The Extended Aluthge Transform

(joint work with Chafiq Benhida)

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Overview

Aluthge Transform

Let T be a Hilbert space operator, let $P := |T|$ be its positive part, and let $T = VP$ denote the canonical polar decomposition of T, with V a partial isometry and $\ker V = \ker T = \ker P$.

We define the Aluthge transform of T as

$$A(T) \equiv \hat{T} := \sqrt{P}V\sqrt{P}.$$

The iterates are

$$A^{(n+1)}(T) := A(A^{(n)}(T)) \; (n \geq 1).$$

The Aluthge transform has been extensively studied, in terms of algebraic, structural and spectral properties.

For instance,

- (i) $T = \hat{T} \Leftrightarrow T$ is quasinormal;
- (ii) (Aluthge, 1990) If $0 < p < \frac{1}{2}$ and T is p-hyponormal, then $\hat{T}$ is $(p + \frac{1}{2})$-hyponormal;
- (iii) (Jung, Ko & Pearcy, 2000) If $\hat{T}$ has a n.i.s., then T has a n.i.s.
- (iv) (Kim-Ko, 2005; Kimura, 2004) T has property (β) if and only if $\hat{T}$ has property (β); and
- (v) (Ando, 2005) $\left\|(T-\lambda)^{-1}\right\| \geq \left\|(\hat{T}-\lambda)^{-1}\right\|$ $(\lambda \notin \sigma(T))$.
- (vi) Observe that if $A := \sqrt{P}$ and $B := V\sqrt{P}$, then $\hat{T} = AB$ and $T = BA$, and therefore

$$\sigma(\hat{T}) \setminus \{0\} = \sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\} = \sigma(T) \setminus \{0\}.$$

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift
- In general, $W_\alpha = U_+ D_\alpha$ (polar decomposition)
- G. Exner (IWOTA 2006 Lecture): subnormality is not preserved under the Aluthge transform.
- (S.H. Lee, W.Y. Lee and J. Yoon, 2012) For $k \geq 2$, the Aluthge transform, when acting on weighted shifts, need not preserve k-hyponormality.

Note that the Aluthge transform of a weighted shift is again a weighted shift.

Concretely, the weights of $\widehat{W_\alpha}$ are

$$\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_2\alpha_3}, \sqrt{\alpha_3\alpha_4}, \cdots .$$

Define

$$W_{\sqrt{\alpha}} := \text{shift } \left(\sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \cdots\right).$$

Then $\widehat{W_\alpha}$ is the Schur product of $W_{\sqrt{\alpha}}$ and its restriction to the subspace $\bigvee\{e_1, e_2, \cdots\}$. Thus, a sufficient condition for the subnormality of $\widehat{W_\alpha}$ is the subnormality of $W_{\sqrt{\alpha}}$.

Iterates of the Aluthge Transform

Observe that if $W_\alpha = \text{shift}(\alpha_0, \alpha_1, \cdots)$, then

$$A(W_\alpha) = \text{shift}(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots),$$

$$A^{(2)}(W_\alpha) = \text{shift}(\sqrt{\sqrt{\alpha_0\alpha_1}\sqrt{\alpha_1\alpha_2}}, \sqrt{\sqrt{\alpha_1\alpha_2}\sqrt{\alpha_2\alpha_3}}, \cdots),$$

$$A^{(3)}(W_\alpha) = \text{shift}((\alpha_0\alpha_1^3\alpha_2^3\alpha_3)^{\frac{1}{8}}, (\alpha_1\alpha_2^3\alpha_3^3\alpha_4)^{\frac{1}{8}}, \cdots)$$

$$A^{(4)}(W_\alpha) = \text{shift}((\alpha_0\alpha_1^4\alpha_2^6\alpha_3^4\alpha_4)^{\frac{1}{16}}, (\alpha_1\alpha_2^4\alpha_3^6\alpha_4^4\alpha_5)^{\frac{1}{16}}, \cdots).$$

Iterates of the Aluthge Transform

Let W_ω be a unilateral weighted shift, and let $\omega^{(n)}$ be the weight sequence of $A^{(n)}(W_\omega)$. Then

$$\omega_k^{(n+1)} = \sqrt{\omega_k^{(n)}\omega_{k+1}^{(n)}}.$$

Therefore,

$$\omega_k^{(n)} = (\prod_{j=0}^{n} \omega_{k+j}^{\binom{n}{j}})^{\frac{1}{2^n}}.$$

The Spherical Aluthge Transform

Consider a (joint) polar decomposition of the form

$$(T_1, T_2) \equiv (V_1 P, V_2 P),$$

or equivalently,

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P,$$

as an operator from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$, with

$$P := \sqrt{T_1^* T_1 + T_2^* T_2}.$$

Assume this is the unique canonical polar decomposition of $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$.

Then $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is a partial isometry from $(\ker P)^\perp$ onto $\overline{\operatorname{Ran} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}}$.

Multivariable Weighted Shifts

$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^{\infty}(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

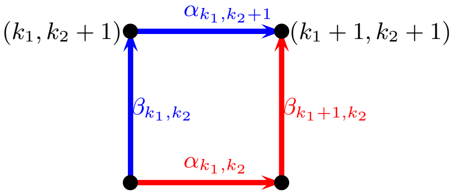
$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

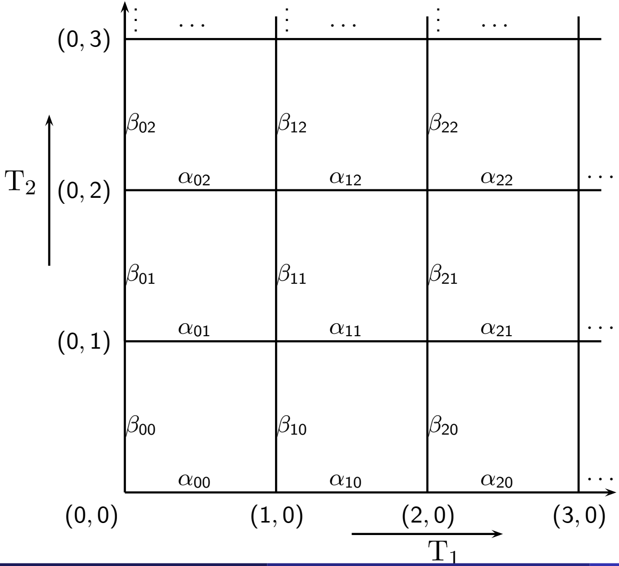
We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \iff \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \ \text{(all } \mathbf{k}\text{)}.$$





(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T2
T1
β02
β12
β22
α02
α12
α22
β01
β11
β21
α01
α11
α21
β00
β10
β20
α00
α10
α20

Spherically Quasinormal Pairs

We say that a commuting pair of Hilbert space operators $\mathbf{T} \equiv (T_1, T_2)$ is **spherically quasinormal** if T_i commutes with P, for $i = 1, 2$.

Equivalently, $\mathbf{T}$ is spherically quasinormal if $\widehat{\mathbf{T}} = \mathbf{T}$.

Proposition

(RC-J. Yoon; 2015) A 2-variable weighted shift $\mathbf{T}$ is spherically quasinormal if and only if there exists $C > 0$ such that $\frac{1}{C}\mathbf{T}$ is a **spherical isometry** (S_1, S_2), *that is,*

$$S_1^* S_1 + S_2^* S_2 = I.$$

Theorem

(Athavale; JOT, 1990) A spherical isometry is always subnormal.

Corollary

(RC-J. Yoon; 2016) A spherically quasinormal 2-variable weighted shift is subnormal.

Theorem

(A. Athavale - S. Poddar; 2015 and S. Chavan - V. Sholapurkar; 2013) Let $\mathbf{T}$ *be a spherically quasinormal pair. Then* $\mathbf{T}$ *is subnormal.*

Theorem

(V. Müller - M. Ptak; 1999) Spherical isometries are hyperreflexive.

Theorem

(J. Eschmeier - M. Putinar; 2000) For every $n \geq 3$ there exists a non-normal spherical isometry $\mathbf{T}$ such that the polynomially convex hull of $\sigma_T(\mathbf{T})$ is contained in the unit sphere.

Theorem

(C. Benhida - RC, 2018) Let $T \equiv (T_1, \cdots, T_n)$ be a commuting n-tuple of Hilbert space operators, and let $\widehat{T}$ be its spherical Aluthge transform. Then

$$\sigma_T(\widehat{T}) = \sigma_T(T).$$

The Extended Aluthge Transform

We propose a unifying approach that brings the classical Aluthge transform and the spherical Aluthge transform as special cases of a new transform.

Starting with the canonical polar decomposition $T \equiv V\,|T|$, we consider the class of positive operators P such that

$$VP = V\,|T|\,.$$

For each such P we then define the *extended* Aluthge transform as

$$\Delta_P(T) := \sqrt{P}V\sqrt{P}.$$

Naturally, the classical Aluthge transform is simply $\Delta_{|T|}(T)$.

Lemma

For P as above,

- $|T| \leq P$.
- $\ker P \subseteq \ker |T|$.
- $\overline{\mathrm{Ran}\,|T|} \subseteq \overline{\mathrm{Ran}\,P}$.
- $|T|$ *commutes with* P.
- $P|_{\mathrm{Ran}|T|} = |T|\,|_{\mathrm{Ran}|T|}$.

Lemma

Write $\mathcal{H} = \overline{\operatorname{Ran}|T|} \oplus \ker T$. *Then*

$$|T| = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A := |T|\,|_{\overline{\operatorname{Ran}|T|}}$ *and* $B := P|_{\ker T}$.

Consider now the orthogonal decomposition

$$\mathcal{H} = \overline{\text{Ran}\,|T|} \oplus (\overline{\text{Ran}\,B} \oplus \ker P),$$

where the orthogonal sum in parentheses equals $\ker|T|$. Then

$$|T| = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} A & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that $P = |T|$ if and only if $C = 0$.

We wish to find the matrix for V.

Recall that $\ker V = \ker |T|$. Therefore,

$$V = \begin{pmatrix} X & 0 & 0 \\ Y & 0 & 0 \\ Z & 0 & 0 \end{pmatrix}.$$

Since V^*V is the projection onto $(\ker V)^\perp = \overline{\mathrm{Ran}\,|T|}$, we must have

$$X^*X + Y^*Y + Z^*Z = I_{\overline{\mathrm{Ran}|T|}}.$$

Since $T = VP = V\,|T|$, it follows that

$$T = \begin{pmatrix} XA & 0 & 0 \\ YA & 0 & 0 \\ ZA & 0 & 0 \end{pmatrix}.$$

Then

$$\Delta(T) = |T|^{\frac{1}{2}} V |T|^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}} X A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

while

$$\Delta_P(T) = P^{\frac{1}{2}} V P^{\frac{1}{2}} = \begin{pmatrix} A^{\frac{1}{2}} X A^{\frac{1}{2}} & 0 & 0 \\ C^{\frac{1}{2}} Y A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$\Delta(T)^*\Delta(T) = \begin{pmatrix} A^{\frac{1}{2}}X^*AXA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Delta_P(T)^*\Delta_P(T) = \begin{pmatrix} A^{\frac{1}{2}}X^*AXA^{\frac{1}{2}} + A^{\frac{1}{2}}Y^*CYA^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a consequence,

$$|\Delta_P(T)| \geq |\Delta(T)|$$

and

$$\|\Delta_P(T)\| \geq \|\Delta(T)\|.$$

Proposition

Let $T \equiv V|T|$ be the canonical polar decomposition of T, and let P be a positive operator such that $T = VP$. Let U be a unitary operator on $\mathcal{H}$. Then

$$\Delta_{UPU^*}(UTU^*) = U\Delta_P(T)U^*.$$

Proposition

With T, V and P as above, we have

$$\Delta_{VPV^*}(T^*) = V\Delta_P(T)^* V^*.$$

Corollary

In the case when $P = |T|$ *we have*

$$\Delta(T^*)^* = V\Delta(T)V^*.$$

Proposition

Let $T_i \equiv V_i\,|T_i|$ *be the canonical polar decomposition of* T_i $(i = 1, 2)$, *and let* P_i *be a positive operator such that* $T_i = V_i P_i$ $(i = 1, 2)$. *Then*

$$\Delta_{P_1 \oplus P_2}(T_1 \oplus T_2) = \Delta_{P1}(T_1) \oplus \Delta_{P_2}(T_2).$$

Fixed Points of the Ext'd Aluthge Transform

It is well known that the fixed points of the classical Aluthge transform are the quasinormal operators, that is, those $T \equiv V|T|$ such that V and $|T|$ commute. Here's the analogous result for the extended Aluthge transform.

Theorem

Let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator such that $VP = T$, and assume that $\Delta_P(T) = T$. Then T commutes with P, V commutes with P, and $\ker P$ *reduces T and V. If $P = |T|$ then T is quasinormal.*

The Case of T Idempotent; i.e., $T^2 = T$

$$T^2 = T \implies \begin{cases} XAXA &= XA \\ YAXA &= YA \\ ZAXA &= ZA. \end{cases}$$

Since $\text{Ran}\, A$ is dense in $\overline{\text{Ran}\, |T|}$, we have

$$\begin{cases} XAX &= X \\ YAX &= Y \\ ZAX &= Z, \end{cases}$$

and therefore

$$\begin{cases} X^*XAX &= X^*X \\ Y^*YAX &= Y^*Y \\ Z^*ZAX &= Z^*Z. \end{cases}$$

Since $X^*X + Y^*Y + Z^*Z = I_{\overline{\text{Ran}|T|}}$, we readily obtain

$$AX = I_{\overline{\text{Ran}|T|}}.$$

As a result,

$$A^{\frac{1}{2}} X A^{\frac{1}{2}} = I_{\overline{\text{Ran}|T|}}.$$

We now see that $\Delta(T)$ is the projection from $\mathcal{H}$ onto $\overline{\text{Ran}\,|T|}$, and therefore

$$\Delta_P(T) = \begin{pmatrix} I & 0 & 0 \\ C^{\frac{1}{2}} Y A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $AX = I_{\overline{\text{Ran}|T|}}$, we know that A is right invertible on $\overline{\text{Ran}\,|T|}$, therefore invertible (as an operator on $\overline{\text{Ran}\,|T|}$).

Theorem

Let T be an idempotent. Then

$$\Delta(T) = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and

$$\Delta_P(T) = \begin{pmatrix} I & 0 & 0 \\ C^{\frac{1}{2}} Y A^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with A invertible.

This result is, in some ways, optimal.

Some Useful Identities

First, a direct matrix calculation shows that

$$\Delta(T)P = \Delta(T)\,|T|\,.$$

More is true, however.

Proposition

(Intertwining Property) For T, P, $\Delta(T)$ and $\Delta_P(T)$ be as before, we have:

$$|T|^{\frac{1}{2}}\,\Delta_P(T)P^{\frac{1}{2}} = P^{\frac{1}{2}}\Delta(T)\,|T|^{\frac{1}{2}}\,.$$

Theorem

Let T be a Hilbert-Schmidt operator. Then

$$\|\Delta_P(T)\|_2^2 = \|\Delta(T)\|_2^2 + \left\| C^{\frac{1}{2}}\,YA^{\frac{1}{2}} \right\|_2^2.$$

Application: The Spherical Aluthge Transf.

Recall that

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} Q = \begin{pmatrix} V_1 Q \\ V_2 Q \end{pmatrix},$$

as operators from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$, with $Q := \sqrt{T_1^* T_1 + T_2^* T_2}$ and $\begin{pmatrix} V_1 Q \\ V_2 Q \end{pmatrix}$ a joint partial isometry.

The spherical Aluthge transform of $\mathbf{T} \equiv (T_1, T_2)$ is $\widehat{\mathbf{T}} \equiv (\widehat{T_1}, \widehat{T_2})$, where

$$\widehat{T}_i := Q^{\frac{1}{2}} V_i Q^{\frac{1}{2}} \; (i = 1, 2).$$

Lemma

$\widehat{\mathbf{T}}$ *is commutative.*

We now let

$$\Phi(\mathbf{T}) := \begin{pmatrix} T_1 & 0 \\ T_2 & 0 \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{H}).$$

It is clear that

$$|\Phi(\mathbf{T})| = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}.$$

We also let $\mathbf{V} := (V_1, V_2)$. (Notice that $\mathbf{V}$ is not necessarily commuting.)

Finally, let

$$\Phi(\mathbf{V}) := \begin{pmatrix} V_1 & 0 \\ V_2 & 0 \end{pmatrix}.$$

Lemma

With $\mathbf{T}$ *and* $\mathbf{V}$ *as above,* $\Phi(\mathbf{T}) = \Phi(\mathbf{V})\,|\Phi(\mathbf{T})|$ *is the canonical polar decomposition of* $\Phi(\mathbf{T})$*.*

Consider now the positive operator

$$P \equiv P(Q) := \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

Then

$$\Phi(\mathbf{V})P = \Phi(\mathbf{V})\,|\Phi(\mathbf{T})| = \Phi(\mathbf{T}).$$

We wish to study the extended Aluthge transform $\Delta_P(\Phi(\mathbf{T}))$.

Theorem

With $\mathbf{T}$ *and* P *as above,*

$$\Delta_P(\Phi(\mathbf{T})) = \Phi(\widehat{\mathbf{T}}).$$

Remark

For the spherical Aluthge transform, the operator P is uniquely determined by the commuting pair $\mathbf{T}$; that is, the pair $\mathbf{T}$ determines Q, which in turn determines P. □

In view of this Remark, we believe that the Extended Aluthge Transform will be particularly useful in the study of the iterates of the spherical Aluthge transform.

The Numerical Range and the Extended Aluthge Transform

For an operator $A \in \mathcal{B}(\mathcal{H})$, recall that the numerical range $W(A)$ of A is defined as

$$W(A) := \{\langle Ax, x \rangle : x \in \mathcal{H} \text{ with } \|x\| = 1\}$$

Theorem

(P.Y. Wu, 2002) Let T be an operator on $\mathcal{H}$. Then

$$\overline{W(\Delta(T))} \subseteq \overline{W(T)}$$

We now turn our attention to the extended Aluthge transform.

We begin with a natural question.

QUESTION

Is the previous Theorem still true for the extended Aluthge transform?

We'll show here that this Theorem is not true for all extended Aluthge transforms.

However, there is nice relationship connecting the numerical ranges.

Recall the following decompositions

$$T = \begin{pmatrix} XA & 0 & 0 \\ YA & 0 & 0 \\ ZA & 0 & 0 \end{pmatrix} \quad \Delta(T) = \begin{pmatrix} \sqrt{A}X\sqrt{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_P(T) = \begin{pmatrix} \sqrt{A}X\sqrt{A} & 0 & 0 \\ \sqrt{C}Y\sqrt{A} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, we have

$$\overline{W(\Delta(T))} \subseteq \overline{W(\Delta_P(T))} \quad \text{and} \quad \overline{W(\Delta(T))} \subseteq \overline{W(T)}.$$

D.S. Keeler, L. Rodman and I.M. Spitkovsky proved in 1997 that the numerical range of an upper triangular matrix of the form

$$A = \begin{pmatrix} 0 & 0 & 0 \\ x & p & 0 \\ y & z & p \end{pmatrix},$$

when $xyz = 0$, is the closed disc centered at p and with radius $\frac{1}{2}\sqrt{|x|^2 + |y|^2 + |z|^2}$.

So, if

$$T = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix},$$

and we recall that

$$P_\gamma = \begin{pmatrix} |\alpha| & 0 & 0 \\ 0 & |\beta| & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

it follows that

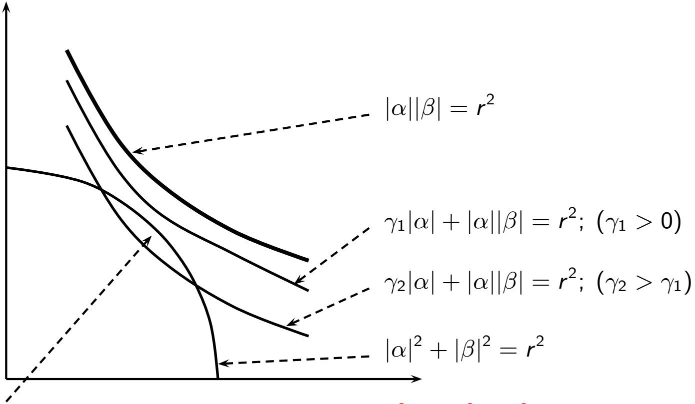
1. $W(T) = \bar{\mathbf{D}}\left(0, \frac{1}{2}\sqrt{|\alpha|^2 + |\beta|^2}\right).$
2. $W(\Delta(T)) = \bar{\mathbf{D}}\left(0, \frac{1}{2}\sqrt{|\alpha||\beta|}\right).$
3. $W(\Delta_{P_\gamma}(T)) = \bar{\mathbf{D}}\left(0, \frac{1}{2}\sqrt{\gamma|\alpha| + |\alpha||\beta|}\right).$

As expected, we have

$$\begin{cases} W(\Delta(T)) \subseteq W(T) \\ W(\Delta(T)) \subseteq W(\Delta_{P_\gamma}(T)). \end{cases}$$

Remark

1. We may choose α, β and γ in the previous example such that the inclusions above are strict, as we will see on the next slide.
2. $\overline{W(\Delta_{P_\gamma}(T))}$ and $\overline{W(T)}$ are not comparable in general, unless we impose restrictions on γ. For example, observe that if $\gamma \leq |\beta|$ in the previous discussion, then $W(\Delta_{P_\gamma}(T)) \subseteq W(T)$, with equality holding if and only if $\gamma = |\alpha| = |\beta|$. □



In this region there exist α, β, γ such that $|\alpha|^2 + |\beta|^2 < r^2$ and $\gamma|\alpha| + |\alpha||\beta| > r^2$; as a consequence, $W(\Delta(T)) \subseteq W(\Delta_{P_\gamma}(T))$.

FIGURE: Graphs of radii in previous Remark

Muito obrigado pela sua atenção!