Toral and Spherical Aluthge Transforms

(joint work with Jasang Yoon)

Raúl E. Curto

University of Iowa

NEAM, University at Albany

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Overview

Hyponormality and Subnormality

$\mathcal{L}(\mathcal{H})$: algebra of operators on a Hilbert space $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$ (We say that N is a lifting of T, or an extension of T.
- hyponormal if $T^*T \geq TT^*$

normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal

For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.

- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- (Bram-Halmos):

T subnormal $\Leftrightarrow$ T is k-hyponormal for all $k \geq 1$.

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift
- In general, $W_\alpha = U_+ D_\alpha \quad$ (polar decomposition)
- $\|W_\alpha\| = \sup_k \alpha_k$

  $W_\alpha^n e_k = \alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1} e_{k+n}$, so

$$W_\alpha^n \cong \bigoplus_{i=0}^{n-1} W_{\beta(i)},$$

Weighted Shifts and Berger's Theorem

Given a bounded sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, \ldots$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0).$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)

Berger Measures

- (Berger; Gellar-Wallen) W_α is subnormal if and only if there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \;\; (\text{all } k \geq 0).$$

  ξ is the Berger measure of W_α.

- For $0 < a < 1$ we let $S_a := \text{shift}\,(a, 1, 1, ...)$.
- The Berger measure of U_+ is δ_1.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

WLOG, assume $\|W_\alpha\| = 1$. Observe that $r(W_\alpha) = 1 = \sup \alpha$. Thus,

$$\begin{cases} \sigma(W_\alpha) = \bar{\mathbb{D}} \\ \sigma_e(W_\alpha) = \mathbb{T} \\ \operatorname{ind}(W_\alpha - \lambda) = -1 \;\text{ for }\; |\lambda| < 1. \end{cases}$$

Therefore, all norm-one hyponormal weighted shifts are spectrally equivalent.

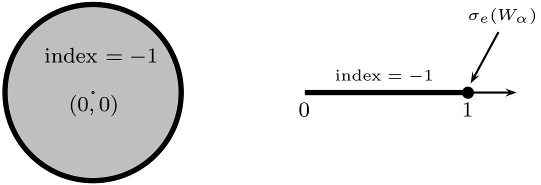


FIGURE 1. Spectral picture of a norm-one hyponormal weighted shift

(RC, 1990) W_α is k-hyponormal if and only if the following Hankel moment matrices are positive for $m = 0, 1, 2, \dots$:

$$\begin{pmatrix} \gamma_m & \gamma_{m+1} & \gamma_{m+2} & \cdots & \gamma_{m+k} \\ \gamma_{m+1} & \gamma_{m+2} & & \cdots & \gamma_{m+k+1} \\ \gamma_{m+2} & \cdots & & \cdots & \gamma_{m+k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_{m+k} & \gamma_{m+k+1} & & \cdots & \gamma_{m+2k} \end{pmatrix} \geq 0.$$

(Thus, an operator matrix condition is replaced by a scalar matrix condition.)

Aluthge Transform

Let T be a Hilbert space operator, let $P := |T|$ be its positive part, and let $T = VP$ denote the canonical polar decomposition of T, with V a partial isometry and $\ker V = \ker T = \ker P$.

We define the Aluthge transform of T as

$$\hat{T} := \sqrt{P} V \sqrt{P}.$$

The iterates are

$$\hat{T}^{n+1} := \widehat{(\hat{T})^n} \; (n \geq 1).$$

The Aluthge transform has been extensively studied, in terms of algebraic, structural and spectral properties.

For instance,

- (i) $T = \hat{T} \Leftrightarrow T$ is quasinormal;
- (ii) (Aluthge, 1990) If $0 < p < \frac{1}{2}$ and T is p-hyponormal, then $\hat{T}$ is $(p + \frac{1}{2})$-hyponormal;
- (iii) (Jung, Ko & Pearcy, 2000) If $\hat{T}$ has a n.i.s., then T has a n.i.s.
- (iv) (Kim-Ko, 2005; Kimura, 2004) T has property (β) if and only if $\hat{T}$ has property (β); and
- (v) (Ando, 2005) $\left\|(T-\lambda)^{-1}\right\| \geq \left\|(\hat{T}-\lambda)^{-1}\right\|$ $(\lambda \notin \sigma(T))$.
- (vi) Observe that if $A := \sqrt{P}$ and $B := V\sqrt{P}$, then $\hat{T} = AB$ and $T = BA$, and therefore

$$\sigma(\hat{T}) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

On the other hand,

G. Exner (IWOTA 2006 Lecture): subnormality is not preserved under the Aluthge transform. Concretely, Exner proved that the Aluthge transform of the weighted shift in the following example is not subnormal.

Example

(RC, Y. Poon and J. Yoon, 2005) Let

$$\alpha \equiv \alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n + \frac{1}{2}}{2^n + 1}}, & \text{if } n \geq 1 \end{cases},$$

Then W_α is subnormal, with 3-atomic Berger measure

$$\mu = \frac{1}{3}(\delta_0 + \delta_{1/2} + \delta_1).$$

(S.H. Lee, W.Y. Lee and J. Yoon, 2012) For $k \geq 2$, the Aluthge transform, when acting on weighted shifts, need not preserve k-hyponormality.

Note that the Aluthge transform of a weighted shift is again a weighted shift.

Concretely, the weights of $\widehat{W_\alpha}$ are

$$\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_2\alpha_3}, \sqrt{\alpha_3\alpha_4}, \cdots .$$

Define

$$W_{\sqrt{\alpha}} := \text{shift}\ \left(\sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \cdots\right).$$

Then $\widehat{W_\alpha}$ is the Schur product of $W_{\sqrt{\alpha}}$ and its restriction to the subspace $\bigvee\{e_1, e_2, \cdots\}$. Thus, a sufficient condition for the subnormality of $\widehat{W_\alpha}$ is the subnormality of $W_{\sqrt{\alpha}}$.

Agler Shifts

For $j = 2, 3, \ldots$, the j-th Agler shift A_j is given by

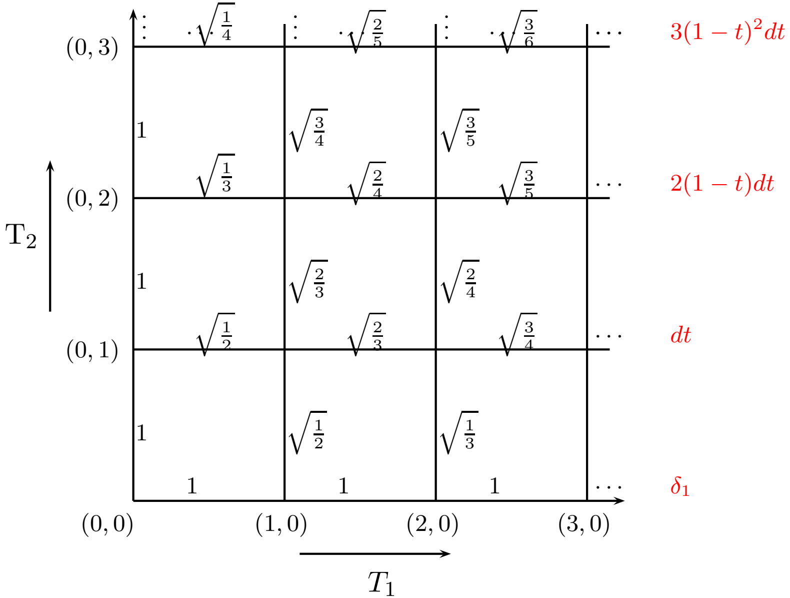
$$\alpha^j := \sqrt{\frac{1}{j}}, \sqrt{\frac{2}{j+1}}, \sqrt{\frac{3}{j+2}}, \ldots.$$

It is well known that A_j is subnormal, with Berger measure

$$d\mu^j(t) = (j-1)(1-t)^{j-2}dt.$$

Clearly, A_2 is the Bergman shift, and the remaining Agler shifts are the upper row shifts of the Drury-Arveson 2-variable weighted shift, which incidentally is a spherical complete hyperexpansion.

Weight Diagram of the Drury-Arveson Shift



$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^{\infty}(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

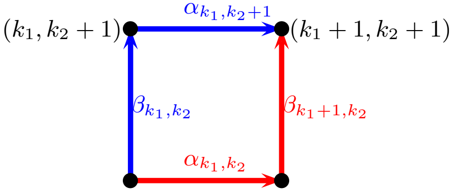
$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

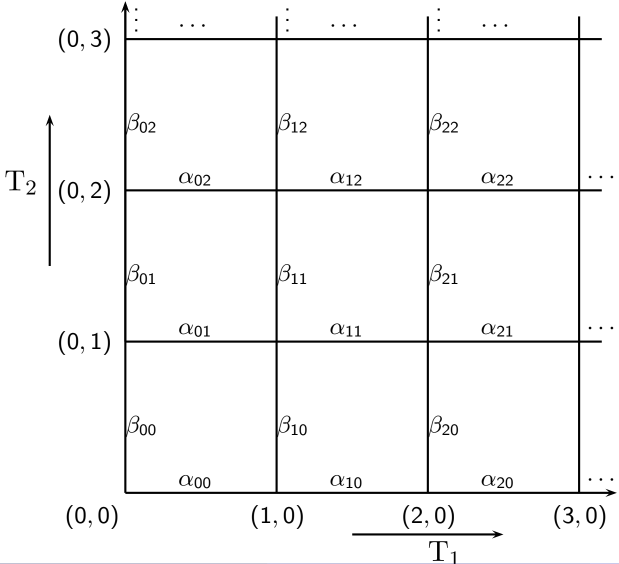
We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad (\text{all } \mathbf{k}).$$





(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T_2
T_1
β_{02} β_{12} β_{22}
α_{02} α_{12} α_{22}
β_{01} β_{11} β_{21}
α_{01} α_{11} α_{21}
β_{00} β_{10} β_{20}
α_{00} α_{10} α_{20}

Recall the definition of joint hyponormality.

An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ldots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

To detect hyponormality, there is a simple criterion:

Theorem

(RC, 1988) (Six-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ *be a 2-variable weighted shift, with weight sequences* α *and* β. *Then*

$$\mathbf{T} \text{ is hyponormal } \Leftrightarrow \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geq 0$$

(all $\mathbf{k} \in \mathbb{Z}^2_+$*).*

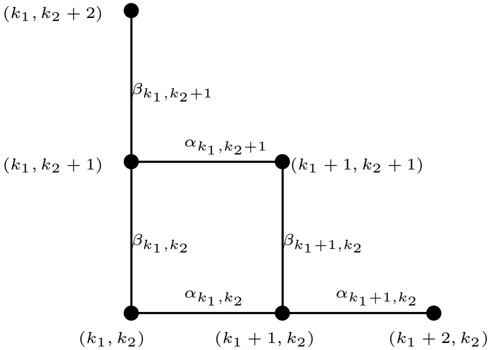


FIGURE 3. Weight diagram used in the Six-point Test

We now recall the notion of moment of order $\mathbf{k}$ for a commuting pair (α, β). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is $\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta)$

$$:= \begin{cases} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

By commutativity, $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).

- (Jewell-Lubin)

$$W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} := \prod_{i=0}^{k_1-1} \alpha_{(i,0)}^2 \cdot \prod_{j=0}^{k_2-1} \beta_{(k_1-1,j)}^2$$
$$= \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \;\; (\text{all } \mathbf{k} \geq \mathbf{0}).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

The Spectral Picture of Subnormal 2-variable Weighted Shifts

For subnormal 2-variable weighted shifts, RC-K. Yan gave in 1995 a complete description of the spectral picture, by exploiting the groupoid machinery in Muhly-Renault and RC-Muhly, and the presence of the Berger measure, which was used to analyze the asymptotic behavior of sequences of weights.

Notation

μ : compactly supported finite positive Borel measure on $\mathbb{C}^n$ $(n \geqslant 1)$

$P^2(\mu)$: norm closure in $L^2(\mu)$ of $\mathbb{C}[z_1, \cdots, z_n]$

$M_{\mathbf{z}} \equiv M_{\mathbf{z}}^{(\mu)} := (M_{z_1}^{(\mu)}, \cdots, M_{z_n}^{(\mu)})$: multiplication operators acting on $P^2(\mu)$

$M_{\mathbf{z}}$ on $P^2(\mu)$ is the universal model for cyclic subnormal n-tuples

Theorem

(RC-K. Yan, 1995) Let μ be a Reinhardt measure on $\mathbb{C}^2$, and let $C := \log \left| \widehat{K} \right|$. *Assume that* $\partial \widehat{K} \bigcap (z_1 z_2 = 0)$ *contains no 1-dimensional open disks. Then*

(i) $\sigma_T(M_{\mathbf{z}}, P^2(\mu)) = \sigma_r(M_{\mathbf{z}}, P^2(\mu)) = \widehat{K}$

(ii) $\sigma_{Te}(M_{\mathbf{z}}, P^2(\mu)) = \sigma_{r_e}(M_{\mathbf{z}}, P^2(\mu)) = \partial \widehat{K}$

(iii) $index(M_{\mathbf{z}} - \lambda) = \begin{cases} 1 & \text{if } \lambda \in int.(\widehat{K}) \\ 0 & \text{if } \lambda \notin int.(\widehat{K}) \end{cases}$

(vi) $kerD^1_{M_{\mathbf{z}}-\lambda} = ranD^0_{M_{\mathbf{z}}-\lambda}$ for all $\lambda \in int.(\widehat{K})$.

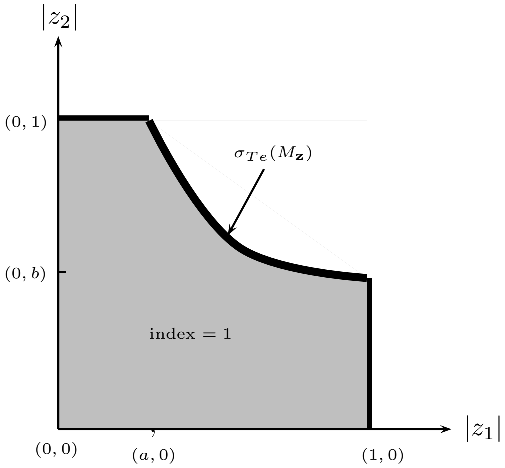


FIGURE 5. Spectral picture of a typical subnormal 2-variable weighted shift

Toral Aluthge Transform

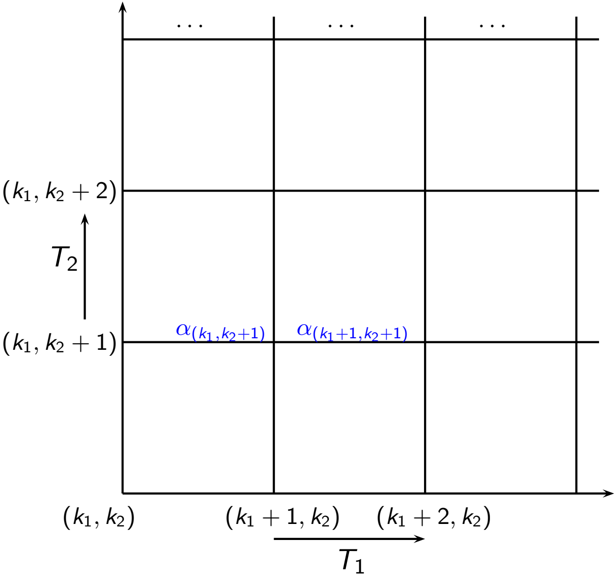
We introduce the toral Aluthge transform of 2-variable weighted shifts $W_{(\alpha,\beta)} \equiv (T_1, T_2)$.

For $i = 1, 2$, consider the polar decomposition $T_i \equiv U_i\,|T_i|$. Then for a 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$, we define the toral Aluthge transform of $W_{(\alpha,\beta)}$ as follows:

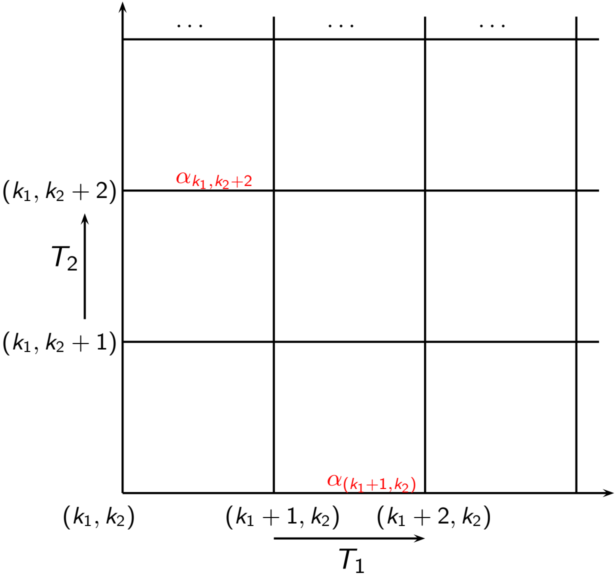
$$\widetilde{W}_{(\alpha,\beta)} := (\widetilde{T_1, T_2}) := \left(|T_1|^{\frac{1}{2}} U_1 |T_1|^{\frac{1}{2}}, |T_2|^{\frac{1}{2}} U_2 |T_2|^{\frac{1}{2}}\right). \quad (1)$$

Observe: commutativity of $\widetilde{W}_{(\alpha,\beta)}$ does not automatically follow from the commutativity of $W_{(\alpha,\beta)}$; actually, the necessary and sufficient condition to preserve commutativity is

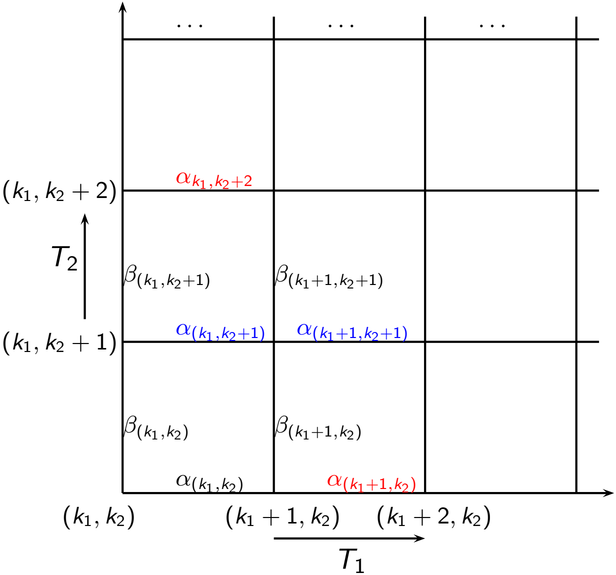
$\alpha_{(k_1,k_2+1)}\alpha_{(k_1+1,k_2+1)} = \alpha_{(k_1+1,k_2)}\alpha_{(k_1,k_2+2)}$ (for all $k_1, k_2 \geq 0$).



. . .
. . .
. . .
(k_1, k_2+2)
T_2
$\alpha_{(k_1,k_2+1)}$
$\alpha_{(k_1+1,k_2+1)}$
(k_1, k_2+1)
(k_1, k_2)
(k_1+1, k_2)
(k_1+2, k_2)
T_1



. . .
$(k_1, k_2 + 2)$
α_{k_1,k_2+2}
T_2
$(k_1, k_2 + 1)$
$\alpha_{(k_1+1,k_2)}$
(k_1, k_2)
$(k_1 + 1, k_2)$
$(k_1 + 2, k_2)$
T_1


$\cdots$
$\cdots$
$\cdots$
α_{k_1,k_2+2}
(k_1, k_2+2)
T_2
$\beta_{(k_1,k_2+1)}$
$\beta_{(k_1+1,k_2+1)}$
$\alpha_{(k_1,k_2+1)}$
$\alpha_{(k_1+1,k_2+1)}$
(k_1, k_2+1)
$\beta_{(k_1,k_2)}$
$\beta_{(k_1+1,k_2)}$
$\alpha_{(k_1,k_2)}$
$\alpha_{(k_1+1,k_2)}$
(k_1, k_2)
(k_1+1, k_2)
(k_1+2, k_2)
T_1

Consider a (joint) polar decomposition of the form

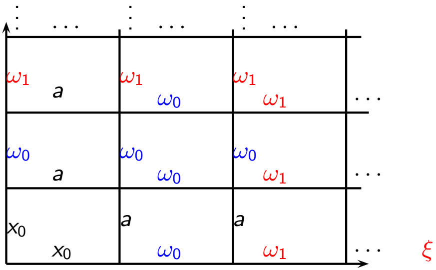
$$(T_1, T_2) \equiv (U_1 P, U_2 P).$$

where $P := \sqrt{T_1^* T_1 + T_2^* T_2}$. Now let

$$\widehat{W}_{(\alpha,\beta)} := \left(\sqrt{P} U_1 \sqrt{P}, \sqrt{P} U_2 \sqrt{P}\right), \qquad (2)$$

One can prove that $U_1^* U_1 + U_2^* U_2$ is a (joint) partial isometry, and that $\widehat{W}_{(\alpha,\beta)}$ is commutative whenever $W_{(\alpha,\beta)}$ is commutative.

The toral Aluthge transform does not preserve hyponormality:



Let ξ be the Berger measure of shift $(\omega_0, \omega_1, \cdots)$ and let $\rho := \int \frac{1}{s}\, d\xi(s)$.

Theorem

Assume $\omega_1^2 \rho < 2$. Then: (i) (T_1, T_2) is subnormal; (ii) $\widetilde{(T_1, T_2)}$ is not hyponormal.

(The condition $\omega_1^2 \rho < 2$ can be satisfied with a 2-atomic measure ξ.)

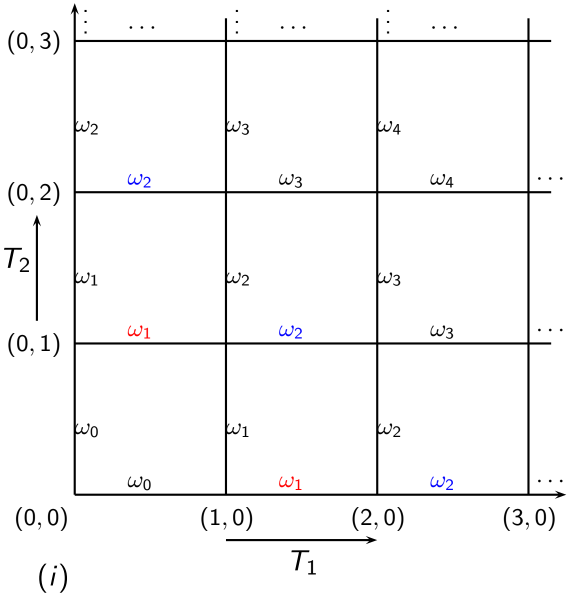
Question

When is hyponormality invariant under the toral Aluthge transform?

Given a 1-variable weighted shift W_ω, let $\Theta\left(W_\omega\right) \equiv W_{(\alpha,\beta)}$ be the 2-variable weighted shift given by

$$\alpha_{(k_1,k_2)} = \beta_{(k_1,k_2)} := \omega_{k_1+k_2} \text{ for } k_1, k_2 \geq 0.$$

We will say that $\Theta\left(W_\omega\right)$ is a 2-variable *embedding* of the unilateral weighted shift W_ω.


(0, 3)
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(0, 0)
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T_2
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ω_0
ω_1
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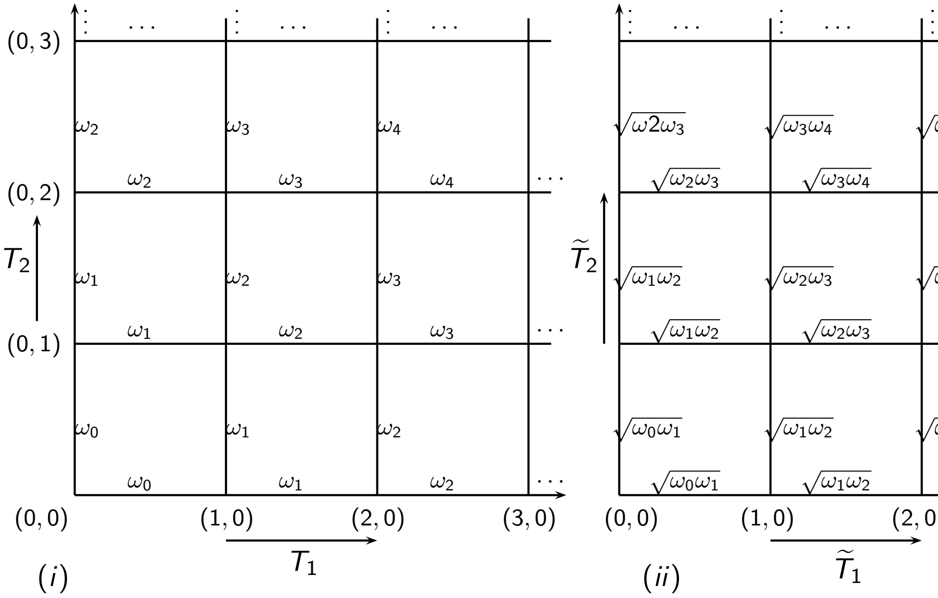
ω_3
ω_4
(i)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T_2
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ω_0
ω_1
ω_2
ω_3
ω_4
(i)
$\sqrt{\omega 2\omega_3}$
$\sqrt{\omega_3\omega_4}$
$\sqrt{\omega_2\omega_3}$
$\sqrt{\omega_1\omega_2}$
$\sqrt{\omega_0\omega_1}$
$\widetilde{T}_2$
$\widetilde{T}_1$
(ii)

Proposition

Consider $\Theta(W_\omega) \equiv (T_1, T_2)$ *given as above. Then for* $k \geq 1$

W_ω *is k-hyponormal if and only if* $\Theta(W_\omega)$ *is k-hyponormal.*

Theorem

Suppose that $\Theta(W_\omega)$ *is hyponormal. Then the toral Aluthge transform* $\widetilde{\Theta(W_\omega)} \equiv \Theta\left(\widetilde{W_\omega}\right)$ *is also hyponormal. The same result holds for the spherical Aluthge transform.*

Spherically quasinormal pairs

Let $\mathbf{T}$ be a commuting pair with joint polar decomposition $T_i \equiv U_i P$, $(i = 1, 2)$. Recall that the spherical Aluthge transform preserves commutativity for 2-variable weighted shifts. We say that $\mathbf{T}$ is spherically quasinormal if $\widehat{\mathbf{T}} = \mathbf{T}$.

Lemma

Assume P injective. Then $\mathbf{T}$ is spherically quasinormal if and only if $T_i P = P T_i$ $(i = 1, 2)$ if and only if $U_i P = P U_i$ $(i = 1, 2)$. As a consequence, if $\mathbf{T}$ is spherically quasinormal then (U_1, U_2) is commuting.

Proposition

(RC-J. Yoon; 2015) A 2-variable weighted shift $\mathbf{T}$ is spherically quasinormal if and only if there exists $C > 0$ such that $\frac{1}{C}\mathbf{T}$ is a spherical isometry, that is, $T_1^* T_1 + T_2^* T_2 = I$.

Definition

A commuting pair **T** is a spherical isometry if $T_1^* T_1 + T_2^* T_2 = I$.

Lemma

(RC-J. Yoon; 2015) $W_{\alpha,\beta}$ *is a spherical isometry if and only if*

$\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$ *for all* $\mathbf{k} \in \mathbb{Z}_+^{\mathbf{2}}$.

Theorem

(Athavale; JOT, 1990) A spherical isometry is always subnormal.

Corollary

(RC-J. Yoon; 2016) A spherically quasinormal 2-variable weighted shift is subnormal.

Corollary

(RC-J. Yoon; 2016) Let $\mathbf{T}$ *be a spherically quasinormal pair, and assume that* P *is injective. Then* $\mathbf{T}$ *is hyponormal.*

Theorem

(A. Athavale - S. Poddar; 2015 and S. Chavan - V. Sholapurkar; 2013) Let $\mathbf{T}$ *be a spherically quasinormal pair. Then* $\mathbf{T}$ *is subnormal.*

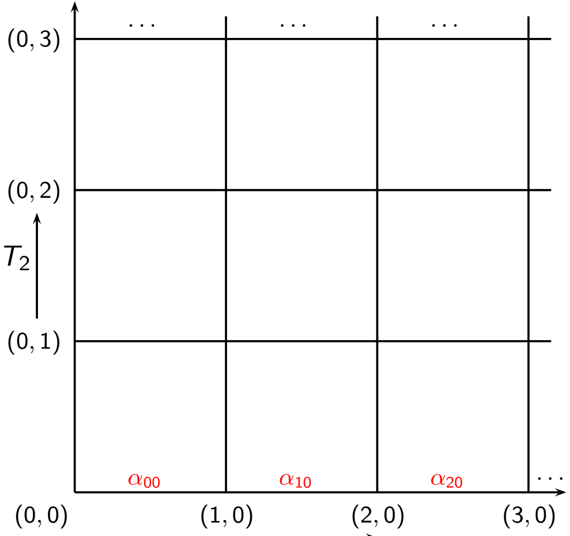
Theorem

(V. Müller - M. Ptak; 1999) Spherical isometries are hyporeflexive.

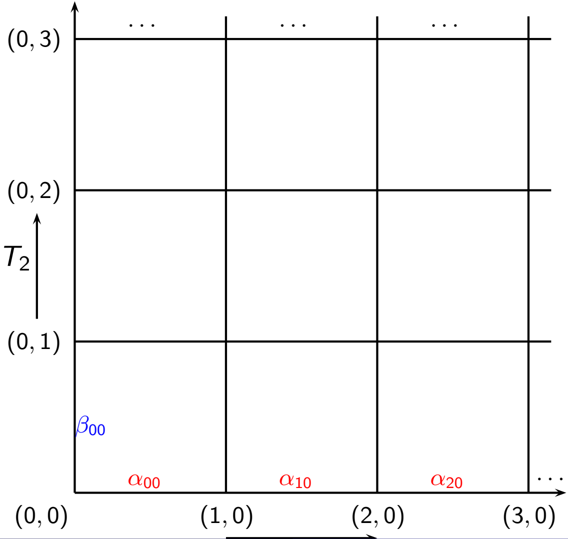
Theorem

(J. Eschmeier - M. Putinar; 2000) For every $n \geq 3$ *there exists a non-normal spherical isometry* $\mathbf{T}$ *such that the polynomially convex hull of* $\sigma_T(\mathbf{T})$ *is contained in the unit sphere.*

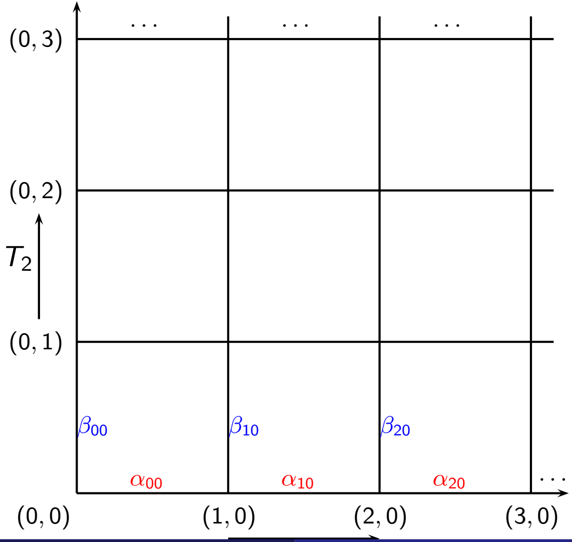
Construction of spher. isom. 2-var. w.s.



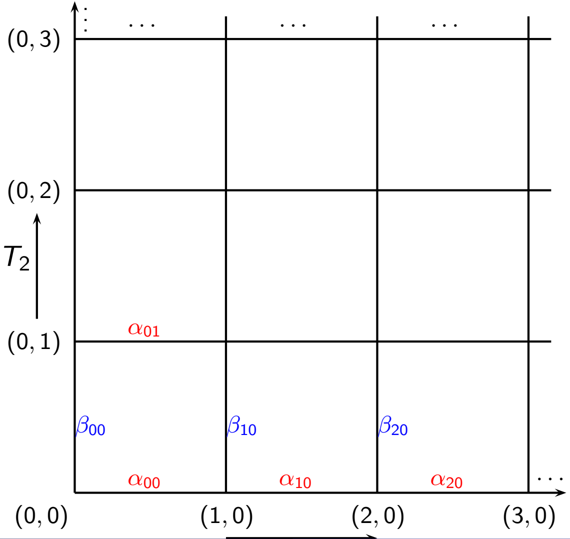
Construction of spher. isom. 2-var. w.s.



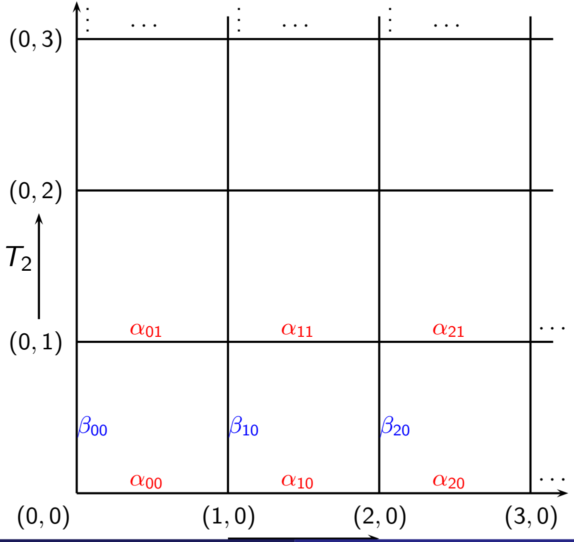
Construction of spher. isom. 2-var. w.s.



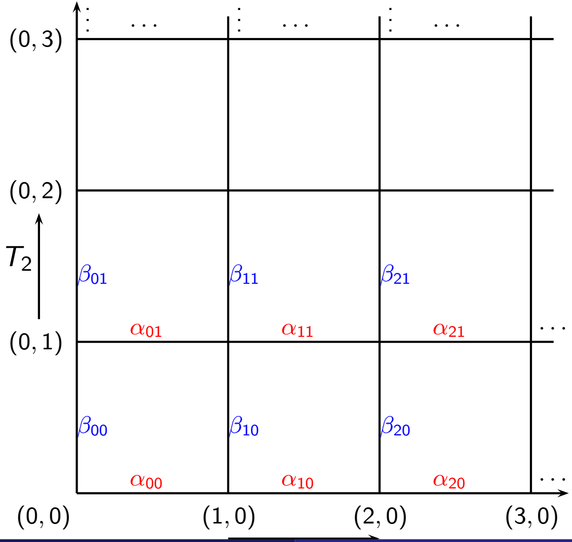
Construction of spher. isom. 2-var. w.s.



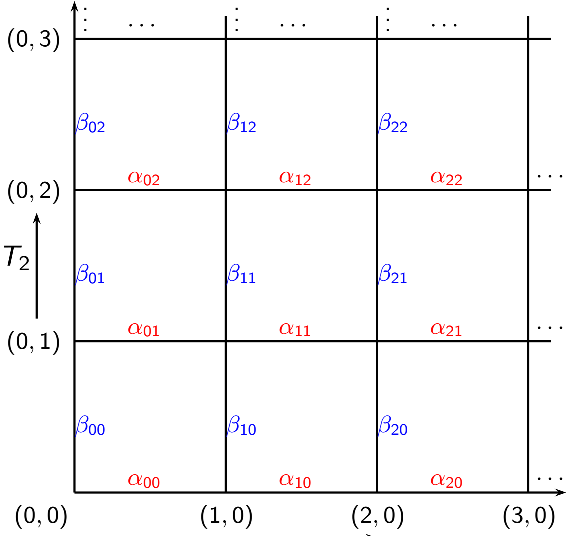
Construction of spher. isom. 2-var. w.s.



CONSTRUCTION OF SPHER. ISOM. 2-VAR. W.S.



Construction of spher. isom. 2-var. w.s.

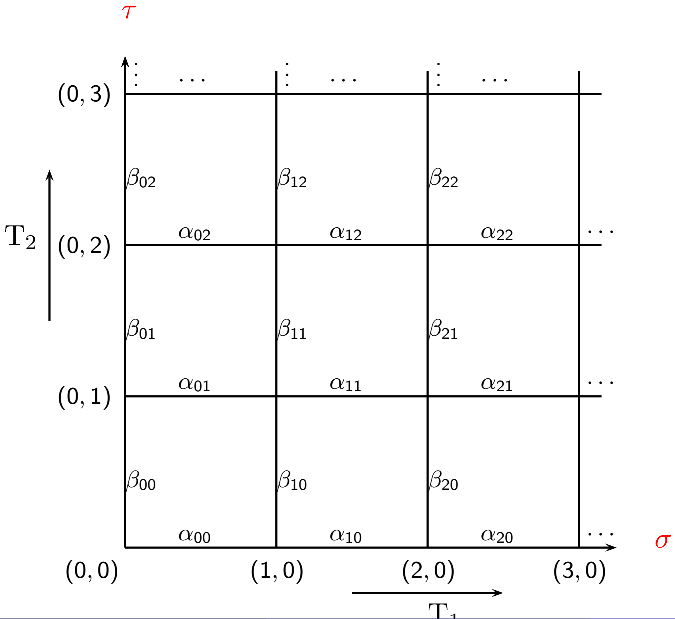


Recall that a unilateral weighted shift is *recursively generated* if the sequence of moments satisfy a linear relation

$$\gamma_{n+k} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} + \cdots + \varphi_{k-1} \gamma_{n+k-1} \quad (k \geq 1,\ n \geq 0).$$

THEOREM

(RC-Fialkow; IEOT, 1993) A subnormal weighted shift is recursively generated if and only if its Berger measure is finitely atomic.


τ
(0, 3)
T₂
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
σ
T₁
β02
β12
β22
α02
α12
α22
β01
β11
β21
α01
α11
α21
β00
β10
β20
α00
α10
α20

Theorem

(RC-Yoon; 2016) Let $W_{(\alpha,\beta)}$ be a spherical isometry, and assume that the zero-th row is subnormal with finitely atomic Berger measure σ.

(i) Each horizontal row is recursively generated, and its moments satisfy the same linear relation as the zero-th row.

(ii) Each vertical column is recursively generated, and its moments satisfy the linear relation obtained from (ii) which appropriately reflects the condition $\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$ ($\mathbf{k} \in \mathbb{Z}_+^2$).

(iii) The Berger measure of $W_{(\alpha,\beta)}$ is finitely atomic, with support contained in the Cartesian product of σ and τ, where τ is the Berger measure of the zero-th column of $W_{(\alpha,\beta)}$.

Theorem (cont.)

(iv) If $\Lambda^{(0)}$ *and* ${}^{(0)}\Lambda$ *are the Riesz functional of the zero-th row and zero-th column of* $W_{(\alpha,\beta)}$*, resp., then*

$$ {}^{(0)}\Lambda(p(t)) = \Lambda^{(0)}(p(1-t)) $$

for every polynomial p. As a result,

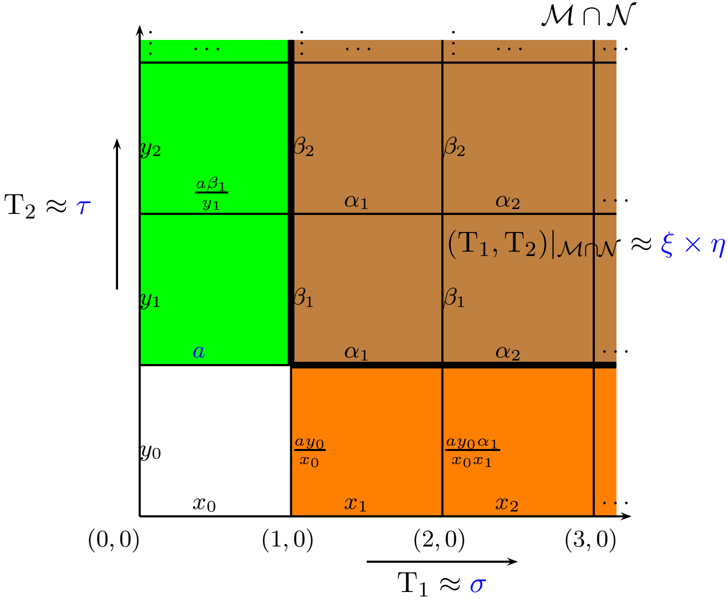
$$ \mathit{supp}\ \tau = 1 - \mathit{supp}\ \sigma. $$

Spectral Properties

We aim to calculate the spectral picture of $W_{\alpha,\beta} \equiv (T_1, T_2)$ and of its toral and spherical Aluthge transforms. This entails finding the Taylor spectrum, the Taylor essential spectrum, and the Fredholm index. We focus on the case when the pair has a core of tensor form.

The class of pairs with core of tensor form is large, and has been used to exhibit structural and spectral behavior of 2-variable weighted shifts, not found in the classical theory of unilateral weighted shifts.

Special Case (tensor core): Given $\xi, \eta, \sigma, \tau, a$, study sp. picture of $W_{\alpha,\beta}$.



Spectral Properties, cont.

Lemma

(i) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $A_i \in \mathcal{B}(\mathcal{H}_1)$, $C_i \in \mathcal{B}(\mathcal{H}_2)$ and $B_i \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), (i = 1, \cdots, n)$ be such that

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ 0 & \mathbf{C} \end{pmatrix} := \left(\begin{pmatrix} A_1 & 0 \\ B_1 & C_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ B_n & C_n \end{pmatrix} \right)$$

is commuting. Assume that $\mathbf{A}$ and $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ 0 & \mathbf{C} \end{pmatrix}$ are Taylor invertible. Then, $\mathbf{C}$ is Taylor invertible. Furthermore, if $\mathbf{A}$ and $\mathbf{C}$ are Taylor invertible, then $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ 0 & \mathbf{C} \end{pmatrix}$ is Taylor invertible.

Lemma (cont.)

(ii) For **A** *and* **B** *two commuting n-tuples of bounded operators on Hilbert space, we have:*

$$\sigma_T(\mathbf{A} \otimes I, I \otimes \mathbf{B}) = \sigma_T(\mathbf{A}) \times \sigma_T(\mathbf{B})$$

and

$$\sigma_{Te}(\mathbf{A} \otimes I, I \otimes \mathbf{B}) = \sigma_{Te}(\mathbf{A}) \times \sigma_T(\mathbf{B}) \cup \sigma_T(\mathbf{A}) \times \sigma_{Te}(\mathbf{B}).$$

To use the Lemma, we split $\ell^2(\mathbb{Z}_+^2)$ as the orthogonal direct sum of the 0-th row and the rest. For the Taylor essential spectrum, we use the fact that compressions of $W_{(\alpha,\beta)}$ and $\widetilde{W_{(\alpha,\beta)}}$ differ by a compact perturbation, when $W_{(\alpha,\beta)}$ is hyponormal.

Theorem

Consider a hyponormal 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$. *Then*

$$\sigma_T\left(W_{(\alpha,\beta)}\right) = \left(\|W_\omega\| \cdot \overline{\mathbb{D}} \times \|W_\tau\| \cdot \overline{\mathbb{D}}\right) \text{ and}$$

$$\sigma_{Te}\left(W_{(\alpha,\beta)}\right) = \left(\|W_\omega\| \cdot \mathbb{T} \times \|W_\tau\| \cdot \overline{\mathbb{D}}\right) \cup \left(\|W_\omega\| \cdot \overline{\mathbb{D}} \times \|W_\tau\| \cdot \mathbb{T}\right). \tag{3}$$

Here $\overline{\mathbb{D}}$ *denotes the closure of the open unit disk* $\mathbb{D}$ *and* $\mathbb{T}$ *the unit circle.*

We next consider the Taylor essential spectrum $\sigma_{Te}(T_1, T_2)$ of $W_{(\alpha,\beta)} \equiv (T_1, T_2)$. To prove the result for σ_{Te}, observe that $W_{\omega^{(2)}}$ is a compact perturbation of $W_{\omega^{(1)}}$ and $W_{\omega^{(0)}}$. $\frac{\omega_0 y_0}{x_0 x_1} I$ and $\tau_0 I$ are also compact perturbations of I_1 and I_2, respectively.

Theorem

Consider a hyponormal 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$. Then, we have

$$\sigma_T\left(\widetilde{W}_{(\alpha,\beta)}\right) = \left(\|W_\omega\| \cdot \overline{\mathbb{D}} \times \|W_\tau\| \cdot \overline{\mathbb{D}}\right) \text{ and}$$

$$\sigma_{Te}\left(\widetilde{W}_{(\alpha,\beta)}\right) = \left(\|W_\omega\| \cdot \mathbb{T} \times \|W_\tau\| \cdot \overline{\mathbb{D}}\right) \cup \left(\|W_\omega\| \cdot \overline{\mathbb{D}} \times \|W_\tau\| \cdot \mathbb{T}\right). \quad (4)$$

A similar result holds for the spherical Aluthge transform.

Consider now the Drury-Arveson 2-shift, denoted by DA. As usual, $\widetilde{DA}$ is the toral Aluthge transform of DA and $\widehat{DA}$ is the spherical Aluthge transform of DA. Also, it is well known that DA is essentially normal.

Theorem

(i) $\widetilde{DA}$ *is a compact perturbation of* DA.

(ii) $\widehat{DA}$ *is a compact perturbation of* DA.

Corollary

DA, $\widetilde{DA}$ *and* $\widehat{DA}$ *all share the same Taylor spectral picture; that is,*

$$(i)\ \sigma_T(DA) = \bar{\mathbb{B}}^2, \quad (ii)\ \sigma_{Te}(DA) = \partial\mathbb{B}^2, \quad \text{and}$$

$$(iii)\ index\ DA = index\ \widetilde{DA} = index\ \widehat{DA}.$$

(Here $\mathbb{B}^2$ *denotes the open unit ball in* $\mathbb{C}^2$*, and* $\partial\mathbb{B}^2$ *its topological*

Thank you!