Bivariate Truncated Moment Problems with Algebraic Variety in the Nonnegative Quadrant in $\mathbb{R}^2$

(joint work with Sang Hoon Lee and Jasang Yoon)

Raúl E. Curto, University of Iowa

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Overview

Hyponormality and Subnormality

$\mathcal{L}(\mathcal{H})$: algebra of operators on a Hilbert space $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- quasinormal if T commutes with T^*T
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$

normal $\Longrightarrow$ quasinormal $\Longrightarrow$ subnormal $\Longrightarrow$ hyponormal

For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.

- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- (Bram-Halmos):

T subnormal $\Leftrightarrow$ T is k-hyponormal for all $k \geq 1$.

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift
- In general, $W_\alpha = U_+ D_\alpha$ (polar decomposition)
- $\|W_\alpha\| = \sup_k \alpha_k$

Weighted Shifts and Berger's Theorem

Given a bounded sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, \ldots$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0).$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)

Berger Measures

- (Berger; Gellar-Wallen) W_α is subnormal if and only if there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \ \ (\text{all } k \geq 0).$$

  ξ is the Berger measure of W_α.

- The Berger measure of U_+ is δ_1.
- For $0 < a < 1$ we let $S_a := \text{shift}\,(a, 1, 1, \ldots)$.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

Multivariable Weighted Shifts

$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

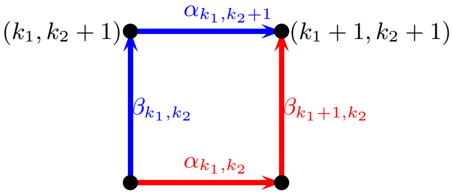
$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

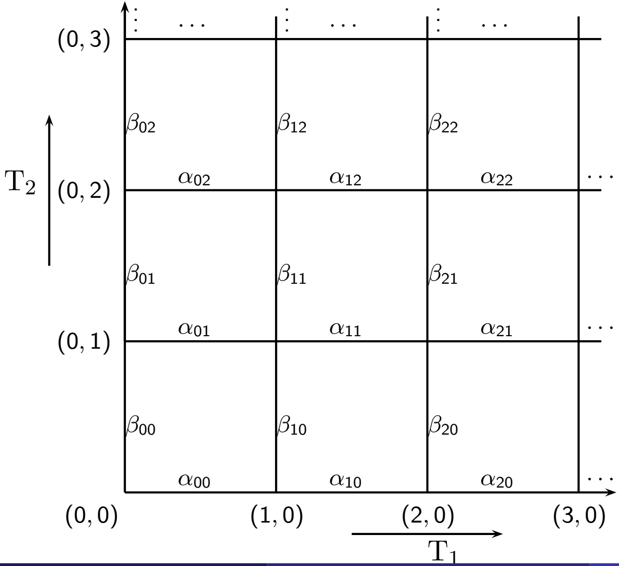
We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}} \text{ (all } \mathbf{k}\text{)}.$$

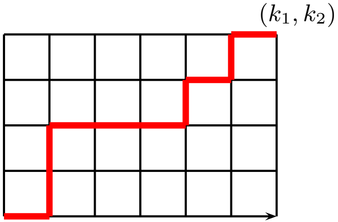



(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T_2
T_1
β_{02}
β_{12}
β_{22}
α_{02}
α_{12}
α_{22}
β_{01}
β_{11}
β_{21}
α_{01}
α_{11}
α_{21}
β_{00}
β_{10}
β_{20}
α_{00}
α_{10}
α_{20}

We now recall the notion of moment of order $\mathbf{k}$ for a commuting pair (α, β). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is $\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta)$

$$:= \begin{cases} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

By commutativity, $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).



- (Jewell-Lubin)

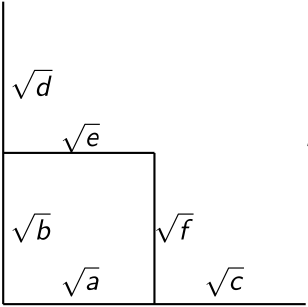
$$W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} := \prod_{i=0}^{k_1-1} \alpha_{(i,0)}^2 \cdot \prod_{j=0}^{k_2-1} \beta_{(k_1-1,j)}^2$$

$$= \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \;\; (\text{all } \mathbf{k} \geq \mathbf{0}).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

The Subnormal Completion Problem

Consider the following completion problem: Given



$be = af$ (commutativity)

Figure: The initial family of weights Ω_1

we wish to add infinitely many weights and generate a subnormal 2-variable weighted shift, whose Berger measure interpolates the initial family of weights.

Strategy: The initial family needs to satisfy an obvious necessary condition, that is,

$$\mathcal{M}(\Omega_1) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{01} & \gamma_{20} \end{pmatrix} \equiv \begin{pmatrix} 1 & a & b \\ a & ac & be \\ b & be & bd \end{pmatrix} \geq 0.$$

We use tools and techniques from the theory of TMP to solve SCP in the foundational case of six prescribed initial weights; these weights give rise to the linear and quadratic moments. For Ω_1, the natural necessary conditions for the existence of a subnormal completion are also sufficient.

To calculate explicitly the associated Berger measure, we compute the *algebraic variety* of the associated truncated moment problem; it turns out that this algebraic variety is precisely the support of the Berger measure of the subnormal completion.

In this case, solving the SCP consists of finding a probability measure μ supported on $\mathbb{R}_+^2$ such that

$$\int_{\mathbb{R}_+^2} y^i x^j \, d\mu(x,y) = \gamma_{ij} \; (i,j \geq 0, \; i+j \leq 2).$$

To ensure that the support of μ remains in $\mathbb{R}_+^2$ we will use the *localizing matrices* $\mathcal{M}_x(2)$ and $\mathcal{M}_y(2)$; each of these matrices will need to be appropriately defined and positive semidefinite.

Definition

Given $m \geq 0$ and a finite family of positive numbers $\Omega_m \equiv \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$, we say that a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ with weight sequences $\alpha_{\mathbf{k}}^{\mathbf{T}}$ and $\beta_{\mathbf{k}}^{\mathbf{T}}$ is a subnormal completion of Ω_m if

(i) $\mathbf{T}$ is subnormal, and

(ii) $(\alpha_{\mathbf{k}}^{\mathbf{T}}, \beta_{\mathbf{k}}^{\mathbf{T}}) = (\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$

whenever $|\mathbf{k}| \leq m$.

Definition

Given $m \geq 0$ and $\Omega_m \equiv \{(\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})\}_{|\mathbf{k}| \leq m}$, we say that $\hat{\Omega}_{m+1} \equiv \{(\hat{\alpha}_{\mathbf{k}}, \hat{\beta}_{\mathbf{k}})\}_{|\mathbf{k}| \leq m+1}$ is an *extension* of Ω_m if $(\hat{\alpha}_{\mathbf{k}}, \hat{\beta}_{\mathbf{k}}) = (\alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ whenever $|\mathbf{k}| \leq m$. When $m = 1$, we say that Ω_1 is quadratic.

Recall that a commuting pair (T_1, T_2) is *2-hyponormal* if the 5-tuple $(T_1, T_2, T_1^2, T_1T_2, T_2^2)$ is (jointly) hyponormal. For 2-variable weighted shifts and $m = 2$, this is equivalent to the condition

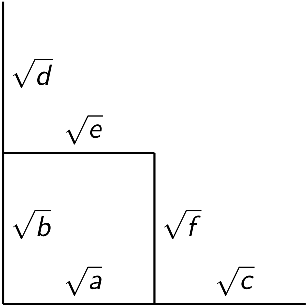
$$M_{\mathbf{u}}(2) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{0 \le p+q \le 2, 0 \le m+n \le 2} \ge 0 \quad (\text{ all } \mathbf{u} \in \mathbb{Z}_+^2);$$

that is,

$$\begin{pmatrix} \gamma_{\mathbf{u}} & \gamma_{\mathbf{u}+(0,1)} & \gamma_{\mathbf{u}+(1,0)} & \gamma_{\mathbf{u}+(0,2)} & \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(2,0)} \\ \gamma_{\mathbf{u}+(0,1)} & \gamma_{\mathbf{u}+(0,2)} & \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(0,3)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(2,1)} \\ \gamma_{\mathbf{u}+(1,0)} & \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(2,0)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(2,1)} & \gamma_{\mathbf{u}+(3,0)} \\ \gamma_{\mathbf{u}+(0,2)} & \gamma_{\mathbf{u}+(0,3)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(0,4)} & \gamma_{\mathbf{u}+(1,3)} & \gamma_{\mathbf{u}+(2,2)} \\ \gamma_{\mathbf{u}+(1,1)} & \gamma_{\mathbf{u}+(1,2)} & \gamma_{\mathbf{u}+(2,1)} & \gamma_{\mathbf{u}+(1,3)} & \gamma_{\mathbf{u}+(2,2)} & \gamma_{\mathbf{u}+(3,1)} \\ \gamma_{\mathbf{u}+(2,0)} & \gamma_{\mathbf{u}+(2,1)} & \gamma_{\mathbf{u}+(3,0)} & \gamma_{\mathbf{u}+(2,2)} & \gamma_{\mathbf{u}+(3,1)} & \gamma_{\mathbf{u}+(4,0)} \end{pmatrix} \ge 0$$

(all $\mathbf{u} \in \mathbb{Z}_+^2$). $M_{\mathbf{u}}(2)$ detects the 2-hyponormality of (T_1, T_2).

Recall the initial weight diagram:



$be = af$ (commutativity)

FIGURE: The initial family of weights Ω_1

with its associated moment matrix

$$M(\Omega_1) := \begin{pmatrix} 1 & a & b \\ a & ac & be \\ b & be & bd \end{pmatrix}.$$

Notation

We also need the notion of **localizing matrix**; in this case, these are $M_x(2)$ and $M_y(2)$.

Giving a 6×6 moment matrix $M(2)$, with rows and columns labeled 1, X, Y, X^2, XY and Y^2, the localizing matrix $M_x(2)$ is a 3×3 matrix obtained from $M(2)$ by selecting the three columns X, X^2 and XY and the top three rows. Similarly, the localizing matrix $M_y(2)$ consists of the columns Y, XY and Y^2 and the top three rows.

Theorem

(RC, S.H. Lee and J. Yoon; 2010) Let Ω_1 *be a quadratic, commutative, initial set of positive weights, and assume* $\mathcal{M}(\Omega_1) \geq 0$. *Then there exists a quartic commutative extension* $\hat{\Omega}_3$ *of* Ω_1 *such that* $\mathcal{M}(\hat{\Omega}_3)$ *is a flat extension of* $\mathcal{M}(\Omega_1)$, *and* $\mathcal{M}_x(\hat{\Omega}_3) \geq 0$ *and* $\mathcal{M}_y(\hat{\Omega}_3) \geq 0$. *As a consequence,* Ω_1 *admits a subnormal completion* $\mathbf{T}_{\hat{\Omega}_\infty}$.

Sketch of Proof.

- Six new weights, $\hat{\alpha}_{20}, \hat{\beta}_{20}, \hat{\alpha}_{11}, \hat{\beta}_{11}, \hat{\alpha}_{02}$ and $\hat{\beta}_{02}$ can be chosen in such a way that $M_x(\hat{\Omega}_3) \geq 0$ and $M_y(\hat{\Omega}_3) \geq 0$.
- Once this is done, the next step is to employ techniques from truncated moment problems to establish the existence of a flat extension $M(\hat{\Omega}_3)$ of $M(\Omega_1)$.
- Using the Flat Extension Theorem, there exists a representing measure μ for $M(1)$, and the positivity of the localizing matrices $M_x(2)$ and $M_y(2)$ means that $\operatorname{supp} \mu \subseteq \mathbb{R}^2_+$.
- Thus, μ will be the Berger measure of a subnormal 2-variable weighted shift $\mathbf{T}_{\Omega_\infty}$, which will be the desired subnormal completion of Ω_1.

Let us build $M(2)$.

Since $M(1) \equiv M(\Omega_1) \geq 0$, it follows that $\det \begin{pmatrix} ac & be \\ be & bd \end{pmatrix} \geq 0$, i.e.,

$$acd \geq be^2.$$

By the commutativity of Ω_1, we have

$$af = be,$$

and therefore

$$cd \geq ef.$$

A straightforward calculation shows that

$$\det\ M(1) = acbd - b^2e^2 - a^2bd + 2ab^2e - b^2ac$$

and that

$$\det\ M(1) > 0 \Longrightarrow cd - ef > 0.$$

WLOG, $c \geq e$. We also assume that $a < c$, since otherwise a trivial solution exists.

To build $M(2)$, we first need six new weights (the quadratic weights).

Since the extension $\hat{\Omega}_3$ will also be commutative, two of these weights will be expressible in terms of other weights.

We thus denote $\hat{\alpha}_{20}$ by $\sqrt{p}$, $\hat{\alpha}_{11}$ by $\sqrt{q}$, $\hat{\alpha}_{02}$ by $\sqrt{r}$, and $\hat{\beta}_{02}$ by $\sqrt{s}$ ($\hat{\beta}_{20}$ and $\hat{\beta}_{11}$ can be written in terms of the other four new weights). It follows that

$$M(2) = \begin{pmatrix} 1 & a & b & ac & be & bd \\ a & ac & be & acp & beq & bdr \\ b & be & bd & beq & bdr & bds \\ ac & acp & beq & & & \\ be & beq & bdr & & & \\ bd & bdr & bds & & & \end{pmatrix}$$

(with the lower right-hand 3×3 corner yet undetermined).

We now focus on the top three rows of $M(2)$:

$$\begin{pmatrix} 1 & X & Y & X^2 & XY & Y^2 \\ 1 & a & b & ac & be & bd \\ a & ac & be & acp & beq & bdr \\ b & be & bd & beq & bdr & bds \end{pmatrix}$$

from which we get:

$$M_x(2) = \begin{pmatrix} X & X^2 & XY \\ a & ac & be \\ ac & acp & beq \\ be & beq & bdr \end{pmatrix}$$

$$M_y(2) = \begin{pmatrix} Y & XY & Y^2 \\ b & be & bd \\ be & beq & bdr \\ bd & bdr & bds \end{pmatrix}.$$

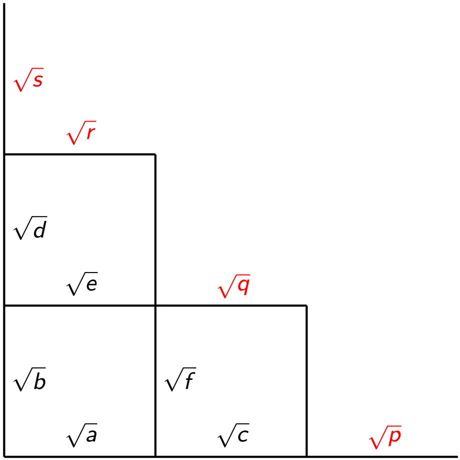


FIGURE: The family of weights $\hat{\Omega}_3$

Since the zero-th row of a subnormal completion of Ω_1 will be a subnormal completion of the zero-th row of Ω_1, which is given by the weights $a < c$, we let $p := c$. By L-shaped propagation, having $\alpha_{10} = \hat{\alpha}_{20}$ immediately implies that $\hat{\alpha}_{11} = \sqrt{c}$, that is, $q := c$. Thus,

$$M_x(2) = \begin{pmatrix} a & ac & be \\ ac & ac^2 & bce \\ be & bce & bdr \end{pmatrix}.$$

By Choleski's Algorithm (or its generalization, proved by J.L. Smul'jan in 1959), $M_x(2) \geq 0$ if and only if $bdr \geq \frac{(be)^2}{a}$, so that we need $r \geq \frac{ef}{d}$. Thus, provided we take $r \geq \frac{ef}{d}$, the positivity of $M_x(2)$ is guaranteed. It remains to show that we can choose s in such a way that $s \geq d$ and $M_y(2) \equiv M_y(2)(s) \geq 0$. This can certainly be done:

$$s = \frac{a^2cd^2 - 2abde^2 + b^2e^3}{a^2d(c-e)}.$$

To complete the proof, we need to define the 3×3 lower right-hand corner of $M(2)$, and then show that $M(2)$ is a flat extension of $M(1)$, and therefore $M(2) \geq 0$. This is done by examining the rank of $M(1)$.

Case 1: $e = c$. We have $d \geq f$, so we can take $r := c$ and guarantee that $M_x(2) \geq 0$. We also let $s := d$. We then have

$$M_y(2) = \begin{pmatrix} b & bc & bd \\ bc & bc^2 & bcd \\ bd & bcd & bd^2 \end{pmatrix}.$$

It follows at once that rank $M_y(2) = 1$, and therefore $M_y(2) \geq 0$ (and of course $s \geq d$).

Case 2: $e < c$. We define r by this extremal value, i.e., $r := \frac{ef}{d}$. This immediately implies that $\hat{\beta}_{11} := \sqrt{f}$, and by propagation, $\hat{\beta}_{1j} := \sqrt{f}$ (all $j \geq 2$) in any subnormal completion. The resulting weight diagram is shown below.

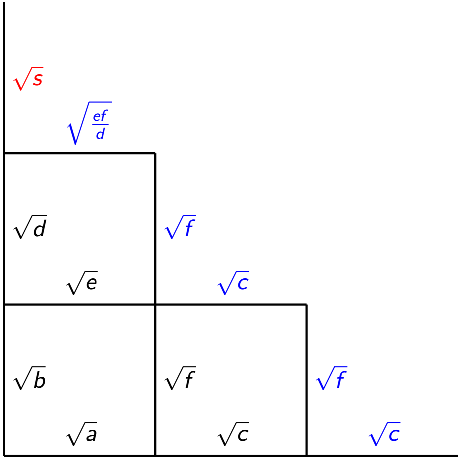


FIGURE: The family Ω_1 augmented with the inclusion of the quadratic weights

Of significant importance is the calculation of the associated algebraic variety, which arises from the column relations in $M(1)$, particularly the column relation

$$Y = \frac{b(c-e)}{c-a} \cdot 1 + \frac{f-b}{c-a} X.$$

It is actually possible to provide a concrete description of the Berger measure for the subnormal completion in terms of the initial data.

Remark

Flat extensions may not exist for bigger families of initial weights. That is, one can build an example of a quartic family of initial weights Ω_2 for which the associated moment matrix $\mathcal{M}(2)$ admits a representing measure, but such that $\mathcal{M}(2)$ has no flat extension $\mathcal{M}(3)$.

Here's a concrete example:

$$
\begin{array}{lllll}
\gamma_{00}=1 & & & & \\
\gamma_{01}=1 & \gamma_{10}=1 & & & \\
\gamma_{02}=2 & \gamma_{11}=0 & \gamma_{20}=3 & & \\
\gamma_{03}=4 & \gamma_{12}=0 & \gamma_{21}=0 & \gamma_{30}=9 & \\
\gamma_{04}=9 & \gamma_{13}=0 & \gamma_{22}=0 & \gamma_{31}=0 & \gamma_{40}=28
\end{array}
\qquad
\begin{array}{ll}
\tilde{\gamma}_{00}=1 & \\
\tilde{\gamma}_{01}=4 & \tilde{\gamma}_{10}=5 \\
\tilde{\gamma}_{02}=17 & \tilde{\gamma}_{11}=19 \\
\tilde{\gamma}_{03}=76 & \tilde{\gamma}_{12}=77 \\
\tilde{\gamma}_{04}=354 & \tilde{\gamma}_{13}=331
\end{array}
$$

Remark

The SCP in the previous Example does admit a solution, and the subnormal completion has a 6-atomic Berger measure. It turns out that $M(2)$ has rank 5, and admits an extension $M(3)$ of rank 6, and this $M(3)$ admits a flat extension $M(4)$.

- D. Kimsey (2014) has a very nice paper in IEOT, in which he describes generalizations of these results:
  The cubic complex moment problem, IEOT 80(2014), 353–378.
- Similarly, K. Idrissi and E.H. Zerouali extend the notion of recursively generated weighted shift and discuss an alternative approach to the SCP:
  K. Idrissi and E.H. Zerouali, Multivariable recursively generated weighted shifts and the 2-variable subnormal completion problem, Kyungpook Math. J. 58(2018), 711–732.

One-Step Extensions of Subnormal 2-Variable Weighted Shifts

Consider the following reconstruction-of-the-measure problem:

Given two probability measures μ_1 and μ_2 on $\mathbb{R}_+^2$, find necessary and sufficient conditions for the existence of a probability measure μ on $\mathbb{R}_+^2$ with $supp\mu \not\subseteq (\mathbb{R}_+ \times 0) \cup (0 \times \mathbb{R}_+)$ such that

$$\frac{s\ d\mu(s,t)}{\int s\ d\mu(s,t)} = d\mu_1(s,t) \text{ and } \frac{t\ d\mu(s,t)}{\int t\ d\mu(s,t)} = d\mu_2(s,t).$$

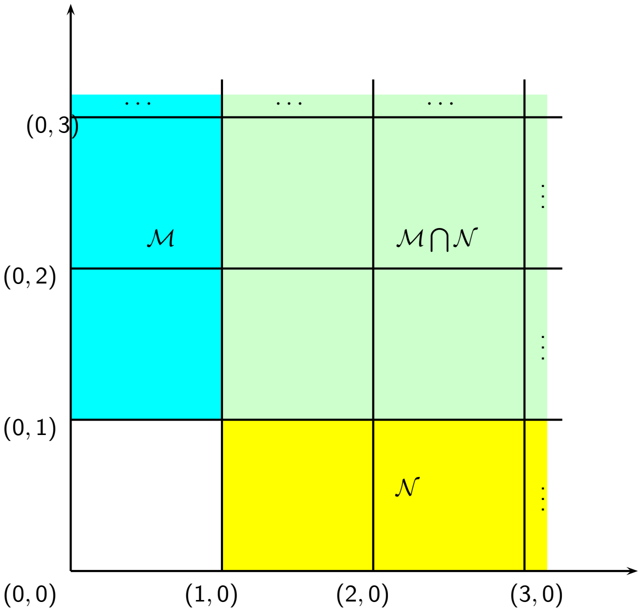
This readily implies that

$$td\mu_1(s,t) = \lambda s d\mu_2(s,t)$$

for some $\lambda > 0$; this condition, while clearly necessary for the existence of μ, is by no means sufficient.

Problem

Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ *and* $W_{(\alpha,\beta)}|_{\mathcal{N}}$ *are subnormal with Berger measures* $\mu_{\mathcal{M}}$ *and* $\mu_{\mathcal{N}}$*, respectively. Find necessary and sufficient conditions on* $\mu_{\mathcal{M}}$*,* $\mu_{\mathcal{N}}$ *and* β_{00} *for the subnormality of* $W_{(\alpha,\beta)}$.



(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
$\mathcal{M}$
$\mathcal{M} \bigcap \mathcal{N}$
$\mathcal{N}$

The following result provides a concrete solution in terms of $\mu_{\mathcal{M}}$, $\mu_{\mathcal{N}}$ and β_{00}.

Theorem

Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively, and let $c := \frac{\int s \, d\mu_{\mathcal{M}}}{\int t \, d\mu_{\mathcal{N}}} \equiv \frac{\alpha_{01}^2}{\beta_{10}^2}$. Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following four conditions hold:

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\frac{1}{s} \in L^1(\mu_{\mathcal{N}})$;

(iii) $c\beta_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} \leq 1$;

(iv) $\beta_{00}^2 \left\{ \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X + c \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} \delta_0 - \frac{c}{s} (\mu_{\mathcal{N}})^X \right\} \leq \delta_0$.

To state the following result, recall that when the core of a 2-variable weighted shift $W_{(\alpha,\beta)}$ is of tensor form, it follows that the Berger measure of the restriction of $W_{(\alpha,\beta)}$ to $\mathcal{M} \cap \mathcal{N}$ splits as a Cartesian product of two 1-variable measures. As a special case, we now have:

Theorem

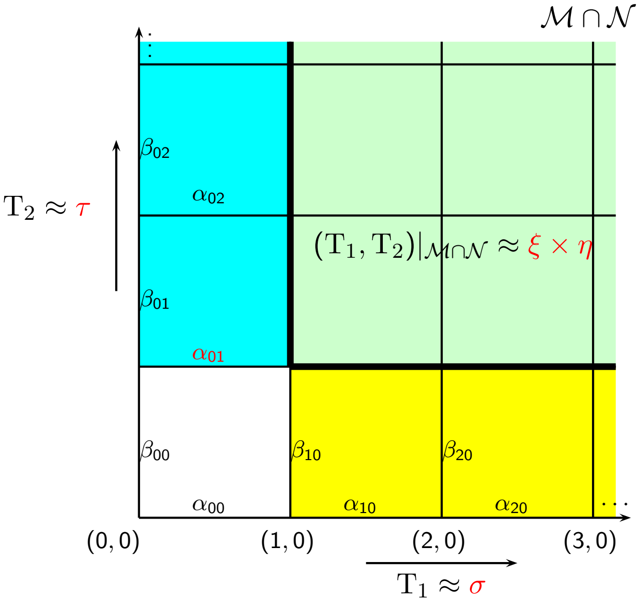
($W_{(\alpha,\beta)}$ has a core of tensor form.) Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively, and let $\rho := \mu_{\mathcal{M}}^{X}$, i.e., ρ is the Berger measure of shift$(\alpha_{01}, \alpha_{11}, \cdots)$. Also assume that $\mu_{\mathcal{M}\cap\mathcal{N}} = \xi \times \eta$ for some 1-variable probability measures ξ and η.

Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following three conditions hold:

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$;

(iii) $\left(\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\tau_1)}\right) \rho = \left(\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}\right) \rho \leq \sigma$.



$\mathcal{M} \cap \mathcal{N}$
$T_2 \approx \tau$
β_{02}
α_{02}
$(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}} \approx \xi \times \eta$
β_{01}
α_{01}
β_{00}
β_{10}
β_{20}
α_{00}
α_{10}
α_{20}
$(0,0)$
$(1,0)$
$(2,0)$
$(3,0)$
$T_1 \approx \sigma$

An Application

We will now see that one-step extensions may not exist, even under very favorable assumptions of subnormality for the restriction of $W_{(\alpha,\beta)}$ to $\mathcal{M} \vee \mathcal{N}$. For instance, both $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ can be unitarily equivalent, and yet for no β_{00} is $W_{(\alpha,\beta)}$ subnormal. To see this, let us assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with the Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively. Assume also that $Y = X$. Let $\mu_{\mathcal{M}} = \mu_{\mathcal{N}}$ be a diagonal measure ϵ on $X \times X$, that is, $supp\, \epsilon \subseteq \{(s,t) \in X \times X : s = t\}$; we loosely describe this by $d\epsilon(s,t) = d\epsilon(s,s) = d\epsilon(t,t)$.

Then by the techniques of disintegration of measures, we can see that

$$\epsilon^X = \epsilon^Y, \qquad \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} = \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = \left\|\frac{1}{s}\right\|_{L^1(\epsilon^X)} = \left\|\frac{1}{t}\right\|_{L^1(\epsilon^Y)}$$

and

$$(\mu_{\mathcal{M}})^X_{ext} = (\epsilon)^X_{ext} = \epsilon^X.$$

Thus, in the Theorem we have $c = 1$ and therefore

$$c\beta_{00}^2 \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} \leq 1 \Longleftrightarrow \beta_{00}^2 \left\|\frac{1}{s}\right\|_{L^1(\epsilon^X)} \leq 1$$

and

$$\beta_{00}^2 \left\{ \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X + \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} d\delta_0(s) - \frac{d(\mu_{\mathcal{N}})^X}{s} \right\} \leq d\delta_0(s)$$

$$\Longleftrightarrow \beta_{00}^2 \left\{ \left\|\frac{1}{t}\right\|_{L^1(\epsilon^Y)} \epsilon^X + \left\|\frac{1}{t}\right\|_{L^1(\epsilon^Y)} \delta_0 - \frac{\epsilon^X}{s} \right\} \leq \delta_0.$$

We can summarize these calculations as follows.

Proposition

Let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift given above. Then $W_{(\alpha,\beta)}$ is subnormal if and only if

(i) $\beta_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\epsilon^X)} \leq 1$;

(ii) $\beta_{00}^2 \left\{ \left\| \frac{1}{t} \right\|_{L^1(\epsilon^Y)} \epsilon^X + \left\| \frac{1}{t} \right\|_{L^1(\epsilon^Y)} \delta_0 - \frac{\epsilon^X}{s} \right\} \leq \delta_0$.

We now present a concrete example.

Example

Let $\mu_{\mathcal{M}} = \mu_{\mathcal{N}}$ be the 2-variable probability measure on $[0,1]^2$ with moments $\gamma_{(k_1,k_2)} := \frac{1}{k_1+k_2+1} \quad (k_1, k_2 \geq 0)$. It is easy to see that $\mu_{\mathcal{M}} = \mu_{\mathcal{N}} = \epsilon$ is a diagonal measure on $[0,1]^2$; specifically, ϵ is normalized Lebesgue measure on the diagonal of $[0,1]^2$. It follows that $\epsilon^X = \epsilon^Y$ is the Lebesgue measure on $[0,1]$. Therefore, we have: $W_{(\alpha,\beta)}$ is never subnormal for any choice of β_{00}. For, $\frac{1}{s} \notin L^1(\epsilon^X)$, which is a necessary condition for subnormality.

Muito obrigado pela sua atenção!