Limits of iterates of spherical Aluthge transforms

(joint work with Chafiq Benhida and Jasang Yoon)

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2. Aluthge Transforms
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Hyponormality and Subnormality

$\mathcal{L}(\mathcal{H})$: algebra of operators on a Hilbert space $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- quasinormal if T commutes with T^*T
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$

normal $\Rightarrow$ quasinormal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal

For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.

- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- (Bram-Halmos):

T subnormal $\Leftrightarrow$ T is k-hyponormal for all $k \geq 1$.

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift
- In general, $W_\alpha = U_+ D_\alpha$ (polar decomposition)
- $\|W_\alpha\| = \sup_k \alpha_k$

  $W_\alpha^n e_k = \alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1} e_{k+n}$, so

$$W_\alpha^n \cong \bigoplus_{i=0}^{n-1} W_{\beta(i)},$$

Weighted Shifts and Berger's Theorem

Given a bounded sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, \ldots$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0).$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)

Berger Measures

- (Berger; Gellar-Wallen) W_α is subnormal if and only if there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

  $$\gamma_k = \int t^k \, d\xi(t) \text{ (all } k \geq 0).$$

  ξ is the Berger measure of W_α.
- For $0 < a < 1$ we let $S_a :=$ shift $(a, 1, 1, \ldots)$.
- The Berger measure of U_+ is δ_1.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

(RC, 1990) W_α is k-hyponormal if and only if the following Hankel moment matrices are positive for $m = 0, 1, 2, \dots$:

$$\begin{pmatrix} \gamma_m & \gamma_{m+1} & \gamma_{m+2} & \cdots & \gamma_{m+k} \\ \gamma_{m+1} & \gamma_{m+2} & & \cdots & \gamma_{m+k+1} \\ \gamma_{m+2} & \cdots & & \cdots & \gamma_{m+k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_{m+k} & \gamma_{m+k+1} & & \cdots & \gamma_{m+2k} \end{pmatrix} \geq 0.$$

(Thus, an operator matrix condition is replaced by a scalar matrix condition.)

Aluthge Transform

Let T be a Hilbert space operator, let $P := |T|$ be its positive part, and let $T = VP$ denote the canonical polar decomposition of T, with V a partial isometry and $\ker V = \ker T = \ker P$.

We define the Aluthge transform of T as

$$A(T) \equiv \hat{T} := \sqrt{P} V \sqrt{P}.$$

The iterates are

$$A^{(n+1)}(T) := A(A^{(n)}(T)) \; (n \geq 1).$$

The Aluthge transform has been extensively studied, in terms of algebraic, structural and spectral properties.

For instance,

- (i) $T = \hat{T} \Leftrightarrow T$ is quasinormal;
- (ii) (Aluthge, 1990) If $0 < p < \frac{1}{2}$ and T is p-hyponormal, then $\hat{T}$ is $(p + \frac{1}{2})$-hyponormal;
- (iii) (Jung, Ko & Pearcy, 2000) T and $\hat{T}$ share many spectral properties.
- (iv) (Jung, Ko & Pearcy, 2000) If $\hat{T}$ has a n.i.s., then T has a n.i.s.
- (v) (Kim-Ko, 2005; Kimura, 2004) T has property (β) if and only if $\hat{T}$ has property (β); and
- (vi) (Ando, 2005) $\left\|(T-\lambda)^{-1}\right\| \geq \left\|(\hat{T}-\lambda)^{-1}\right\|$ $(\lambda \notin \sigma(T))$.

G. Exner (IWOTA 2006 Lecture): subnormality is not preserved under the Aluthge transform.

S.H. Lee, W.Y. Lee and J. Yoon (2012) For $k \geq 2$, the Aluthge transform, when acting on weighted shifts, need not preserve k-hyponormality.

The Aluthge transform of a weighted shift is again a weighted shift. Concretely, the weights of $\widehat{W_\alpha}$ are

$$\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_2\alpha_3}, \sqrt{\alpha_3\alpha_4}, \cdots .$$

$$W_{\sqrt{\alpha}} := \text{shift } (\sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \cdots).$$

Then $\widehat{W_\alpha}$ is the Schur product of $W_{\sqrt{\alpha}}$ and its restriction to the subspace $\bigvee\{e_1, e_2, \cdots\}$. Thus, a sufficient condition for the subnormality of $\widehat{W_\alpha}$ is the subnormality of $W_{\sqrt{\alpha}}$.

Iterates of the Aluthge Transform

Observe that if $W_\alpha = \text{shift}(\alpha_0, \alpha_1, \cdots)$, then

$$A(W_\alpha) = \text{shift}(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots),$$

$$A^{(2)}(W_\alpha) = \text{shift}(\sqrt{\sqrt{\alpha_0\alpha_1}\sqrt{\alpha_1\alpha_2}}, \sqrt{\sqrt{\alpha_1\alpha_2}\sqrt{\alpha_2\alpha_3}}, \cdots),$$

$$A^{(3)}(W_\alpha) = \text{shift}((\alpha_0\alpha_1^3\alpha_2^3\alpha_3)^{\frac{1}{8}}, (\alpha_1\alpha_2^3\alpha_3^3\alpha_4)^{\frac{1}{8}}, \cdots)$$

$$A^{(4)}(W_\alpha) = \text{shift}((\alpha_0\alpha_1^4\alpha_2^6\alpha_3^4\alpha_4)^{\frac{1}{16}}, (\alpha_1\alpha_2^4\alpha_3^6\alpha_4^4\alpha_5)^{\frac{1}{16}}, \cdots).$$

Iterates of the Aluthge Transform

Let W_ω be a unilateral weighted shift, and let $\omega^{(n)}$ be the weight sequence of $A^{(n)}(W_\omega)$. Then

$$\omega_k^{(n+1)} = \sqrt{\omega_k^{(n)} \omega_{k+1}^{(n)}}.$$

Therefore,

$$\omega_k^{(n)} = \Big(\prod_{j=0}^{n} \omega_{k+j}^{\binom{n}{j}}\Big)^{\frac{1}{2^n}}.$$

- I.B. Jung, E. Ko and C. Pearcy conjectured in 2000 that for every operator $T \in B(\mathcal{H})$ the sequence $A^{(n)}(T)$ converges in norm to a quasinormal operator.
- J. Antezana, E. Pujals and D. Stojanoff (2011) proved the conjecture true for $\dim \mathcal{H} < \infty$.
- D. Thompson (2003) found an example of an operator for which the sequence converges to 0 in the SOT, but it does not converge in norm.
- M. Yanagida (2001) found an example of a unilateral weighted shift for which the sequence of iterates does not converge in the WOT.
- M. Chō, I.B. Jung and W.Y. Lee (2005) proved that for any $0 < a < b$ there exists a unilateral weighted shift W_ω such that the sequence $\{\omega_0^{(n)}\}_{n \geq 0}$ clusters at both a and b.

Proposition

(K. Rion, 2016) WOT and SOT convergence of $T_{\omega^{(n)}}$ is equivalent to the pointwise convergence of $\{\omega^{(n)}\}_n$.

Theorem

(K. Rion, 2016) Assume ω is bounded below. Then the set $\mathcal{S}$ of SOT subsequential limits of $\{A^{(n)}(T_\omega)\}$ is nonempty. Moreover, S is a closed interval of quasinormal shifts; that is, $S = [a, b]U_+$.

Multivariable Weighted Shifts

$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

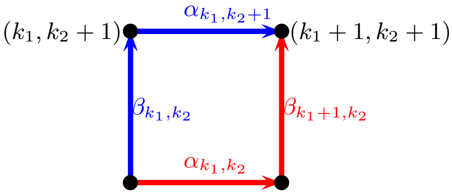
$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

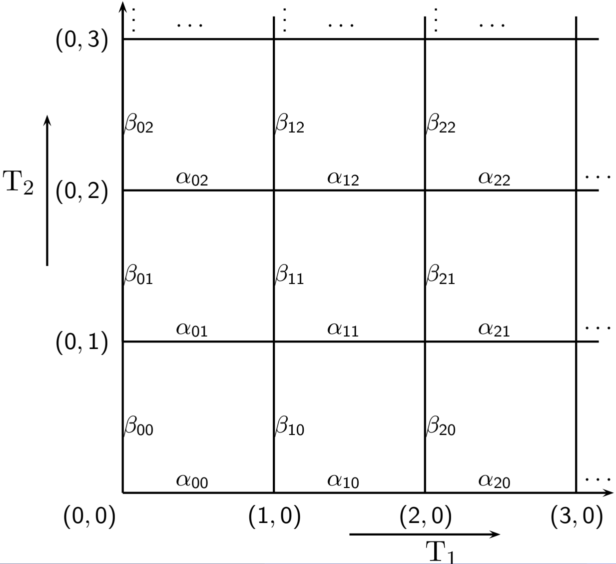
We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad \text{(all } \mathbf{k}\text{).}$$





(0,3)
(0,2)
(0,1)
(0,0)
(1,0)
(2,0)
(3,0)
T2
T1
β02 β12 β22
α02 α12 α22
β01 β11 β21
α01 α11 α21
β00 β10 β20
α00 α10 α20

To detect hyponormality, there is a simple criterion:

Theorem

(RC, 1988) (Six-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ *be a 2-variable weighted shift, with weight sequences* α *and* β. *Then*

$$\mathbf{T} \text{ is hyponormal } \Leftrightarrow \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geq 0$$

(all $k \in \mathbb{Z}^2_+$*).*

We now recall the notion of moment of order $\mathbf{k}$ for a commuting pair (α, β). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is $\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta)$

$$:= \begin{cases} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

By commutativity, $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).

- (Jewell-Lubin)

$$W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} := \prod_{i=0}^{k_1-1} \alpha^2_{(i,0)} \cdot \prod_{j=0}^{k_2-1} \beta^2_{(k_1-1,j)}$$

$$= \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \ \ (\text{all } \mathbf{k} \geq \mathbf{0}).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

Following A. Athavale-S. Poddar and J. Gleason, we say that **T** is **(jointly) quasinormal** if T_i commutes with $T_j^* T_j$ for all $i, j = 1, 2$; and **spherically quasinormal** if T_i commutes with

$$P := T_1^* T_1 + T_2^* T_2$$

for $i = 1, 2$. One has

$$\begin{aligned} \text{normal} \quad &\Longrightarrow \quad \text{(jointly) quasinormal} \Longrightarrow \text{spherically quasinormal} \\ &\Longrightarrow \quad \text{subnormal} \Longrightarrow k\text{-hyponormal.} \Longrightarrow \text{hyponormal.} \qquad (1) \end{aligned}$$

On the other hand, results of RC-S.H. Lee-J. Yoon and J. Gleason show that the reverse implications in (1) do not necessarily hold.

Spherical Aluthge Transform

Consider a (joint) polar decomposition of the form

$$(T_1, T_2) \equiv (V_1 P, V_2 P),$$

or equivalently,

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P,$$

as operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$. Moreover, this is the unique canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$. It follows that $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $(\ker P)^{\perp}$ onto $\overline{\operatorname{Ran} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}$. where $P := \sqrt{T_1^* T_1 + T_2^* T_2}$.

Now let

$$\widehat{\mathbf{T}} := \left(\sqrt{P}V_1\sqrt{P}, \sqrt{P}V_2\sqrt{P}\right). \qquad (2)$$

One can prove that $V_1^* V_1 + V_2^* V_2$ is a (joint) partial isometry, and that $\widehat{\mathbf{T}}$ is commutative whenever $\mathbf{T}$ is commutative.

Spherically quasinormal pairs

Recall that the spherical Aluthge transform preserves commutativity for 2-variable weighted shifts. Also, we say that $\mathbf{T}$ is spherically quasinormal if T_i commutes with P, for $i = 1, 2$. Equivalently, $\mathbf{T}$ is spherically quasinormal if $\widehat{\mathbf{T}} = \mathbf{T}$.

Lemma

Assume P injective. Then $\mathbf{T}$ is spherically quasinormal if and only if $T_i P = P T_i$ $(i = 1, 2)$ if and only if $V_i P = P V_i$ $(i = 1, 2)$. As a consequence, if $\mathbf{T}$ is spherically quasinormal then (V_1, V_2) is commuting.

Proposition

(RC-J. Yoon; 2015) A 2-variable weighted shift $\mathbf{T}$ is spherically quasinormal if and only if there exists $C > 0$ such that $\frac{1}{C}\mathbf{T}$ is a spherical isometry, that is, $T_1^* T_1 + T_2^* T_2 = I$.

Definition

A commuting pair $\mathbf{T}$ is a spherical isometry if $T_1^* T_1 + T_2^* T_2 = I$.

Lemma

(RC-J. Yoon; 2015) $W_{\alpha,\beta}$ *is a spherical isometry if and only if*

$\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$ *for all* $\mathbf{k} \in \mathbb{Z}_+^{\mathbf{2}}$.

Theorem

(Athavale; JOT, 1990) A spherical isometry is always subnormal.

Corollary

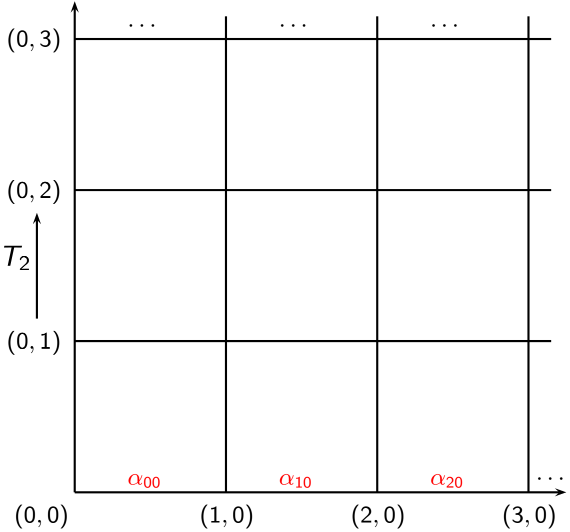
(RC-J. Yoon; 2016) A spherically quasinormal 2-variable weighted shift is subnormal.

Theorem

(C. Benhida - RC, 2018) Let $T \equiv (T_1, \cdots, T_n)$ be a commuting n-tuple of Hilbert space operators, and let $\widehat{T}$ be its spherical Aluthge transform. Then

$$\sigma_T(\widehat{T}) = \sigma_T(T).$$

Construction of spher. isom. 2-var. w.s.



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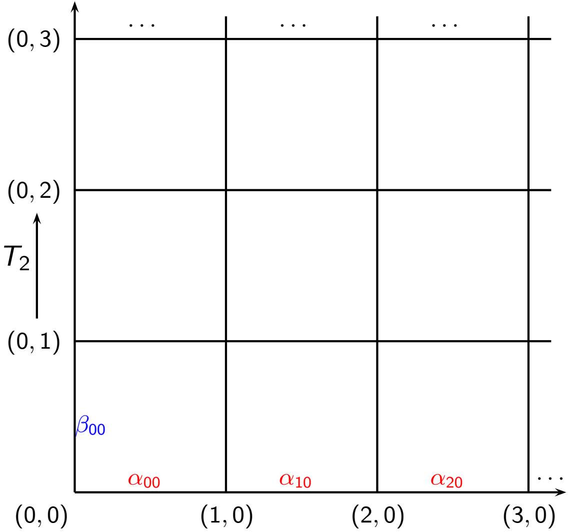

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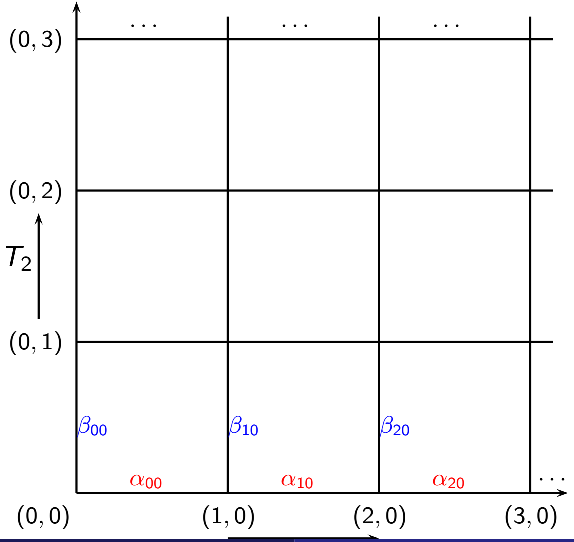
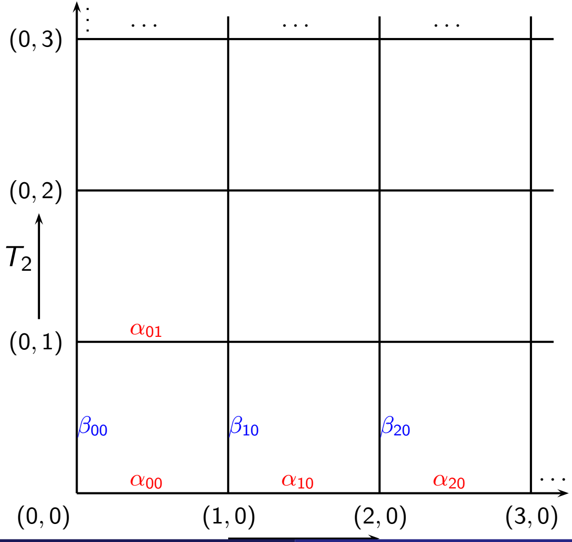
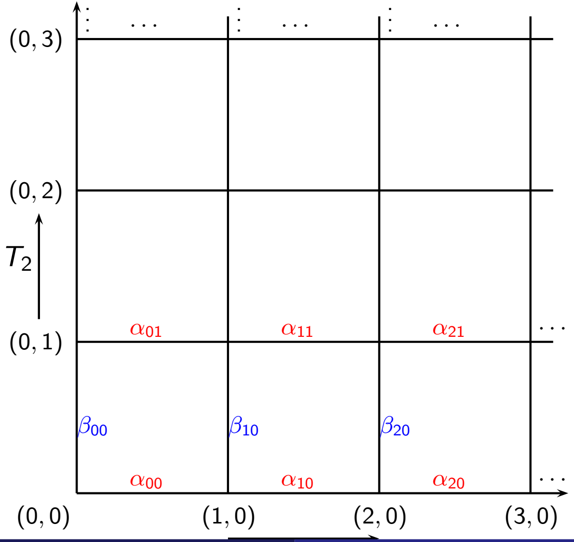


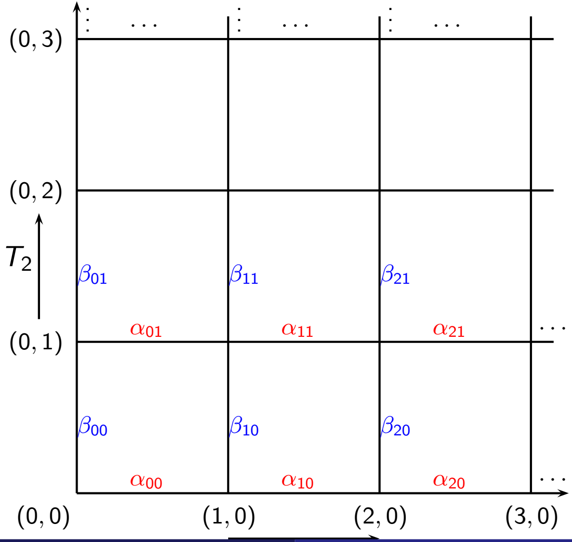
Construction of spher. isom. 2-var. w.s.



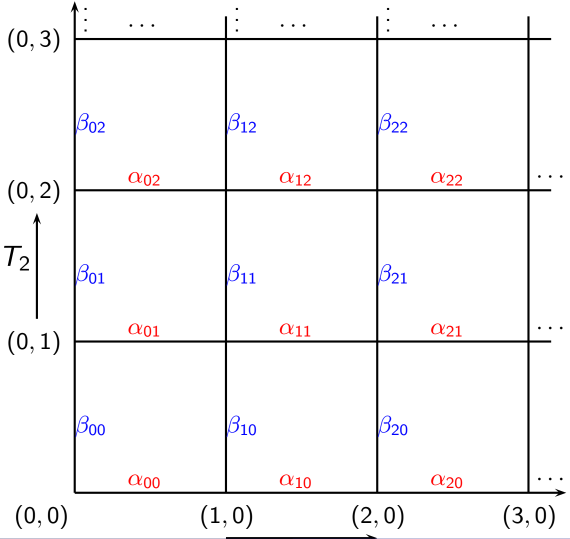
Construction of spher. isom. 2-var. w.s.



Construction of spher. isom. 2-var. w.s.



Construction of spher. isom. 2-var. w.s.



The Above Construction May Stall if $\{\alpha_{(k,0)}\}_{k\geq 0}$ Is Not Strictly Increasing.

Proposition

Let

$$\alpha_{(0,0)} := \sqrt{p}, \;\; \alpha_{(1,0)} := \sqrt{q}, \;\; \alpha_{(2,0)} := \sqrt{r} \;\; \textit{and } \alpha_{(3,0)} := \sqrt{r},$$

and assume that $0 < p < q < r < 1$. *Then the algorithm described in this section fails at some stage. As a consequence, there does not exist a spherical isometry interpolating these initial data.*

Recursively Generated Weighted Shifts

Recall that a unilateral weighted shift is *recursively generated* if the sequence of moments satisfy a linear relation

$$\gamma_{n+k} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} + \cdots + \varphi_{k-1} \gamma_{n+k-1} \quad (k \geq 1,\ n \geq 0).$$

Theorem

(RC-Fialkow; IEOT, 1993) A subnormal weighted shift W_α is recursively generated if and only if its Berger measure is finitely atomic.

Theorem

(RC-Yoon; 2016) Let $W_{(\alpha,\beta)}$ be a spherical isometry, and assume that the zero-th row is subnormal with finitely atomic Berger measure σ.

(i) Each horizontal row is recursively generated, and its moments satisfy the same linear relation as the zero-th row.

(ii) Each vertical column is recursively generated, and its moments satisfy the linear relation obtained from (ii) which appropriately reflects the condition $\alpha_{\mathbf{k}}^2 + \beta_{\mathbf{k}}^2 = 1$ ($\mathbf{k} \in \mathbb{Z}_+^2$).

(iii) The Berger measure of $W_{(\alpha,\beta)}$ is finitely atomic, with support contained in the Cartesian product of σ and τ, where τ is the Berger measure of the zero-th column of $W_{(\alpha,\beta)}$.

Theorem (cont.)

(iv) If $\Lambda^{(0)}$ *and* ${}^{(0)}\Lambda$ *are the Riesz functional of the zero-th row and zero-th column of* $W_{(\alpha,\beta)}$*, resp., then*

$$ {}^{(0)}\Lambda(p(t)) = \Lambda^{(0)}(p(1-t)) $$

for every polynomial p. As a result,

$$ supp\ \tau = 1 - supp\ \sigma. $$

Theorem

For spherically quasinormal 2-variable weighted shifts, the property of being recursively generated **transfers** *from the 0-th row in the weight diagram to the 0-th column.*

Theorem

Let $W_{(\alpha,\beta)}$ be a spherically quasinormal 2-variable weighted shift, with constant $c > 0$, and assume that the unilateral weighted shift W_0 (which corresponds to the 0-th row in the weight diagram of $W_{(\alpha,\beta)}$) is recursively generated. Then the unilateral weighted shift V_0 ((which corresponds to the 0-th column) is also recursively generated.

Theorem

Let $W_{(\alpha,\beta)}$ *be a spherically quasinormal* 2*-variable weighted shift, with constant* $c > 0$*, and assume that the unilateral weighted shift* W_0 *(which corresponds to the* 0*-th row in the weight diagram of* $W_{(\alpha,\beta)}$*) is recursively generated. Let* σ *be the Berger measure of* W_0*, and let* μ *be the Berger measure of* $W_{(\alpha,\beta)}$*. Then*

(i) $\operatorname{supp}\mu \subseteq \operatorname{supp}\sigma \times (c - \operatorname{supp}\sigma)$*; and*

(ii) μ *is finitely atomic.*

(RC - C. Benhida; 2018–2019) For 2-variable weighted shifts, recall that if $(S, T) \equiv W_{\alpha,\beta}$ is a 2-variable weighted shift with weight sequences α and β, then the spherical Aluthge transform is given by

$$\Delta(S)e_{\mathbf{k}} = \alpha_{\mathbf{k}} \frac{(\alpha^2_{\mathbf{k}+\epsilon_1} + \beta^2_{\mathbf{k}+\epsilon_1})^{1/4}}{(\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}})^{1/4}} e_{\mathbf{k}+\epsilon_1}$$

and

$$\Delta(T)e_{\mathbf{k}} = \beta_{\mathbf{k}} \frac{(\alpha^2_{\mathbf{k}+\epsilon_2} + \beta^2_{\mathbf{k}+\epsilon_2})^{1/4}}{(\alpha^2_{\mathbf{k}} + \beta^2_{\mathbf{k}})^{1/4}} e_{\mathbf{k}+\epsilon_2}$$

for all $\mathbf{k} \in \mathbb{Z}^2_+$.

Define a recursive sequence of 2-variable weighted shifts by

$$\begin{aligned} S_{n+1}(i,j) &:= \Delta(S_n, T_n)_1(i,j) \\ T_{n+1}(i,j) &:= \Delta(S_n, T_n)_2(i,j). \end{aligned}$$

It follows that

$$S_{n+1}e_{(0,0)} = S_n(0,0)\frac{(S_n(1,0)^2 + T_n(1,0)^2)^{1/4}}{(S_n(0,0)^2 + T_n(0,0)^2)^{1/4}}e_{(1,0)}$$

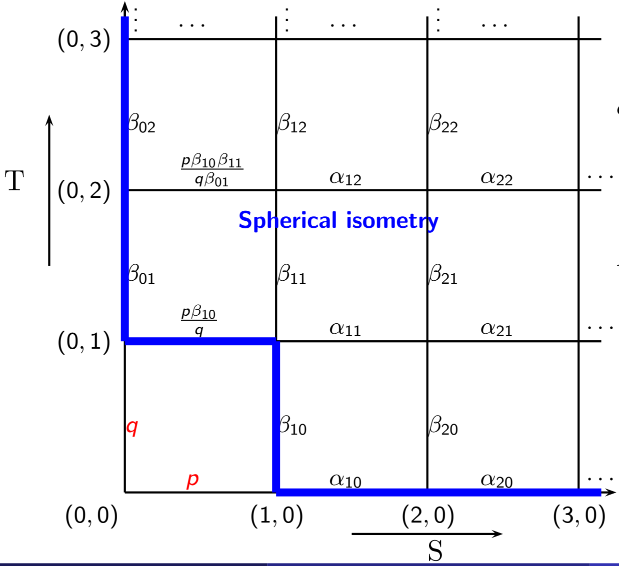
and

$$T_{n+1}e_{(0,0)} = T_n(0,0)\frac{(S_n(0,1)^2 + T_n(0,1)^2)^{1/4}}{(S_n(0,0)^2 + T_n(0,0)^2)^{1/4}}e_{(0,1)}.$$

As in the 1 variable case (Rion 2016), one observes that the asymptotic behavior anywhere is reflected by the asymptotic behavior at the origin $(0,0)$; that is, WLOG we can focus attention on the recursively defined sequences

$$S_{n+1}e_{(0,0)} = S_n(0,0)\frac{(S_n(1,0)^2 + T_n(1,0)^2)^{1/4}}{(S_n(0,0)^2 + T_n(0,0)^2)^{1/4}}e_{(1,0)}$$

$$T_{n+1}e_{(0,0)} = T_n(0,0)\frac{(S_n(0,1)^2 + T_n(0,1)^2)^{1/4}}{(S_n(0,0)^2 + T_n(0,0)^2)^{1/4}}e_{(0,1)}.$$



$$\alpha_{i,j}^2 + \beta_{i,j}^2 = 1 \quad (i,j) \neq (0,0)$$

Also, $\left(\frac{p\beta_{10}}{q}\right)^2 + \beta_{01}^2 = 1, \cdots$

It is not hard to see that $\Delta(S,T)$ has the same structure, and the same is true of $\Delta^2(S,T)$, $\Delta^3(S,T)$, ...

Thus, for this special case, the asympotic behavior of the spherical Aluthge iterates is controlled by the pair

$$\begin{cases} p_n & := & S_n(0,0) \\ q_n & := & T_n(0,0). \end{cases}$$

Observe that

$$\begin{cases} p_1 & = & p(p^2+q^2)^{-1/4} \\ q_1 & = & q(p^2+q^2)^{-1/4} \end{cases}$$

$$\begin{cases} p_2 & = & p(p^2+q^2)^{-3/8} \\ q_2 & = & q(p^2+q^2)^{-3/8} \end{cases}$$

and, in general, for $n > 2$,

$$\begin{cases} p_n &= p(p^2+q^2)^{-\sum_{k=2}^{n+1}(\frac{1}{2})^k} \\ q_n &= q(p^2+q^2)^{-\sum_{k=2}^{n+1}(\frac{1}{2})^k} \end{cases}$$

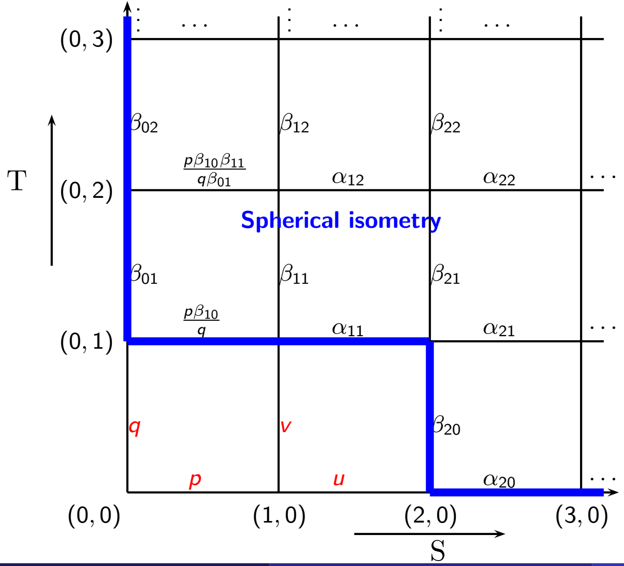
From this it readily follows that, in the limit, we obtain

$$\begin{cases} p_\infty &= p(p^2+q^2)^{-\frac{1}{2}} \\ q_\infty &= q(p^2+q^2)^{-\frac{1}{2}} \end{cases}$$

Since

$$p_\infty^2 + q_\infty^2 = 1$$

we see that the sequence of iterates does converge to a spherical isometry.



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Spherical isometry
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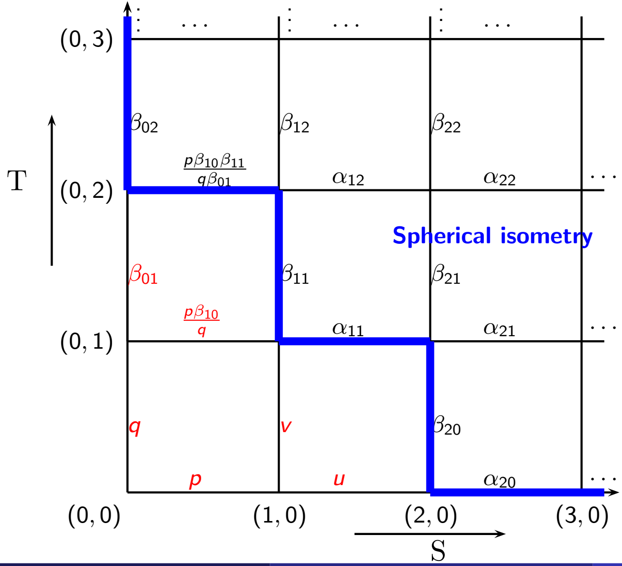
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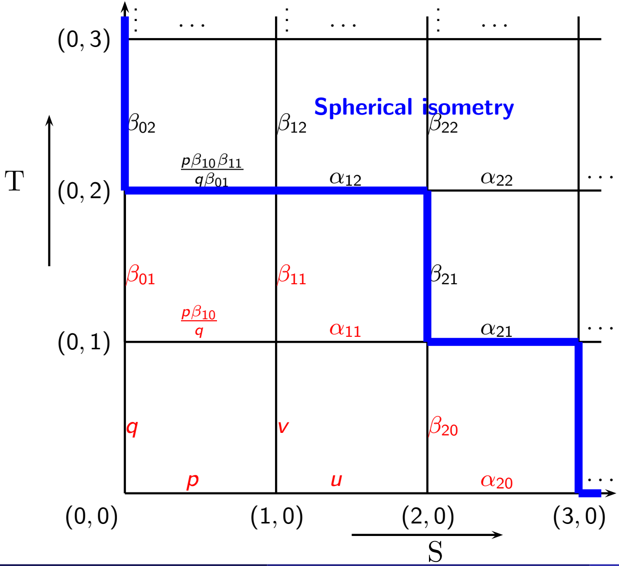
Spherical isometry
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Question

Let (T_1, T_2) be a commuting pair of operators on a finite dimensional Hilbert space. Does the sequence of iterates $\Delta^n(T_1, T_2)$ converge in the norm?

Merci beaucoup
pour votre attention!

The Above Construction May Stall if $\{\alpha_{(k,0)}\}_{k\geq 0}$ Is Not Strictly Increasing.

Proposition

Let

$$\alpha_{(0,0)} := \sqrt{p}, \;\; \alpha_{(1,0)} := \sqrt{q}, \;\; \alpha_{(2,0)} := \sqrt{r} \;\text{ and }\; \alpha_{(3,0)} := \sqrt{r},$$

and assume that $0 < p < q < r < 1$*. Then the algorithm described in this section fails at some stage. As a consequence, there does not exist a spherical isometry interpolating these initial data.*

Sketch of Proof. Assume that the algorithm works, and let

$$\beta_{(0,0)} := \sqrt{1-\alpha_{(0,0)}^2} = \sqrt{1-p},$$

$$\beta_{(1,0)} := \sqrt{1-\alpha_{(1,0)}^2} = \sqrt{1-q},$$

$$\beta_{(2,0)} := \sqrt{1-\alpha_{(2,0)}^2} = \sqrt{1-r},$$

and

$$\beta_{(3,0)} := \sqrt{1-\alpha_{(3,0)}^2 = \sqrt{1-r}}.$$

Also, let

$$\alpha_{(0,1)} := \frac{\alpha_{(0,0)}\beta_{(1,0)}}{\beta_{(0,0)}} = \sqrt{\frac{p(1-q)}{1-p}}.$$

We recursively define two sequences a and b of real numbers by

$$\begin{aligned} a_0 &:= p && (3) \\ a_1 &:= \frac{p(1-q)}{1-p} && (4) \\ b_0 &:= q && (5) \\ b_1 &\equiv \alpha^2_{(1,1)} := \frac{\alpha^2_{(1,0)}\beta^2_{(2,0)}}{\beta^2_{(1,0)}} = \frac{q(1-r)}{1-q} && (6) \\ a_{n+1} &:= \frac{a_n(1-b_n)}{1-a_n} \quad (n \geq 1) && (7) \\ b_{n+1} &:= \frac{b_n(1-r)}{1-b_n} \quad (n \geq 1). && (8) \end{aligned}$$

Claim 1. For all $n \geq 0$, we have $0 < b_{n+1} < b_n < r$.

Claim 2. The sequence b converges to 0.

Using *Mathematica*, one conjectures that

$$b_n = \frac{qr(1-r)n}{q[(1-r)^n - 1] + r}$$

and

$$a_n = \frac{pr[q(1-r)^n + r - q]}{pq[(1-r)^n - 1] + r(r + pqn - prn)} =: \frac{K_n}{L_n}.$$

Both formulas can be proved by mathematical induction.

Claim 3. The inequality $0 < a_n < 1$ is not true for every $n \geq 0$; that is, for some $n > 1$ the algorithm that constructs spherical isometries fails.

Proof of Claim 3. Assume to the contrary that $0 < K_n < L_n$ for all $n \geq 0$. It follows that

$$pr[q(1-r)^n + r - q] < pq[(1-r)^n - 1] + r(r + pqn - prn).$$

then

$$(pqr - pq)(1-r)^n + pr^2 - pqr < -pq + r^2 + pqrn - pr^2 n,$$

so that

$$-pq(1-r)^{n+1} < -pr(r-q)n - pr(r-q) + r^2 - pq.$$

This leads to a contradiction.