Berger measures for transformations of subnormal weighted shifts

(joint work with George Exner)

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Abstract

- A subnormal weighted shift may be transformed to another shift in various ways, such as taking the p-th power of each weight or forming the Aluthge transform.
- We determine in a number of cases whether the resulting shift is subnormal; if it is, we find a concrete representation of the associated Berger measure.
- We do this directly for finitely atomic measures, and using both Laplace and Fourier transform methods for other measures.
- The problem may be viewed in purely measure theoretic terms: solve moment matching equations such as

$$\left(\int t^n \, d\mu(t)\right)^2 = \int t^n \, d\nu(t) \quad (n = 0, 1, \ldots)$$

for one measure given the other.

Some Key Words and Phrases

- subnormal
- weighted shift
- Berger measure
- Aluthge transform
- Schur products
- other transformations
- moment problem
- finitely atomic measures
- Laplace and Fourier transform methods

$\mathcal{L}(\mathcal{H})$: algebra of operators on a Hilbert space $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$ (We say that N is a lifting of T, or an extension of T.
- hyponormal if $T^*T \geq TT^*$
- For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.

normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal

If $\dim \mathcal{H} < \infty$, then hyponormal $\Rightarrow$ normal.

For, if $T^*T - TT^* \geq 0$, then

$$0 \leq \text{trace}\,(T^*T - TT^*) = \text{trace}\,(T^*T) - \text{trace}\,(TT^*) = 0,$$

so $\text{trace}\,(T^*T - TT^*) = 0$, which readily implies that $T^*T - TT^* = 0$, that is, T is normal.

- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- (Bram-Halmos)

$$\begin{aligned} T \text{ subnormal} \quad &\Leftrightarrow \quad T \text{ is } k\text{-hyponormal for all } k \geq 1 \\ &\Leftrightarrow \quad (T, T^2, ..., T^k) \text{ is hyponormal for all } k \geq 1. \end{aligned}$$

- An operator T is polynomially hyponormal if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$.
- T subnormal $\Rightarrow$ T polynomially hyponormal.
- The converse is false (RC & M. Putinar, 1993): indeed, there exists a Hilbert space operator T which is polynomially hyponormal and not 2-hyponormal, and there exists a polynomially hyponormal weighted shift which is not subnormal.

n-CONTRACTIVITY

For $n = 1, 2, \ldots$, T is said to be n-contractive if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \geq 0.$$

- Agler-Embry Characterization of Subnormality: Assume that T is a contraction. Then T is subnormal if and only if T is n-contractive for all $n \geq 1$.

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift
- In general, $W_\alpha = U_+ D_\alpha \quad$ (polar decomposition)
- $\|W_\alpha\| = \sup_k \alpha_k$

  $W_\alpha^n e_k = \alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1} e_{k+n}$, so

$$W_\alpha^n \cong \bigoplus_{i=0}^{n-1} W_{\beta(i)},$$

Five classical topics:

- (i) weighted shifts as fertile ground to test hypotheses
- (ii) weighted shifts as models for positivity and dilation theories (hyponormality, subnormality, n-contractivity)
- (iii) C^*-algebra generated by W_α
- (iv) spectral properties of W_α
- (v) nonselfadjoint properties (cyclicity, reflexivity)

Given a bounded sequence of positive numbers (weights) $\alpha \equiv \alpha_0, \alpha_1, \alpha_2, \ldots$, the unilateral weighted shift on $\ell^2(Z_+)$ associated with α is

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0).$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)

Berger Measures

- (Berger; Gellar-Wallen) W_α is subnormal iff there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \ \ (\text{all } k \geq 0).$$

  ξ is the Berger measure of W_α.
- For $0 < a < 1$ we let $S_a := shift\{a, 1, 1, ...\}$.
- The Berger measure of U_+ is δ_1.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

Spectral Pictures of Hyponormal U.W.S.

WLOG, assume $\|W_\alpha\| = 1$. Observe that $r(W_\alpha) = 1 = s(\alpha)$. Thus,

$$\begin{cases} \sigma(W_\alpha) = \bar{\mathbb{D}} \\ \sigma_e(W_\alpha) = \mathbb{T} \\ \operatorname{ind}(W_\alpha - \lambda) = -1 \ \text{ for } \ |\lambda| < 1. \end{cases}$$

Therefore, all norm-one hyponormal weighted shifts are spectrally equivalent.

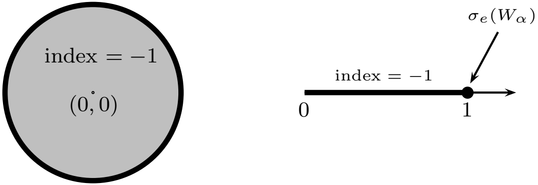


FIGURE 1. Spectral picture of a norm-one hyponormal weighted shift

Now consider $C^*(W_\alpha)$, W_α hyponormal and $\|W_\alpha\| = 1$. Since $\alpha_0 \le \alpha_1 \le ... \le \alpha_k \nearrow 1$, it follows that $I - D_\alpha = \text{diag}\ \{1 - \alpha_k\}$ is compact, so $U_+ - W_\alpha = U_+(I - D_\alpha)$ is compact, and therefore $C^*(W_\alpha)$ gives rise to an element of $Ext(\mathbb{T})$:

$$0 \to K(\ell^2(\mathbb{Z}_+)) \to C^*(W_\alpha) \to C(\mathbb{T}) \to 0.$$

k-hyponormal and n-contractive U.W.S.

(RC, 1990) W_α is k-hyponormal iff the following Hankel moment matrices are positive for $m = 0, 1, 2, \ldots$:

$$\begin{pmatrix} \gamma_m & \gamma_{m+1} & \gamma_{m+2} & \cdots & \gamma_{m+k} \\ \gamma_{m+1} & \gamma_{m+2} & & \cdots & \gamma_{m+k+1} \\ \gamma_{m+2} & \cdots & & \cdots & \gamma_{m+k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_{m+k} & \gamma_{m+k+1} & & \cdots & \gamma_{m+2k} \end{pmatrix} \geq 0.$$

(An operator matrix condition is replaced by a scalar matrix condition.)

(G. Exner, 2006) W_α is n-contractive iff

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \gamma_{m+j} \geq 0, \qquad m = 0, 1, \ldots .$$

Aluthge Transform

Let T be a Hilbert space operator, let $P := |T|$ be its positive part, and let $T = VP$ denote the canonical polar decomposition of T, with V a partial isometry and $\ker V = \ker T = \ker P$.

We define the Aluthge transform of T as

$$AT(T) := \sqrt{P}V\sqrt{P}.$$

The iterates of AT are

$$AT^{n+1} := AT(AT^n(T)) \quad (n \geq 1).$$

The Aluthge transform has been extensive studied, in terms of algebraic, structural and spectral properties.

For instance,

- (i) $T = AT(T) \Leftrightarrow T$ is quasinormal;
- (ii) (Aluthge, 1990) If $0 < p < \frac{1}{2}$ and T is p-hyponormal, then $AT(T)$ is $(p + \frac{1}{2})$-hyponormal;
- (iii) (Jung, Ko & Pearcy, 2000) If $AT(T)$ has a n.i.s., then T has a n.i.s.
- (iv) (Kim-Ko, 2005; Kimura, 2004) T has property (β) if and only if $AT(T)$ has property (β); and
- (v) (Ando, 2005) $\left\|(T - \lambda)^{-1}\right\| \geq \left\|(AT(T) - \lambda)^{-1}\right\|$ $(\lambda \notin \sigma(T))$.

On the other hand,

G. Exner (IWOTA 2006 Lecture): subnormality is not preserved under AT. Concretely, Exner proved that the Aluthge transform of the weighted shift in the following example is not subnormal.

Example

(RC, Y. Poon and J. Yoon, 2005) Let

$$\alpha \equiv \alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n + \frac{1}{2}}{2^n + 1}}, & \text{if } n \geq 1 \end{cases},$$

Then W_α is subnormal, with 3-atomic Berger measure

$$\mu = \frac{1}{3}(\delta_0 + \delta_{1/2} + \delta_1).$$

Note that the Aluthge transform of a weighted shift is again a weighted shift.

Concretely, the weights of $AT(W_\alpha)$ are

$$\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_2\alpha_3}, \sqrt{\alpha_3\alpha_4}, \cdots .$$

Define

$$W_{\sqrt{\alpha}} := \text{shift}\,(\sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \cdots).$$

Then $AT(W_\alpha)$ is the Schur product of $W_{\sqrt{\alpha}}$ and its restriction to the subspace $\bigvee\{e_1, e_2, \cdots\}$. Thus, a sufficient condition for the subnormality of $AT(W_\alpha)$ is the subnormality of $W_{\sqrt{\alpha}}$.

Moreover, shift $(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_2\alpha_3}, \cdots)$ and shift $(\sqrt{\alpha_1\alpha_2}, \sqrt{\alpha_3\alpha_4}, \cdots)$ are directly related, via Schur squaring, to shift $(\alpha_0\alpha_1, \alpha_2\alpha_3, \cdots)$ and shift $(\alpha_1\alpha_2, \alpha_3\alpha_4, \cdots)$, which are precisely the two direct summands in the canonical representation of W_α^2.

Basic Assumption: $W_{\sqrt{\alpha}}$ is subnormal.

This guarantees that both W_α and $AT(W_\alpha)$ are subnormal. Denote the Berger measures of W_α and $AT(W_\alpha)$ by μ and $AT(\mu)$, resp.

Problem

How is $AT(\mu)$ related to μ? In Exner's example, is the fact that card supp $\mu = 3$ *of significance, since it is an odd number, and the Aluthge transform of W_α is related to W_α^2?*

Other Transformations

Given W_α and $0 < p < 1$, consider the weighted shift W_{α^p} with weights

$$\alpha_i^{(p)} := (\alpha_i)^p \ (i \geq 0).$$

If W_α is subnormal, with Berger measure μ, under what conditions is W_{α^p} subnormal?

If it is, how is the Berger measure of W_{α^p} related to μ ?

For $j = 2, 3, \ldots$, the j-th Agler shift A_j is given by

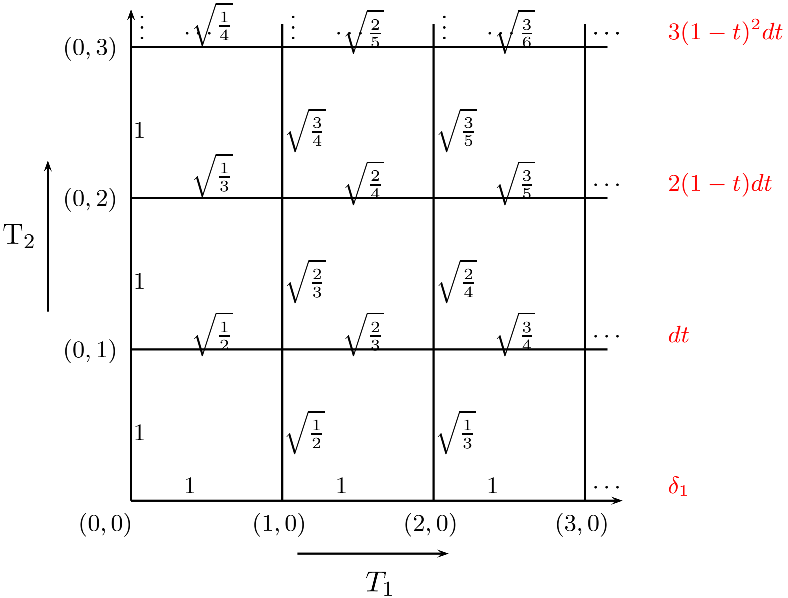
$$\alpha^j := \sqrt{\frac{1}{j}}, \sqrt{\frac{2}{j+1}}, \sqrt{\frac{3}{j+2}}, \ldots .$$

It is well known that A_j is subnormal, with Berger measure

$$d\mu^j(t) = (j-1)(1-t)^{j-2}dt.$$

Clearly, A_2 is the Bergman shift, and the remaining Agler shifts are the upper row shifts of the Drury-Arveson 2-variable weighted shift, which incidentally is a spherical complete hyperexpansion.

Weight Diagram of the Drury-Arveson Shift



Transformations of the Agler shifts under powers and Aluthge transforms turn out to be subnormal, via an approach based upon completely monotone functions. A function $f : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ is *completely monotone* if its derivatives alternate in sign: $f^{(2j)}$ is non-negative for $j \geq 1$, and $f^{(2j+1)}$ is non-positive for $j \geq 0$. Fact: If some completely monotone function f interpolates the moments of a shift in the sense that $f(m) = \gamma_m$ for all m, then the shift is subnormal. We then have:

THEOREM

(Exner, 2009) For $j = 2, 3, \ldots$, let A_j be the j-th Agler shift. Any p-th power transformation ($p > 0$) of A_j is subnormal, as is any n-th iterated Aluthge transform of A_j.

However, this mode of proof offers no information about the Berger measure of the resulting shift.

Formulation of the Central Question

Operator-theoretic formulation of the Square Root Problem

Given a subnormal weighted shift W_α, under what conditions is its "square root" $W_{\sqrt{\alpha}}$ subnormal?

By the uniqueness in the Hamburger moment problem, we may phrase the Square Root Problem as follows.

Measure-theoretic formulation of the Square Root Problem

Consider the following "moment matching" equation.

$$\int t^n \, d\mu(t) = \left(\int t^n \, d\nu(t) \right)^2, \quad n = 0, 1, 2, \ldots \quad . \tag{1}$$

The **Square Root Problem** can be stated as follows:

Given a probability measure μ (supported on a compact interval in $\mathbb{R}_+$), does there exist ν satisfying (1) ? If ν exists, can one find it?

Observe that we also have the **Square Problem**:

Given a probability measure ν (supported on a closed interval in $\mathbb{R}_+$), what is the measure μ so that (1) holds?

Obviously, there is no operator-theoretic formulation of the Square Problem, because the Schur product of two subnormal operators is always subnormal.

A Basic Case

Problem

What is the Berger measure for the square root of the Bergman shift, $\sqrt{B_+}$, with weight sequence

$$\sqrt[4]{\frac{1}{2}}, \sqrt[4]{\frac{2}{3}}, \sqrt[4]{\frac{3}{4}}, \ldots ?$$

Equivalently, what is the measure ν with the following moment matching equations:

$$\int t^n dt = \left(\int t^n d\nu(t) \right)^2 \; (n = 0, 1, 2, \ldots)?$$

A Previous Reconstruction Result

Let δ_z denote the Dirac point mass at z.

Theorem

(RC & J. Yoon, 2006) Suppose W_α *is a subnormal weighted shift with Berger measure* μ. *Then there exists a subnormal weighted shift with weight sequence* $x, \alpha_0, \alpha_1, \ldots$ *(a subnormal "back step extension") if and only if*

(i) $\frac{1}{t} \in L^1(\mu)$, *and*

(ii) $x^2 \leq (\|\frac{1}{t}\|_{L^1(\mu)|})^{-1}$. *In this case, the Berger measure for the back step extension is*

$$\frac{x^2}{t} d\mu(t) + (1 - x^2 \|\frac{1}{t}\|_{L^1(\mu)|}) d\delta_0(t).$$

Associated Moment Problem for Other Transformations

For $0 < p < 1$ consider the moment matching equations

$$\int t^n \, d\mu(t) = \left(\int t^n \, d\nu(t) \right)^p, \quad n = 0, 1, 2, \ldots \quad . \tag{2}$$

One then tries to find one measure, given the other.

The Case of Finitely Atomic Berger Measures

Definition

Let W_α be a unilateral weighted shift, and for $k \geq 0$ let γ_k denote the k-moment of α; that is,

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

W_α is said to be recursively generated if there exist real numbers $\varphi_0, \ldots, \varphi_{r-1}$ such that

$$\gamma_{r+j} = \varphi_0 \gamma_j + \ldots + \varphi_{r-1} \gamma_{r-1+j} \quad (\text{all } j \geq 0).$$

(RC & L. Fialkow, 1993) Let W_α be a subnormal, with Berger measure μ. Then μ is finitely atomic if and only if W_α is recursively generated.

A Result About the Support

Let us recast the moment matching equations in terms of product measures:

For each n,

$$\begin{aligned}\int t^n \, d\mu(t) &= \left(\int t^n \, d\nu(t) \right)^2 \\ &= \left(\int s^n \, d\nu(s) \right) \cdot \left(\int t^n \, d\nu(t) \right) \\ &= \iint s^n t^n \, d\nu(s) d\nu(t) \\ &= \iint s^n t^n \, d(\nu \times \nu).\end{aligned}$$

If we define $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $p(x, y) := xy$, we obtain that

$$\mu = (\nu \times \nu) \circ p^{-1}.$$

We can then prove the following result, which was obtained independently, and in greater generality, by J. Stochel and J.B. Stochel (2012).

Theorem

(J. Stochel & J.B. Stochel, 2012) Suppose that μ and ν are measures supported in some compact subset of $\mathbb{R}_+$ satisfying (1), and let $p(x, y) \equiv xy$ be as above. Then

(1) $\mu = (\nu \times \nu) \circ p^{-1}$

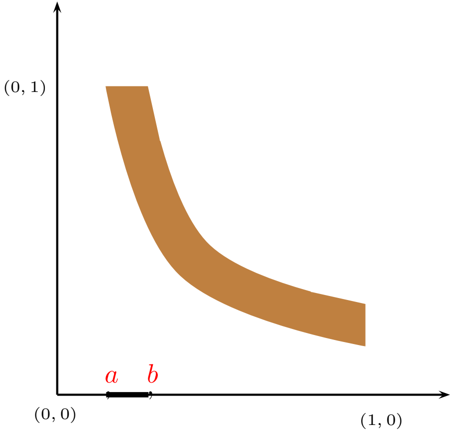
(2) $\mathrm{supp}(\mu) = \overline{(\mathrm{supp}(\nu))^2}$.

When all of the above happens, we will write $\mu = \nu^2$.

Observe that in this situation

$$\mu(E) = (\nu \times \nu)(p^{-1}(E)) = (\nu \times \nu)(\{(s,t) : st \in E\}).$$

One can give a geometrical interpretation, using the following picture (in the case when $E = [a, b]$)



Restrictions of Shifts

Given a subnormal weighted shift W_α, with Berger measure μ, given $n \geq 1$ and the invariant subspace $\mathcal{M}_n$ spanned by the basis vectors $e_n, e_{n+1}, e_{n+2}, \ldots$, it is straightforward to check that the Berger measure of $W_\alpha|_{\mathcal{M}_n}$ is $\frac{1}{\gamma_n} t^n d\mu(t)$.

Proposition

Suppose one has solved $\mu = \nu^2$ and fix $n \geq 1$; then $t^n \mu \approx (t^n \nu)^2$ (up to normalization). More precisely,

$$\left(\frac{1}{\int_0^1 t^n d\mu(t)} \right) t^n d\mu(t) = \left(\left(\frac{1}{\int_0^1 t^n d\mu(t)} \right)^{1/2} t^n d\nu(t) \right)^2 .$$

Turning to atomic measures, the Square Problem is easy. Recall that δ_z denotes the Dirac point mass at z, and, for a set S, let $S^2 = \{s \cdot t : s, t \in S\}$. The following comes simply from comparing moments.

PROPOSITION

Suppose $\nu \equiv \sum_{i=1}^{\infty} \varphi_i \delta_{x_i}$ is a probability measure. Let $X = \{x_i\}_{i=1}^{\infty}$. Then

$$\nu^2 = \sum_{z \in X^2} \Big(\sum_{x_i \cdot x_j = z} \varphi_i \varphi_j \Big) \delta_z .$$

We are interested in contraction operators and their Berger measures, that is, $\|W_\alpha\| \leq 1$; thus, our measures are probability measures supported in $[0,1]$ and, for convenience, we assume 1 is in the support. Also, we assume as well that there is an atom at the point 1.

For finitely atomic measures, there is a solution to the Square Root Problem – in fact, two – although they are not completely satisfactory.

Proposition

Let μ be a finitely atomic probability measure with support contained in $[0,1]$, and having positive mass at an atom at 1, say $\mu \equiv \sum_{i=0}^{N} \rho_i \delta_{x_i}$, where the x_i are in increasing order and $\rho_i > 0$ for all i. (Note $x_N = 1$.) If there exists ν such that $\mu = \nu^2$, then

$$\mathrm{supp}(\nu) = \begin{cases} \{0\} \cup ([\sqrt{x_1}, 1] \cap \mathrm{supp}(\mu)) & \text{if } x_0 = 0, \\ ([\sqrt{x_0}, 1] \cap \mathrm{supp}(\mu)) & \text{if } x_0 \neq 0. \end{cases}$$

If the square of the appropriate set above is not $\mathrm{supp}(\mu)$, then μ has no square root.

Proposition (cont.)

Further, given μ, one may solve "algorithmically" for the only possible "candidate" $\hat{\nu}$, by assigning the needed masses on the support of $\hat{\nu}$ one by one, in decreasing order of the point in the support. (The attempt may fail in various ways at some step, if no (positive) assignment is possible.) Alternatively, with $\{\gamma_n\}$ the moments for μ, and $\hat{S} = \{s_0, s_1, \ldots, s_{M-1}\}$ the required support set, assign masses φ_k to $\hat{\nu}$ by solving a Vandermonde type equation

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ s_0 & s_1 & \ldots & s_{M-1} \\ s_0^2 & s_1^2 & \ldots & s_{M-1}^2 \\ \ldots & \ldots & & \ldots \\ s_0^{M-1} & s_1^{M-1} & \ldots & s_{M-1}^{M-1} \end{pmatrix} \cdot \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{M-1} \end{pmatrix} = \begin{pmatrix} \sqrt{\gamma_0} \\ \sqrt{\gamma_1} \\ \sqrt{\gamma_2} \\ \vdots \\ \sqrt{\gamma_{M-1}} \end{pmatrix}. \qquad (3)$$

Proposition (cont.)

This candidate need not be successful, and all but one of the masses of $\hat{\nu}^2$ must be compared to those of μ to determine if this candidate is satisfactory. Alternatively, we must "match" (at least) $N+1$ (the cardinality of the support of μ) of the square roots of the moments of μ (equivalently, extend the equation in (3) to have the obvious $N+1$ rows).

Corollary

Neither a two-atomic nor a four-atomic measure can have a square root, unless one atom is at zero.

(The case of a two-atomic measure has been obtained independently by S.H. Lee, W.Y. Lee & J. Yoon, 2013.)

Recall

Example

(RC, Y. Poon and J. Yoon, 2005) Let

$$\alpha \equiv \alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n+\frac{1}{2}}{2^n+1}}, & \text{if } n \geq 1 \end{cases},$$

Then W_α is subnormal, with 3-atomic Berger measure $\mu = \frac{1}{3}(\delta_0 + \delta_{1/2} + \delta_1)$.

This 3-atomic measure μ does not have a square root: for, if ν is such that $\nu^2 = \mu$ then

$$\text{supp}(\nu) = \{0\} \cup ([\sqrt{\frac{1}{2}}, 1] \cap \text{supp}(\mu)) = \{0, 1\}.$$

If we now square this set, we get $\{0, 1\} \neq \text{supp}(\mu)$, a contradiction.

Remark

(Measures with a geometric series as support). For some measures whose support is $\{r^n : n \geq 0\}$ for some $0 < r < 1$, we can obtain some results by re-interpreting the questions as one of generating functions. If

$$\mu = \sum_{n=0}^{\infty} \rho_n \delta_{r^n} \quad \text{and} \quad \nu = \sum_{n=0}^{\infty} \varphi_n \delta_{r^n},$$

to say $\mu = \nu^2$ is equivalent to

$$\sum_{n=0}^{\infty} \rho_n x^n = \left(\sum_{n=0}^{\infty} \varphi_n x^n \right)^2 .$$

One solves by coefficient matching.

Example

Sometimes the generating function approach gives an immediate answer. If $\rho_n = \left(\frac{1}{2}\right)^{n+1}$, then the generating function is

$$\frac{1}{2} + \frac{1}{2^2}z + \frac{1}{2^3}z^2 + \ldots = \frac{1}{2-z}.$$

But the expansion for its square root is

$$\frac{1}{\sqrt{2-z}} = \frac{\sqrt{2}}{2} + \frac{1}{8}z + \frac{3}{64}z^2 + \ldots$$

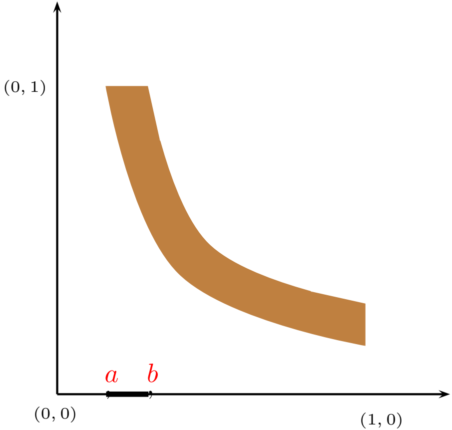
and the coefficients give the masses for the square root measure.

Absolutely Continuous Measures

Assume we wish to solve the Square Problem $\mu = \nu^2$, and we know that ν is absolutely continuous with respect to Lebesgue measure on $[0,1]$. Write $d\nu(t) \equiv g(t)dt$, with g the Radon-Nikodym derivative in L^1. It appears reasonable to hope that μ will also be absolutely continuous, and to pursue its Radon-Nikodym derivative, call it f. Now, for $0 < a < 1$, we have (almost everywhere)

$$f(a) = \lim_{n\to\infty} \frac{\mu\left([a, a+1/n]\right)}{1/n}.$$

Now recall that $\mu(E) = (\nu \times \nu)(p^{-1}(E))$, and that the inverse image of a small interval is a small hyperbolic slice in the plane:



$$\begin{aligned}
\mu([a, a+1/n]) &= \int\int_{p^{-1}(E)} 1\, d\nu \times d\nu \\
&= \int_a^{a+1/n} \int_{a/x}^{1} 1\ g(y)g(x)\, dy\, dx + \\
&\quad + \int_{a+1/n}^{1} \int_{a/x}^{(a+1/n)/x} 1\ g(y)g(x)\, dy\, dx \\
&= \int_a^{1} \int_{a/x}^{(a+1/n)/x} 1\ g(y)g(x)\, dy\, dx,
\end{aligned}$$

(where we have used that g vanishes outside $[0, 1]$). Then with f the Radon-Nikodym derivative of μ, we have:

$$
\begin{aligned}
f(a) &= \lim_{n\to\infty} \frac{\mu([a, a+1/n])}{1/n} \\
&= \lim_{n\to\infty} \left(\frac{1}{1/n}\right) \int_a^1 \int_{a/x}^{(a+1/n)/x} 1 \; g(y) g(x) \, dy \, dx \\
&= \int_a^1 \left(\lim_{n\to\infty} \frac{\int_{a/x}^{(a+1/n)/x} 1 \; g(y) \, dy}{\frac{1}{nx}} \right) \frac{1}{x} g(x) \, dx \\
&= \int_a^1 g(\frac{a}{x}) g(x) \frac{1}{x} \, dx \quad \text{(a.e.).}
\end{aligned}
$$

We thus have:

Proposition

Let $\mu = \nu^2$ and assume that $d\nu(t) = g(t)dt$, with $g \in L^1([0,1])$. Then $d\mu(t) = f(t)dt$, where

$$f(a) = \int_a^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx \quad \text{(a.e.)}.$$

Remark

For future use, we note that, since g vanishes outside $[0,1]$, one has

$$\int_a^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx = \int_0^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx. \tag{4}$$

This displays the integral as a convolution $g * g$ with respect to Haar measure on the multiplicative semigroup $(0,1)$.

As a consequence, we get a concrete expression for f if g is a polynomial.

Theorem

Suppose that g is a polynomial, say $g(x) \equiv \sum_{i=0}^{n} a_i x^i$, positive on $[0,1]$ and inducing a probability measure $d\nu(t) = g(t)\chi_{(0,1)}(t)dt$. Then $\mu = \nu^2$ is absolutely continuous with respect to Lebesgue measure and with Radon-Nikodym derivative $f \cdot \chi_{0,1)}$ where f is given by

$$\begin{aligned} f(x) &= \sum_{i=0}^{n-1} \left(\left(\frac{1-x^{i+1}}{i+1} \right) \sum_{j=0}^{n-i-1} a_j a_{j+i+1} x^j \right) + \\ & -\ln x \cdot \sum_{i=0}^{n} a_i^2 x^i \\ & + \sum_{i=-n-1}^{-2} \left(\left(\frac{1-x^{i+1}}{i+1} \right) \sum_{j=-i-1}^{n} a_j a_{j+i+1} x^j \right). \end{aligned}$$

There are various consequences of this that seem (at least to us) somewhat surprising; here, given the simple nature of $1\,dt$, is one.

Corollary

Let ν be the measure on $[0,1]$ given by $d\nu(t) = 1\,dt$; i.e., ν is Lebesgue measure on $[0,1]$. Then ν^2 is given by $-\ln t\,dt$. In particular, $(1\,dt)^2$ is singular at the origin.

Remark

We also observe that the square of any polynomial measure is singular at the origin and vanishes at 1. As well, of course, the square root of $1\,dt$ cannot possibly be a polynomial measure, which makes it unlikely that it arises even from a function continuous on $[0,1]$. Some numerical experiments using *Mathematica* suggest that what is needed is some function zero at 0 and with a singularity at 1.

We pause to record a curiosity; the only information below not available merely from moment computations is the Berger measure associated with the square of $1\,dt$.

Corollary

The Aluthge transform of the weighted shift associated with the Berger measure $-\ln t\,dt$ *is* A_3*, the third Agler shift; that is,*

$$AT(A_2^{(2)}) = A_3.$$

This result does not hold for other Agler shifts.

Laplace Transform Method

Theorem

(A. Athavale, 2013) The square root measure for $1\,dt$ *(on* $[0,1]$*) is* $\frac{1}{\sqrt{\pi}}(-\ln t)^{(-1/2)}\,dt$.

The key to this and allied results is the Laplace transform and the movement of the moment problem from the interval $[0,1]$ to the interval $[0,\infty)$. Recall that the Laplace transform of a function h is $H \equiv \mathcal{L}\{h\}$ where

$$H(s) := \int_0^\infty e^{-st} h(t)\,dt.$$

Now suppose that $F := \mathcal{L}\{g\}$. Then

$$\begin{aligned} F(s+1) &= \mathcal{L}\{e^{-t}g(t)\}(s) \quad \text{“First Shifting Theorem”} \\ &= \int_0^\infty e^{-st}e^{-t}g(t)\,dt \\ &= \int_0^1 u^{s+1}\frac{1}{u}f(u)\,du \quad (u := e^{-t}, f(u) := g(-\ln u)) \\ &= \int_0^1 u^s f(u)\,du. \end{aligned}$$

Thus, if $\gamma(s) := F(s+1)$ interpolates some sequence (γ_n) we wish to "match," the Inverse Laplace Transform, appropriately shifted to $[0, 1]$, is the Radon-Nikodym derivative f of the measure we seek. For example, we may obtain the qth power of $1\,dt$ for any positive q, subsuming Athavale's result for the square root $(q = 1/2)$ and also the earlier result for $(1\,dt)^2$.

Theorem

The q-th power of $1\,dt$ (on $[0, 1]$) for $q > 0$ is

$$f(u)\,du = \frac{1}{\Gamma(q)}(-\ln u)^{q-1}\,du$$

(where Γ denotes the classical Gamma function). Alternatively, the Berger measure of the weighted shift whose moment sequence is $\left(\sqrt[q]{\frac{1}{n+1}}\right)_{n=1}^{\infty}$ is $\frac{1}{\Gamma(q)}(-\ln u)^{q-1}\,du$.

One obtains some odd results by this route.

Example

For a constant $c > 0$, is

$$\gamma_n = \frac{e^{-c/\sqrt{(n+1)}}}{e^{-c}}$$

the moment sequence of a subnormal shift? Showing that f given by

$$f(x) = \frac{e^{-c/\sqrt{(x+1)}}}{e^{-c}}$$

is completely monotone by examining signs of the derivatives is computationally infeasible. But the answer is "yes," because $e^{-c/\sqrt{s}}$ is the Laplace transform of

$$\frac{c}{2\sqrt{\pi t^3}} e^{-c^2/4t}.$$

Also, while every Laplace transform table entry is potentially interesting, the results need not be very friendly. For example, for the square root of A_3 (corresponding to the measure $2(1-t)\,dt$), the square root having moments $\frac{\sqrt{2}}{\sqrt{n+1}\sqrt{n+2}}$, the appropriate measure involves the "modified Bessel function of the first kind of order 0" and is

$$\sqrt{2(1-t)\,dt} = \sqrt{2}e^{(\ln u)/2} \sum_{m=0}^{\infty} \frac{((\ln u)/2)^{2m}}{2^{2m}(m!)^2}\,du. \tag{5}$$

Note that the relationship here is not between Laplace transforms and moment sequences but between Laplace transforms and functions that interpolate moment sequences. This is conceptually right as shown by a result of Hausdorff-Bernstein-Widder:

Theorem

A function is completely monotone if and only if it is the Laplace transform of a measure.

Note also that if we take the fundamental relationship

$$\gamma_n = \int_0^1 t^n \, d\mu(t)$$

and turn it into

$$\gamma(s) = \int_0^1 t^s \, d\mu(t)$$

the resulting function γ is completely monotone (if perhaps not readily computable) as seen by moving s derivatives inside the integral.

Fourier Transform Method

Recall that the relation

$$f(a) = \int_a^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx = \int_0^1 g(\frac{a}{x})g(x)\frac{1}{x}\,dx$$

between Radon-Nikodym derivatives of a measure and its square may be regarded as the convolution of g with itself with respect to Haar measure on the multiplicative semi-group $(0,1)$. This can be reinterpreted as saying that the back step extension of a shift corresponding to $g\,dt$ is subnormal if and only if $g \in \mathcal{L}^1((0,1), \mathrm{dHaar})$. This is consistent with the RC & Yoon result on reconstruction of the Berger measure.

This also motivates consideration of the Fourier transform (including the transform of a measure), and the well known fact that convolution gets mapped to the product of the Fourier transforms. This looks potentially useful for the Square and Square Root Problems. However, there are two potential drawbacks: (i) harmonic analysis on semigroups is not primarily designed to be computational, and we are interested in concrete expressions for measures; and (ii) results often assume that some function is in $\mathcal{L}^1$, and some of the functions we are particularly interested in are not (for example, "1"). Luckily, by moving the problem to $[0, \infty)$ by the change of variables $t = -\ln x$ as we did to employ the Laplace transform, we may have something of the best of both worlds.

Some Notation:

We will consider two domains, $(0,1)$ and $\mathbb{R}$, and we will reserve t for $(0,1)$ and x for $\mathbb{R}$. We will also use $(0,+\infty)$ and assume that $t = e^{-x}$, or equivalently, $x = -lnt$. We will make use for the Heaviside function, denoted H, given by

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

and occasional use the Signum function Sgn,

$$\mathrm{Sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

We will use "$*$" for the ordinary (additive) convolution on $\mathbb{R}$, and "$*_I$" for the multiplicative convolution on $(0,1)$. To codify this,

(a) for f defined on $(0,\infty)$ we let $L(f)(t) := f(-\ln t)$; and

(b) for f defined on $(0,1)$, let $E(f)(x) := f(e^{-x})$.

(c) Clearly,

$$h = f *_I g \Leftrightarrow E(h) = E(f) * E(g), \qquad f, g, h \text{ on } (0,1),$$

and

$$h = f * g \Leftrightarrow L(h) = L(f) *_I L(g), \qquad f, g, h \text{ on } (0,\infty).$$

Also, $f \cdot \chi_{(0,1)}$ is transformed to $E(f) \cdot H$ and $f \cdot H$ to $L(f) \cdot \chi_{(0,1)}$.

As a consequence, if we wish to solve (in either direction)

$$f\chi_{(0,1)} = (g\chi_{(0,1)}) *_I (g\chi_{(0,1)}),$$

it is equivalent to solve some related equation

$$f \cdot H = (g \cdot H) * (g \cdot H)$$

in the familiar setting of $\mathbb{R}$ and under the appropriate changes of variable using L and E.

To find solutions we will use the Fourier transform. Recall that for $f \in L^1(\mathbb{R})$, its Fourier transform is

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x s} \, dx.$$

Recall also that the Fourier transform is routinely extended to a map from the class of "tempered distributions" to itself. This class includes objects such as the Dirac delta at x (denoted δ_x) and "$\frac{1}{x}$". It is well known that

$$\widehat{f * g} = \hat{f} \hat{g}.$$

Let's examine if we can deal with the specific case of

$$\sqrt{(1\chi_{(0,1)}\,dt)} = \frac{1}{\sqrt{\pi}}(-\ln t)^{-1/2}\chi_{[0,1]}\,dt$$

via a Fourier transform approach. We must

(i) change domains;

(ii) take the Fourier transforms;

(iii) square the right hand side; and

(iv) compare.

This yields

$$\begin{aligned} 1\chi_{(0,1)} \text{ on } (0,1) &\to H(x) \text{ on } (0,\infty) \\ &\to \left(\frac{1}{2}\right)\left(\delta_0(s) - \frac{1}{i\pi s}\right) \end{aligned}$$

in transform space,

and

$$\begin{aligned}
\frac{1}{\sqrt{\pi}}(-\ln t)^{-1/2}\chi_{[0,1]} \text{ on } (0,1) \quad &\rightarrow \quad \frac{1}{\sqrt{\pi}}\frac{1}{(-\ln(e^{-x}))^{(1/2)}}H(x) \text{ on } (0,\infty) \\
&= \quad \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{x}}H(x) \text{ on } (0,\infty) \\
&\rightarrow \quad \left(\frac{1}{2\sqrt{\pi}|s|^{1/2}}\right)(1 - i\mathrm{Sgn}(s))
\end{aligned}$$

in transform space.

We must therefore compare

$$\left(\frac{1}{2}\right)\left(\delta_0(s) - \frac{1}{i\pi s}\right)$$

with

$$\left(\frac{1}{2\sqrt{\pi}|s|^{1/2}}\right)^2 (1 - i\mathrm{Sgn}(s))^2 = \frac{-i\mathrm{Sgn}(s)}{2\pi|s|} + \left(\frac{1}{4\pi|s|}\right)(1 - (\mathrm{Sgn}(s))^2).$$

This presents us with the possible identity

$$\frac{\delta_0(s)}{2} = \left(\frac{1}{4\pi|s|}\right)(1 - (\mathrm{Sgn}(s))^2),$$

and we can find no interpretation of the right hand side that renders it meaningful, even as a tempered distribution.

On the other hand, it is indeed possible to check that

$$2t\chi_{(0,1)}\,dt = \left(\frac{\sqrt{2}}{\sqrt{\pi}}t(-\ln t)^{-1/2}\chi_{[0,1]}\,dt\right)^2,$$

that is, a proof using Fourier transforms works for the restrictions to the subspace spanned by $e_1, e_2, \ldots$.

What has gone wrong? First, the product of two tempered distribution need not be a tempered distribution, e.g., δ_0^2 is not well defined; second, if we wish to view things from the viewpoint of semi-algebras and convolutions, we must have functions in $\mathcal{L}^1$, and $1\chi_{(0,1)}\, dt$ is not in $\mathcal{L}^1 \equiv L^1((0,1), d\mathrm{Haar}) = L^1((0,1), \frac{1}{t}\, dt)$. That is, the computational identity

$$f(a) = \int_a^1 g(\frac{a}{x})g(x)\frac{1}{x}\, dx = \int_0^1 g(\frac{a}{x})g(x)\frac{1}{x}\, dx$$

includes functions (e.g., $1\chi_{(0,1)}$) which might not be suitable for transform methods.

What muddies matters still further, however, is the fact that sometimes things work anyway. Viewing

$$\delta_0(x) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi(x^2 + \epsilon^2)},$$

and the tempered distribution

$$\frac{1}{x^2} = \lim_{\epsilon \to 0} \frac{x^2 - \epsilon^2}{(x^2 + \epsilon^2)},$$

one can then check that

$$(1\chi_{[0,1]}\, dt)^2 = -\ln t \chi_{[0,1]}\, dt$$

via this approach.

Thus, at present the Fourier transform method is heuristic in nature, and it works well to find a good educated guess for the solution, which can later be checked by matching moments.

We end with the following "recognition of a square" result, which we discovered by taking the Fourier transform, squaring, and taking the inverse Fourier transform with the aid of *Mathematica*.

Proposition

Let t_0 *be in* $[0,1]$ *and let* m *be a non-negative integer. Then for* $0 < \lambda < 1$,

$$\begin{aligned}(\lambda\delta_{t_0} &+ (1-\lambda)(m+1)t^m\chi_{(0,1)}(t)\,dt)^2 \\ &= \lambda^2\delta_{2t_0} + (1-\lambda)^2(m+1)^2(-\ln t)t^m\chi_{(0,1)}(t)\,dt \\ &\quad +2\lambda(1-\lambda)(m+1)\frac{t^m}{t_0^{m+1}}\chi_{(0,t_0)}(t)\,dt.\end{aligned}$$

Proof. Compute moments.

Summary

- The central problem is to find ν given μ in the equation $\mu = \nu^2$.
- We successfully solve it for finitely atomic μ, although the solution is laborious.
- We obtain a convolution condition when μ is absolutely continuous with respect to Lebesgue measure on $[0, 1]$.
- We use Laplace transform methods for more general measures.
- Fourier transform methods are used heuristically, to find good guesses for solutions, which then must be verified by computing moments.

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