit circle; } x^2 + y^2 = 1

(e) $W^2 + 2\bar{W}W + \bar{W}^2 = 2W + 2\bar{W}$ \hspace{1cm} \text{two parallel lines; } x(x - 1) = 0.
Theorem QUARTIC

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ be given, and assume $M(2) \geq 0$ and 
\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\} is a basis for $C_{M(2)}$. Then $\gamma^{(4)}$ admits a rep. meas. $\mu$. 
Moreover, it is possible to find $\mu$ with card supp $\mu = \text{rank } M(2)$, except in some cases when $\mathcal{V}(\gamma^{(4)})$ is a pair of intersecting lines, in which cases there exist $\mu$ with card supp $\mu \leq 6$.

Corollary

Assume that $M(2) \geq 0$, $M(2)$ singular, and that $\text{rank } M(2) \leq \text{card } \mathcal{V}(\gamma^{(4)})$. Then $M(2)$ admits a representing measure.
(L. Fialkow and J. Nie, 2010) Consider a quartic moment problem with invertible $M(2)$. Then there exists a representing measure. The proof is abstract, using convex analysis.

(RC and S. Yoo, 2013) Concrete construction of a representing measure, when $M(2)$ is invertible. Moreover, there exists a 6-atomic representing measure, that is, $M(2)$ admits a flat extension $M(3)$.

SAY A BIT MORE!
Recall: The algebraic variety of $\beta$ is

$$V \equiv V_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker M(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$. If $\beta$ admits a rep. measure $\mu$, then

$$p \in \mathcal{P}_n \text{ satisfies } \hat{p} \in \ker M(n) \iff \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus $\text{supp } \mu \subseteq V$, so $r := \text{rank } M(n)$ and $v := \text{card } V$ satisfy

$$r \leq \text{card } \text{supp } \mu \leq v.$$
Basic necessary conditions for the existence of a representing measure

(Consistency) \( p \in \mathcal{P}_{2n}, \ p|_\mathcal{V} \equiv 0 \iff \Lambda(p) = 0 \)

(where \( \Lambda \) is the Riesz functional associated to \( M(n) \))

(Variety Condition) \( r \leq \nu \), i.e., \( \text{rank } M(n) \leq \text{card } \mathcal{V} \).

Consistency implies

(Recursiveness) \( p, q, pq \in \mathcal{P}_n, \ \hat{p} \in \ker M(n) \implies \hat{pq} \in \ker M(n) \).

(ideal-like property)

Consistency is intimately related to J. Stochel’s Type B: A polynomial \( P \in \mathcal{P}_{2n} \) is type B if \( \Phi \geq 0 \), linear and \( \Phi|_{\mathcal{I}(\mathcal{Z}(P))} \equiv 0 \implies \Phi(f) = d\mu \).
(Consistency) $p \in \mathcal{P}_{2n}$, $p|_V \equiv 0 \implies \Lambda(p) = 0$

(Weak Consistency) $p \in \mathcal{P}_n$, $p|_V \equiv 0 \implies \Lambda(p) = 0$

Consistency $\implies$ Weak Consistency $\implies$ Recursively generated

**Theorem**

*(RC, L. Fialkow and M. Möller, 2005)* Suppose $\mathcal{M}(3) \geq 0$, recursively generated, $Y = X^3$ and $r \leq vle7$. Then $\mathcal{M}(3)$ has a rep. measure.

**Theorem**

*(L. Fialkow; TAMS, 2011)* There exists a real moment matrix $\mathcal{M}(3)$ which is positive, consistent, with column relation $Y = X^3$ and no representing measure.

**Theorem**

Raúl Curto (Oberwolfach, 03.07.2017)  
Truncated Moment Problems
Theorem EXT

(RC, L. Fialkow and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ extremal, i.e., $r = v$, the following are equivalent:

(i) $\beta$ has a representing measure;

(ii) $\beta$ has a unique representing measure, which is rank $M(n)$-atomic (minimal);

(iii) There exists $M(n + 1)$ flat extension of $M(n)$;

(iv) There exists a unique flat extension of $M(n)$;

(iii) $M(n) \geq 0$ and $\beta$ is consistent.
RC-Fialkow have used Truncated Moment Theory to estimate the number and location of the zeros of a prescribed polynomial; for example, to show that the polynomial

\[ p(z) \equiv z^{2n} + az^{2n-1} - az - 1 \quad (0 < a < 1) \]

has \(2n\) distinct zeros, all in the unit circle.

Several authors have used techniques from real algebra to develop structure theorems for positive polynomials on certain noncompact sets \(K_Q\): Kuhlmann-Marshall, Powers-Reznick, Powers-Scheiderer, Prestel, Putinar, Scheiderer, Schmüdgen.

These results lead to moment theorems for measures supported on \(K_Q\).
Recall: (C. Bayer and J. Teichmann) If $\beta$ has a representing measure, then it has a finitely atomic representing measure. Also, \[
\text{rank } M(n) \leq \text{rank } M(n + 1) \leq \text{card } \mathcal{V}(n + 1) \leq \text{card } \mathcal{V}(n).
\] Thus, eventually, a soluble TMP must be flat or extremal.

At present, for a general moment matrix, there is no known concrete test for the existence of a flat extension $M(n + k + 1)$.

For the class of bivariate *recursively determinate* moment matrices, we now present a detailed analysis of an algorithm that can be used in numerical examples to determine the existence or nonexistence of flat extensions (and representing measures).
This algorithm determines the existence or nonexistence of positive, recursively generated extensions $\mathcal{M}(n+1), \ldots, \mathcal{M}(2n-1)$, at least one of which must be a flat extension in the case when there is a measure.

One of our main results shows that there are sequences $\beta^{(2n)}$ for which the first flat extension occurs at $\mathcal{M}_{2n-1}$, so all of the above extensions must be computed in order to recognize that there is a measure.

This result stands in sharp contrast to traditional truncated moment theorems (concerning representing measures supported in $\mathbb{R}$, $[a, b]$, $[0, +\infty)$, or in a planar curve of degree 2), which express the existence of a measure in terms of tests closely related to the original moment data.
Here we see that, at least within the framework of moment matrix extensions, we may need to go far from the original data to resolve the existence of a measure.

We will show that under mild additional hypotheses on $\mathcal{M}(d)$, the implementation of each extension step, from $\mathcal{M}(d+j)$ to $\mathcal{M}(d+j+1)$, leading to a flat extension $\mathcal{M}(d+k+1)$, consists of simply verifying a matrix positivity condition.
A bivariate moment matrix $\mathcal{M}(d)$ is *recursively determinate* if there are column dependence relations of the form

$$ X^k = p(X, Y) \quad (p \in \mathcal{P}_{k-1}, \ k \leq n) $$

and

$$ Y^\ell = q(X, Y) \quad (q \in \mathcal{P}_\ell, \ q \text{ has no } y^\ell \text{ term}, \ \ell \leq n), $$

or with similar relations with the roles of $p$ and $q$ reversed.
Theorem

Suppose the bivariate moment matrix $\mathcal{M}(d)(\beta)$ is positive and recursively generated, with column dependence relations generated entirely the monomials $X^k$ and $Y^\ell$ via recursiveness and linearity. Then there exists a unique moment matrix block $B(n+1)$ such that $\begin{pmatrix} \mathcal{M}(d) & B(n+1) \end{pmatrix}$ is recursively generated and $\text{Ran } B(n+1) \subseteq \text{Ran } \mathcal{M}(d)$. 

Raúl Curto (Oberwolfach, 03.07.2017)
**Corollary**

If $\mathcal{M}(n)$ satisfies the hypotheses of the previous Theorem, then there exists a unique moment matrix block $C \equiv C(n+1)$ consistent with the structure of an RG extension $\mathcal{M}(n+1)$.

By combining the previous Theorem and Corollary, we immediately obtain:

**Theorem**

(RC-L. Fialkow, 2013) If $\mathcal{M}(n)$ is positive, with column relations generated entirely by $X^k$ and $Y^\ell$ via recursiveness and linearity, then $\mathcal{M}(n)$ admits a unique RG extension $\mathcal{M}(n+1)$, i.e., $\text{Ran } B(n+1) \subseteq \text{Ran } \mathcal{M}(n)$, and $\mathcal{M}(n+1)$ is recursively generated.
Corollary

If $\mathcal{M}(n)$ satisfies the above mentioned hypotheses and $n = k + \ell - 2$, then $\mathcal{M}(n)$ admits a flat moment matrix extension $\mathcal{M}(n + 1)$ (and $\beta$ admits a rank $\mathcal{M}(n)$-atomic representing measure).
For $n \geq 1$, there exists a moment matrix $\mathcal{M}(n)$, satisfying the conditions of our main Theorem, for which the extension algorithm determines successive positive, recursively generated extensions $\mathcal{M}(n+1), \ldots, \mathcal{M}(2n-1)$, and for which the first flat extension occurs at $\mathcal{M}(2n-1)$. Moreover, each extension $\mathcal{M}(n+i)$ satisfies the conditions of the Theorem, so to continue the sequence it is only necessary to verify that the RG extension $\mathcal{M}(n+i+1)$ is positive semidefinite.

Proof uses the Division Algorithm of Algebraic Geometry in a nontrivial way.
Cubic Column Relations

Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$, and that $Z^3 = p_2(Z, \bar{Z})$, with $\deg p_2 \leq 2$.

Recall

**Theorem A**

(RC-L. Fialkow) If $M(n)$ admits a column relation of the form $Z^k = p_{k-1}(Z, \bar{Z})$ (for $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $\deg p_{k-1} \leq k - 1$), then $M(n)$ admits a flat extension $M(n + 1)$, and therefore a representing measure.

Now, if $k = 3$, Theorem A can be used only if $n \geq 4$. Thus, one strategy is to somehow extend $M(3)$ to $M(4)$ and preserve the column relation $Z^3 = p_2(Z, \bar{Z})$. This requires checking that the $C$ block in the extension satisfies the Toeplitz condition, something highly nontrivial.
Here’s a different approach:

We’d like to study the case of harmonic poly’s: \( q(z, \bar{z}) := f(z) - \overline{g(z)} \), with \( \deg q = 3 \).

Recall that \( \text{rank } M(n) \leq \text{card } \mathcal{Z}(q) \)

so of special interest is the case when \( \text{card } \mathcal{Z}(q) \geq 7 \), since otherwise the TMP admits a flat extension, or has no representing measure. In the case when \( g(z) \equiv z \), we have

\[ \text{Lemma} \]

(Wilmshurst ’98, Sarason-Crofoot, ’99, Khavinson-Swiatek, ’03)

\[ \text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7. \]

Bézout’s Theorem predicts \( \text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 9 \)
We thus consider the harmonic polynomial \( q_7(z, \bar{z}) := z^3 - itz - u\bar{z} \).

**Proposition**

(RC-S. Yoo, ’09) For \( 0 < |u| < t < 2|u| \), we have \( \text{card } \mathcal{Z}(q_7) = 7 \). In fact, for \( 0 < u < t < 2u \),

\[
\mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r + ir, -r - ir\},
\]

where \( p, q, r > 0 \), \( p^2 + q^2 = u \) and \( r^2 = \frac{t-u}{2} \).
Consider the harmonic polynomial $q_7(z, \bar{z}) := z^3 - itz - u\bar{z}$, with $(0 < |u| < t < 2|u|)$:
Since rank $M(3) = 7$, there must be another column relation besides $q_7(Z, \bar{Z}) = 0$. Clearly the columns

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \bar{Z}Z^2$$

must be linearly independent (otherwise $M(3)$ would be a flat extension of $M(2)$), so the new column relation must involve $\bar{Z}Z^2$ and $\bar{Z}^2Z$. An analysis using the properties of the functional calculus shows that, in the presence of a representing measure, the new column relation must be

$$\bar{Z}^2Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0.$$
Define

\[ q_{LC}(z, \bar{z}) := \bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z} = i(z - i\bar{z})(\bar{z}z - u). \]

Observe that the zero set of \( q_{LC} \) is the union of a line and a circle, and that \( \mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC}) \).
Figure 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$
Theorem (RC & S. Yoo, J. Funct. Anal., 2014)

Let \(M(3) \geq 0\), with \(M(2) > 0\) and \(q_7(Z, \bar{Z}) = 0\). There exists a representing measure for \(M(3)\) if and only if

\[
\begin{align*}
\Lambda(q_{LC}) &= 0 \\
\Lambda(zq_{LC}) &= 0.
\end{align*}
\]

where \(\Lambda \equiv \Lambda_{\beta}\) is the Riesz functional. Equivalently,

\[
\begin{align*}
\text{Re} \gamma_{12} - \text{Im} \gamma_{12} &= u(\text{Re} \gamma_{01} - \text{Im} \gamma_{01}) = 0 \\
\gamma_{22} &= (t + u)\gamma_{11} - 2u \text{ Im} \gamma_{02} = 0.
\end{align*}
\]

Equivalently,

\[q_{LC}(Z, \bar{Z}) = 0\]

Proof uses Consistency Property.
**Proposition (Representation of Polynomials)**

Let $\mathcal{P}_6 := \{ p \in \mathbb{C}_6[z, \bar{z}] : p|_{z(q_7)} \equiv 0 \}$ and let

$\mathcal{I} := \{ p \in \mathbb{C}_6[z, \bar{z}] : p = f q_7 + g \bar{q}_7 + h q_{LC} \text{ for some } f, g, h \in \mathbb{C}_3[z, \bar{z}] \}.$

Then $\mathcal{P}_6 = \mathcal{I}.$
The Division Algorithm

Division Algorithm in $\mathbb{R}[x_1, \cdots, x_n]$

Fix a monomial order $>_{\mathbb{Z}_{\geq 0}^n}$ and let $F = (f_1, \cdots, f_s)$ be an ordered $s$-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or $r$ is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $f_1, \cdots, f_s$.

Furthermore, if $a_i f_i \neq 0$, then we have

$$\text{multideg}(f) \geq \text{multideg}(a_i f_i).$$
Key idea: Use the Division Algorithm to establish representation theorems for polynomials vanishing on the algebraic variety of $\beta$.

The Division Algorithm work is as follows: we identify sufficiently many polynomials $f_1, \ldots, f_s$ vanishing on $\mathcal{V}(\beta)$, and simultaneously in the kernel of the Riesz functional $L_\beta$. By the Division Algorithm, any polynomial $f$ vanishing on $\mathcal{V}(\beta)$ can be written as $f = a_1 f_1 + \cdots + a_s f_s + r$, which readily implies that $r$ must also vanish on $\mathcal{V}(\beta)$. Due to the divisibility condition on the monomials of $r$, and the characteristics of $\mathcal{V}(\beta)$, which generate an invertible Vandermonde matrix, we then prove that $r \equiv 0$.

With some additional work, it is then possible to prove that $f \in \ker L_\beta$, which establishes the Consistency of $\beta$.
## Classification of sextic MP

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Observation: \( r = \nu = 9 \) is impossible, because one column relation of degree 3 results in an infinite algebraic variety.

**Theorem**

(RC-L. Fialkow; JOT, 2005) If \( \mathcal{M}(n) \geq 0 \) admits a flat extension \( \mathcal{M}(n+1) \), then \( \text{rank } \mathcal{M}(n) = \mathcal{V}(\mathcal{M}(n+1)) \) and \( \mathcal{V}(\mathcal{M}(n+1)) \) forms the support of the unique representing measure \( \nu \) for \( \mathcal{M}(n+1) \).

**Theorem**

(RC-L. Fialkow; JOT, 2005) Assume that \( \mathcal{M}(n) \geq 0 \) admits a flat extension \( \mathcal{M}(n+1) \). Then \( \mathcal{V}(\mathcal{M}(n+2)) = \mathcal{V}(\mathcal{M}(n+1)) \).

Combining the two Theorems, we find a requirement for a flat extension:

\[
\begin{align*}
r_n = r_{n+1} \quad &\Rightarrow \quad \begin{cases} 
r_n = \nu_{n+1} \\
\nu_{n+1} = \nu_{n+2}
\end{cases} \\
\Rightarrow 
\end{align*}
\]

\[
r_n = \nu_{n+1} = \nu_{n+2}.
\]
Corollary

Let $\mathcal{M}(n)$ be flat, i.e., a flat extension of $\mathcal{M}(n-1)$. Then $\mathcal{M}(n)$ is extremal, i.e., $r_n = v_n$.

For a concrete example, consider $\mathcal{M}(3) \geq 0$ with $r_3 = 8$ and $v_3 = 9$. We see that the preceding argument admits only two feasible cases:

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<th>$r_3$</th>
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Thus, if $\beta$ admits a representing measure then $\mathcal{M}(4)$ must be extremal. The goal is then to check consistency.
Since the columns $1$, $X$, $Y$, $X^2$, $XY$ and $Y^2$ are linearly independent, four subcases arise:

Subcase 1. $B_1 := \{1, X, Y, X^2, XY, Y^2, X^3\}$

Subcase 2. $B_2 := \{1, X, Y, X^2, XY, Y^2, X^2Y\}$

Subcase 3. $B_3 := \{1, X, Y, X^2, XY, Y^2, XY^2\}$

Subcase 4. $B_4 := \{1, X, Y, X^2, XY, Y^2, Y^3\}$
Theorem

(Subcase 1) Suppose \( M(3)(\beta) \) satisfies the above mentioned conditions. Let \( B_1 \) be a basis for \( C_{M(3)} \). Then \( \beta \) has a representing measure if and only if \( M(3) \) is weakly consistent and for \( 0 \leq i + j \leq 2 \),

\[
\Lambda_{\beta}(x^iy^j(x^4 - a_{00} - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{30}x^3)) = 0,
\]

where \((a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30})^T = W_{B_1}^{-1}(x_1^4, \ldots, x_7^4)^T\).
The Case of $r = v = 8$

There are six different bases, but two subcases are subsumed by the other four, when we use a linear transformation of coordinates (which renders equivalent TMP).

We will list only one of the four theorems, corresponding to the basis $B_4 := \{1, X, Y, X^2, XY, Y^2, X^2Y, XY^2\}$.

**Theorem**

(Subcase 4) Suppose $\mathcal{M}(3)(\beta)$ satisfies the above mentioned conditions. Let $B_4$ be a basis for $\mathcal{C}_\mathcal{M}(3)$. Then $\beta$ has a representing measure if and only if $\mathcal{M}(3)$ is weakly consistent and for $0 \leq i + j \leq 2$,

$$\Lambda_\beta(x^i y^j(x^2 y^2 - a_0 - a_1 x - a_2 y - a_3 x^2 - a_4 xy - a_5 y^2 - a_6 x^3 - a_7 x^2 y)) = 0,$$

where $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)^T = W_{B_4}^{-1}(x_1^2 y_1^2, \ldots, x_8^2 y_8^2)^T$.
Given a point \((a, b) \in \mathbb{R}^2\) we let \(v \equiv v_{(a,b)}\) denote the row vector

\[(1, a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3)\]

We also let \(\delta_{(a,b)}\) denote the point mass at \((a, b)\). It is easy to see that the moment matrix associated with \(\delta_{(a,b)}\) is \(vv^T\), that is, the matrix whose entries are \(M(3)_{ij} = a^i b^j\). For this moment matrix, \(r = 1\) and \(\mathcal{V} = \{(a, b)\}\).

**Theorem (RC & S. Yoo, J. Funct. Anal., 2015)**

Assume \(M(3) \succeq 0, M(2) > 0,\) \(\text{rank } M(3) = 7\) and \(\text{card } \mathcal{V} \geq 8\). Assume also that \(M(3)\) satisfies the Consistency Property. Then \(M(3)\) admits a flat extension \(M(4)\); that is, there exists a representing measure \(\mu\) with \(\text{card supp } \mu = 7\).
Sketch of Proof. WLOG, assume 

\[ \mathcal{V} = \{(x_1, y_1), \ldots, (x_8, y_8)\}. \]

Also assume that in \( M(3) \) the first seven columns are linearly independent. Now form the Vandermonde matrix

\[
\begin{pmatrix}
1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & x_1^3 & x_1y_1 & y_1^2 & y_1^3 \\
1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & x_2^3 & x_2y_2 & y_2^2 & y_2^3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & x_8 & y_8 & x_8^2 & x_8y_8 & y_8^2 & x_8^3 & x_8y_8 & y_8^2 & y_8^3
\end{pmatrix}.
\]

This is an \( 8 \times 10 \) matrix, with rank 7. It follows that exactly seven rows are linearly independent, so one of them must be a linear combination of the other seven, say

\[ R_j = \sum_{i \neq j} \lambda_i R_i. \]
The row \( R_j \) must be associated with a point \( (x_j, y_j) \in \mathcal{V} \). To single out this point, we will denote it by \((a, b)\). Now let

\[
\mathcal{V}' := \mathcal{V} \setminus \{(a, b)\}.
\]

**Claim.** No conic goes through \( \mathcal{V}' \). For, let

\[
C(x, y) \equiv C_{00} + C_{10}x + C_{01}y + C_{20}x^2 + C_{11}xy + C_{02}y^2
\]

be such a conic, that is, if \( \hat{C} \) denotes the vector of coefficients of \( C \) regarded as a 10-vector, then

\[
R_i \hat{C} = C(x_i, y_i) = 0,
\]

and therefore

\[
\sum_{i \neq j} \lambda_i R_i \hat{C} = 0;
\]
that is, \( C(a, b) \equiv R_j \hat{C} = 0 \), which implies that \( C \) also vanishes on \( (a, b) \), and a fortiori \( C \) vanishes on the entire algebraic variety \( \mathcal{V} \). Then, by the Consistency Property, \( C(X, Y) = 0 \), that is, the moment matrix \( M(3) \) admits a quadratic column relation, a contradiction to the fact that \( M(2) > 0 \).

We now define

\[
\hat{M}(3) := M(3) - \rho \mathbf{v} \mathbf{v}^T,
\]

where \( \mathbf{v} \) is the row vector associated with the point \( (a, b) \).
We wish to prove that \( \text{rank } \tilde{M}(3) = 6 \) for some positive value of \( \rho \). If we do this, then \( \tilde{M}(3) \) will be a flat extension of \( \tilde{M}(2) \), and we will have a 6-atomic measure for \( \tilde{M}(3) \), and therefore a 7-atomic measure for \( M(3) \), since \( M(3) = \tilde{M}(3) + \rho \mathbf{v} \mathbf{v}^T \). Moreover, one can show that rank \( \tilde{M}(2) = 6 \), using above Claim.

Let \( \lambda \) denote the nonzero eigenvalue of \( \mathbf{v} \mathbf{v}^T \), and let \( \mathcal{B} \) be the basis of the column space of \( M(3) \). Then

\[
\det \tilde{M}(3)_{\mathcal{B}} = \det M(3)_{\mathcal{B}} - \rho \lambda \det (M(3)_{\mathcal{B}} |_{\{2,3,4,5,6,7\}}).
\]

Thus, with

\[
\rho := \frac{\det M(3)_{\mathcal{B}}}{\lambda \det (M(3)_{\mathcal{B}} |_{\{2,3,4,5,6,7\}})},
\]

we successfully reduce the rank. \( \square \)
The following result gives an upper bound on the number of positive moment matrix extensions needed to solve TMP.

**Theorem**

(L. Fialkow (2008)) Suppose $v < \infty$. Then $\beta$ admits a representing measure if and only if $M(n)(\beta)$ has a positive extension $M(n + v - r + 1)$ satisfying rank $M(n + v - r + 1) \leq \text{card } \mathcal{V}_M(n+v-r+1)$. 
Problem

Assume $M(3) \geq 0$, $M(2) > 0$, $\text{rank } M(3) = 8$, $\text{card } \mathcal{V} = 9$. Under what conditions does the moment sequence admit a representing measure?

Remark

If $\mathcal{M}(n)$ has an $r$-atomic measure $\mu \equiv \sum_{i=1}^{r} \rho_{i} \delta(x_{i}, y_{i})$, then we may write $\mathcal{M}(n)$ as

$$\mathcal{M}(n) = \sum_{i=1}^{r} \rho_{i} \mathbf{v}_{i} \mathbf{v}_{i}^{*},$$

where the densities $\rho_{i}$ are positive, the column vector $\mathbf{v}_{i}$ is given by $(1, x_{i}, y_{i}, \ldots, x_{i}^{n}, x_{i}^{n-1} y_{i}, \ldots, x_{i} y_{i}^{n-1}, y_{i}^{n})^T$, and the point $(x_{i}, y_{i})$ is in the algebraic variety $\mathcal{V}$ for all $i = 1, \ldots, r$. 
# Classification of sextic MP

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In the specific case of \( r = 8 \) and \( v = 9 \), one must have the algebraic variety \( \mathcal{V} \) of \( M(3) \) as the intersection of two cubics \( C_1 \) and \( C_2 \) in general position. We then use the Cayley-Bacharach Theorem:

Assume that two cubics \( C_1 \) and \( C_2 \) in the projective plane meet in nine (different) points (that is \( C_1 \cap C_2 = \mathcal{V} \)). Then every cubic \( C \) that passes through any eight of the points in \( \mathcal{V} \) also passes through the ninth point.
Some Matricial Results

When we decompose a moment matrix as a sum $\mathcal{M}(n) = \tilde{\mathcal{M}}(n) + P$, the goal is to both reduce the rank (i.e., $\text{rank } \tilde{\mathcal{M}}(n) < \text{rank } \mathcal{M}(n)$) and obtain a moment matrix $\tilde{\mathcal{M}}(n)$ for which we can use previous known results to solve TMP. In some cases, we can even make $\tilde{\mathcal{M}}(n)$ flat.

**Lemma**

Let $A$ and $B$ be finite matrices. Then

$$\text{rank } (A + B) \leq \text{rank } A + \text{rank } B$$

We must also ensure $\tilde{\mathcal{M}}(n) \geq 0$; that is, the minimum eigenvalue is nonnegative.
Eigenvalue Inequalities

For a self-adjoint $n \times n$ matrix $A$, we list the eigenvalues as

$$\lambda_1(A) \leq \cdots \leq \lambda_n(A).$$

**Theorem**

(Horn-Johnson, 1990) Let $A \in M_n$ be Hermitian and let $z \in \mathbb{C}^n$ be a given vector. If the eigenvalues of $A$ and $A \pm zz^*$ are arranged in increasing order as above, we have for $k = 1, 2, \ldots, n - 2$,

(I) $\lambda_k(A \pm zz^*) \leq \lambda_{k+1}(A) \leq \lambda_{k+2}(A \pm zz^*)$,

(II) $\lambda_k(A) \leq \lambda_{k+1}(A \pm zz^*) \leq \lambda_{k+2}(A)$. 
One easily checks that, for $\alpha \in \mathbb{R}$

$$\det(A - \alpha I_1) = \det(A) - \alpha \det(A_{\{2,3,\ldots,m\}}).$$ \hspace{1cm} (10.1)

On the other hand, the Spectral Theorem guarantees that any rank-one Hermitian matrix is unitarily equivalent to a scalar multiple of $I_1(m)$. If $P$ is rank-one, there exists a unitary operator $U$ such that $U^*PU = \lambda I_1$, where $\lambda$ is the only nonzero eigenvalue of $P$. We now generalize this as follows.

**Proposition**

Let $A$ be an arbitrary square matrix of size $m$, and let $P$, $U$ and $\lambda$ be as above. Then

$$\det(A - \rho P) = \det(A) - \rho \lambda \det\left((U^*AU)_{\{2,3,\ldots,m\}}\right) \text{ for all } \rho \in \mathbb{R}.$$
**M(3) with** \( r = 8 \) **AND** \( v = 9 \)

To date, most concrete solutions of sextic MP include numerical conditions on one or more of the moments; it is generally intricate to express these numerical conditions as specific properties of the moment matrix. Moreover, when we solve a recursively determinate sextic MP \((r = 8 \text{ and } v \geq 8)\), we need to maintain recursiveness of the extension \(M(4)\), and verify the positivity of \(M(4)\).

This leads naturally to an algorithmic approach to TMP.

**Problem**

Suppose \(M(3) \geq 0\) is of rank 8, consistent, with \(M(2) > 0\), and with \(v = 9\). Let \(\mathcal{V} \equiv \{(x_i, y_i)\}_{i=1}^{9}\) be the algebraic variety of \(M(3)\). Under what conditions does the moment sequence admit a representing measure?
Algorithm.

Step 1. Build the generalized Vandermonde matrix of $\mathcal{V}$, namely,

$$
\mathcal{E} := \begin{pmatrix}
1 & x_i & y_i & x_i^2 & x_i y_i & y_i^2 & x_i^3 & x_i^2 y_i & x_i y_i^2 & y_i^3 \\
\end{pmatrix}_{i=1}^9.
$$

Since $\mathcal{E}$ has 8 linearly independent rows, we can pick a point $(a, b) \in \mathcal{V}$ such that the row $R_{(a,b)}$ associated with $(a, b)$ is linearly dependent of the other 8 rows.

Step 2. Let $\mathcal{B}$ be the basis for the column space of $\mathcal{E}$ and let $\mathcal{E}_\mathcal{B}$ denote the resulting matrix after removing the two dependent columns and the row $R_{(a,b)}$ from $\mathcal{E}$. We prove that it is invertible, using the Cayley-Bacharach Theorem, stating that every cubic passing through any eight of the nine points also passes through the ninth point.
Step 3. Once we know that $\mathcal{E}_B$ is invertible, we choose another point $(c, d) \in \mathcal{V}$ ($(c, d) \neq (a, b)$) and eliminate the row $R_{(c,d)}$ associated with $(c, d)$ from $\mathcal{E}_B$; we denote this matrix as $\mathcal{E}_B'$. Note that $\mathcal{E}_B'$ has rank 7 and this fact implies that there is a new cubic polynomial $r(x, y)$ vanishing on $\hat{\mathcal{V}} := \mathcal{V} - \{(a, b), (c, d)\}$, besides $p(x, y)$ and $q(x, y)$. 
Step 4. We will use a rank-one decomposition of $\mathcal{M}(3)$, and try to understand the structure of the decomposition in case a representing measure exists. Suppose $\mathcal{M}(3)$ has a representing measure. Then the variety condition forces a measure to be 8- or 9-atomic. Let 

$$v(x, y) := (1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3)^T.$$ 

We write 

$$\mathcal{M}(3) = \tilde{\mathcal{M}}(3) + m_1 v(a, b)v(a, b)^T + m_2 v(c, d)v(c, d)^T,$$

where $m_1$ and $m_2$ are nonnegative (not simultaneously zero) for $(a, b), (c, d) \in \mathcal{V}$. The moment matrices $v(a, b)v(a, b)^T$ and $v(c, d)v(c, d)^T$, have representing measures $\delta_{(a,b)}$ and $\delta_{(c,d)}$, resp. Thus, in the presence of a measure, we should be able to find a moment matrix $\tilde{\mathcal{M}}(3)$ with a 6- or 7-atomic measure (since rank $\tilde{\mathcal{M}}(3) = 6$ or 7).
Step 4, cont.

Denote such measure by \(\tilde{\mu}\), with \(\text{supp}\ \tilde{\mu} \subseteq \hat{V}\). Since \(\hat{V} \subseteq \mathbb{Z}(r)\), it follows that \(\text{supp}\ \tilde{\mu} \subseteq \mathbb{Z}(r)\), and therefore \(r(X, Y) = 0\). Thus, we must find \(m_1, m_2 \geq 0\) such that \(r(X, Y) = 0\).

Step 5. In order to find such \(m_1\) and \(m_2\), we need to solve a linear system of 10 equations with the two unknowns \(m_1\) and \(m_2\). If no nonnegative solutions exist, \(\mathcal{M}(3)\) does not have a representing measure. In the case when a solution does exist, we must check whether \(\tilde{\mathcal{M}(3)} \geq 0\) with the fixed \(m_1\) and \(m_2\) (equivalently, \(\Lambda_{\tilde{\mathcal{M}(3)}}(x^iy^jr) = 0\) for \(0 \leq i + j \leq 3\)).
Step 6. After checking positive semidefiniteness, we still have the two possible cases based on the values of rank $\widehat{\mathcal{M}}(3)$: If rank $\widehat{\mathcal{M}}(3) = 6$, then $\widehat{\mathcal{M}}(3)$ is a flat extension of $\widehat{\mathcal{M}}(2)$; hence $\widehat{\mathcal{M}}(3)$ has a 6-atomic measure, and so $\mathcal{M}(3)$ has an 8-atomic measure. Finally, to cover the case $\text{rank } \widehat{\mathcal{M}}(3) = 7$, notice that $\text{card } \mathcal{V}(\widehat{\mathcal{M}}(3)) = 7$; if the cardinality of the variety is 7, then $\widehat{\mathcal{M}}(3)$ is extremal, so we use Division Algorithm techniques. If $\text{card } \mathcal{V}(\widehat{\mathcal{M}}(3)) \geq 8$, then it follows that $\widehat{\mathcal{M}}(3)$ admits a representing measure and so does $\mathcal{M}(3)$.

The construction of the Algorithm is therefore complete. □
**Theorem**

(RC-L. Fialkow; Op, Th. Adv. Appl., 1998) Assume that $\mathcal{M}(n) \geq 0$ satisfies (RG) and that $Y = A1 + BX$ for some $A, B \in \mathbb{R}$. Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n + 1)$.

**Problem**

Let $\mathcal{V}$ be the algebraic variety of $\mathcal{M}(3)$. Assume $\mathcal{M}(3) \geq 0$, of rank 8, consistent, with $\mathcal{M}(2) > 0$, and with $\nu = \infty$. Under what conditions, does the moment sequence admits a representing measure?
We need two preliminary results; the proofs use a separation-of-atoms technique.

**Proposition**

Let $\mathcal{M}(3)$ be a positive semidefinite, recursively generated moment matrix satisfying $XY = 0$. Then $\mathcal{M}(3)$ has a 7-atomic representing measure.

**Proposition**

Let $\mathcal{M}(3)$ be a positive semidefinite, recursively generated moment matrix, with column relation $X^2 = X$. Then $\mathcal{M}(3)$ admits a 7-atomic representing measure.
Algorithm

Write \( p(x, y) = \ell_1(x, y)c_1(x, y) \) and \( q(x, y) = \ell_2(x, y)c_2(x, y) \), where \( \ell_i \) is a line and \( c_i \) is a conic for \( i = 1, 2 \).

Case 1. \( c(x, y) \equiv c_1(x, y) = c_2(x, y) \)

Let \((a_0, b_0) \in \mathbb{Z}(\ell_1) \cap \mathbb{Z}(\ell_2)\). Then \((a_0, b_0)\) must be in the support of a representing measure; otherwise, consistency of \( \mathcal{M}(3) \) forces the existence of a quadratic column relation in \( \mathcal{M}(3) \). Notice that \( \mathcal{M}(3) \) has a representing measure if and only if we may write \( \mathcal{M}(3) \) as a sum of two moment matrices, that is, for some \( \rho_0 > 0 \),

\[
\mathcal{M}(3) = \rho_0 \mathbf{v}\mathbf{v}^T + \mathcal{M}_c(3),
\]

where \( \mathbf{v} = (1, a_0, b_0, a_0^2, a_0 b_0, b_0^2, a_0^3, a_0^2 b_0, a_0 b_0^2, b_0^3)^T \) and \( \mathcal{M}_c(3) \) is a moment matrix generated with atoms in the graph of \( c(x, y) = 0 \).
It follows that $\mathcal{M}_c(3)$ has the quadratic column relation $c(X, Y) = 0$, and hence $\mathcal{M}_c(3)$ has at least 3 column relations. Indeed, there is a positive $\rho_0$ such that rank $\mathcal{M}_c(3) = 7$. For, let $B$ be the basis of the column space of $\mathcal{M}(3)$. All leading principal minors of $\mathcal{M}_c(3)_B$ are linear in $\rho$; that is,

$$\det (\mathcal{M}_c(3)_B) = \det (\mathcal{M}(3)_B) - \rho \lambda \det \left((U^* \mathcal{M}(3)_B U)_{\{2,3,\ldots,8\}}\right), \quad (10.3)$$

for some unitary matrix $U$, where $\lambda$ is the only nonzero eigenvalue of $(vv^T)_B$. Positive definiteness of $\mathcal{M}(3)_B$ implies that both $\det (\mathcal{M}(3)_B)$ and $\lambda \det \left((U^* \mathcal{M}(3)_B U)_{\{2,3,\ldots,8\}}\right)$ are positive.
We thus can take $\rho_0$ as $\det(M(3)_B) / (\lambda \det ((U^* M(3)_B U)_{2,3,\ldots,8}))$. Next, if the eigenvalues of $M(3)$ and $M_c(3)$ are arranged in ascending order, then we can see that $0 < \lambda_3(M(3)) \leq \lambda_4(\tilde{M}_c(3))$. Since rank $M_c(3) = 7$, it follows that $M_c(3)$ has the eigenvalue zero with multiplicity 3, through which we can conclude that $\lambda_k(M_c(3)) = 0$ for $k = 1, 2, 3$ and $\lambda_k(M_c(3)) > 0$ for $k = 4, \ldots, 10$. In other words, $M_c(3)$ is positive semidefinite.

In summary, with the specific $\rho_0$, $M(3)$ has a representing measure if and only if $M_c(3)$ has a representing measure. If $c$ is a circle, parabola, or hyperbola, then we know $M_c(3)$ admits a representing measure; if $c$ is a pair of intersecting or parallel lines, then we apply the two specific situations we studied earlier: $XY = 0$ and $X^2 = X$. 

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Case 2. \( \ell(x, y) \equiv \ell_1(x, y) = \ell_2(x, y) \)

Let \((c_i, d_i) \in \mathcal{Z}(c_1) \cap \mathcal{Z}(c_2)\) for \(i = 1, \ldots, m_2\) \((3 \leq m_2 \leq 4)\). Similarly, \(M(3)\) has a representing measure if and only if \(M(3)\) can be written as a sum of two moment matrices:

\[
M(3) = M_\ell(3) + \begin{pmatrix} \beta_{ij}^{(c)} \end{pmatrix} \equiv M_\ell(3) + M_c(3),
\]

(10.4)

where \(\beta_{ij}^{(c)} = \sum_{k=1}^{m_2} \rho_k^{(c)} c_i^k d_j^k\) for some positive \(\rho_i^{(c)}\) \((1 \leq i \leq m_2)\) and \(M_\ell(3)\) is a moment matrix generated by atoms in the line \(\ell\). We next need to see that \(M_\ell(3)\) must have the column relation \(\ell(X, Y) = 0\). Applying the relation to \(M_\ell(3) = M(3) - M_c(3)\), we have a linear system of 10 equations in the unknowns, \(\rho_1^{(c)}, \ldots, \rho_{m_2}^{(c)}\) (at least 3 of them are positive).
If the system does not have a nonnegative solution set, then $\mathcal{M}(3)$ does not have a representing measure. If we can find a solution of the system, then since $\mathcal{M}_c(3)$ obviously has a representing measure, it follows that we just need to check if $\mathcal{M}_\ell(3) \geq 0$ and if $\mathcal{M}_\ell(3)$ satisfies (RG). If $\mathcal{M}_\ell(3)$ passes both tests, then it has a representing measure and consequently, so does $\mathcal{M}(3)$.
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