A New Necessary Condition for the Hyponormality of Toeplitz Operators on the Bergman Space

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Abstract

It is a well known result of C. Cowen that, for a symbol

$$\varphi \in L^\infty,\ \varphi \equiv \bar{f} + g \ \ (f, g \in H^2),$$

the Toeplitz operator T_φ acting on the Hardy space of the unit circle is hyponormal if and only if

$$f = c + T_{\bar{h}} g,$$

for some $c \in \mathbb{C}$, $h \in H^\infty$, $\|h\|_\infty \leq 1$. In this talk we will consider possible versions of this result in the *Bergman space* case.

Concretely, we consider Toeplitz operators on the Bergman space of the unit disk, with symbols of the form

$$\varphi \equiv \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}_+$, $m < n$ and $p < q$. By letting T_φ act on vectors of the form

$$z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

we study the asymptotic behavior of a suitable matrix of inner products, as $k \to \infty$. As a consequence, we obtain a rather sharp inequality involving the above mentioned data:

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 \left| \bar{\alpha}\beta mn - \bar{\gamma}\delta pq \right|.$$

This result is intended to be a precursor of basic necessary conditions for joint hyponormality of commuting tuples of Toeplitz operators acting on Bergman spaces in several complex variables.

Notation and Preliminaries

$T \in \mathcal{L}(\mathcal{H})$: algebra of bounded operators on a Hilbert space $\mathcal{H}$

- normal if $T^*T = TT^*$
- quasinormal if T commutes with T^*T
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$
- 2-hyponormal if (T, T^2) is (jointly) hyponormal ($k \geq 1$)

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] \\ [T^*, T^2] & [T^{*2}, T^2] \end{pmatrix} \geq 0$$

normal $\Rightarrow$ quasinormal $\Rightarrow$ subnormal $\Rightarrow$ 2-hypon. $\Rightarrow$ hypon.

$L^\infty \equiv L^\infty(\mathbb{T})$; $H^\infty \equiv H^\infty(\mathbb{T})$; $L^2 \equiv L^2(\mathbb{T})$; $H^2 \equiv H^2(\mathbb{T})$,
$P : L^2 \to H^2$ orthogonal projection

For $\varphi \in L^\infty$, the Toeplitz operator with symbol φ is
$T_\varphi : H^2 \to H^2$, given by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2)$$

T_φ is said to be *analytic* if $\varphi \in H^\infty$

Halmos's Problem 5 (1970):

Is every subnormal Toeplitz operator either normal or analytic?

C. Cowen and J. Long (1984): No

- C. Cowen (1988)
  $\varphi \in L^\infty$, $\varphi = \bar{f} + g \;\; (f, g \in H^2)$

$$T_\varphi \textit{ is hyponormal } \Leftrightarrow \; f = c + T_{\bar{h}} g,$$

  for some $c \in \mathbb{C}$, $h \in H^\infty$, $\|h\|_\infty \leq 1$.
- Nakazi-Takahashi (1993)
  For $\varphi \in L^\infty$, let

$$\mathcal{E}(\varphi) := \{k \in H^\infty : \|k\|_\infty \leq 1 \textit{ and } \varphi - k\bar{\varphi} \in H^\infty\}.$$

  Then

$$T_\varphi \textit{ is hyponormal } \Leftrightarrow \; \mathcal{E}(\varphi) \neq \emptyset.$$

Natural Questions:

1) When is T_φ subnormal?

At present, there's no known characterization of subnormality in terms of the symbol φ.

2) Characterize 2-hyponormality for Toeplitz operators

Sample Result:

Theorem

(RC and WY Lee, 2001) Every 2-hyponormal **trigonometric** *Toeplitz operator is subnormal.*

Hyponormal Toeplitz Pairs

In 2001, RC and W.Y. Lee completely characterized the hyponormality of Toeplitz pairs $\mathbf{T} \equiv (T_\varphi, T_\psi)$, when both symbols φ and ψ are trigonometric polynomials.

Key building block: For φ and ψ trigonometric polynomials, the hyponormality of $\mathbf{T} \equiv (T_\varphi, T_\psi)$ forces the co-analytic parts of φ and ψ to necessarily coincide up to a constant multiple; equivalently,

$$[ECAP] \qquad \varphi - \beta\psi \in H^2 \text{ for some } \beta \in \mathbb{C}.$$

ECAP: Equality (up to a constant multiple) of Co-Analytic Parts

Hyponormal Toeplitz Operators on the Bergman Space

$L^\infty \equiv L^\infty(\mathbb{D})$; $H^\infty \equiv H^\infty(\mathbb{D})$; $L^2 \equiv L^2(\mathbb{D})$; $A^2 \equiv A^2(\mathbb{D})$,
$P : L^2 \to A^2$ orthogonal projection

For $\varphi \in L^\infty$, the Toeplitz operator on the Bergman space with symbol φ is

$$T_\varphi : A^2(\mathbb{D}) \to A^2(\mathbb{D}),$$

given by

$$T_\varphi f := P(\varphi f) \quad (f \in A^2).$$

T_φ is said to be *analytic* if $\varphi \in H^\infty$.

Trigonometric Symbols in the Hardy Space Case

Proposition

Let φ be a trigonometric polynomial of the form

$$\varphi(z)=\sum_{n=-m}^{N} a_n z^n .$$

(i) *(D. Farenick & W.Y. Lee, 1996) If T_φ is a hyponormal operator then $m \leq N$ and $|a_{-m}| \leq |a_N|$.*

(ii) *(D. Farenick & W.Y. Lee, 1996) If T_φ is a hyponormal operator then $N-m \leq \operatorname{rank}\,[T_\varphi^*, T_\varphi] \leq N$.*

Trigonometric Symbols in the Hardy Space Case

Proposition (cont.)

(iii) *(RC & W.Y. Lee, 2001) The hyponormality of T_φ is independent of the particular values of the Fourier coefficients $a_0, a_1, \ldots, a_{N-m}$ of φ. Moreover, for T_φ hyponormal, the rank of the self-commutator of T_φ is independent of those coefficients.*

Trigonometric Symbols in the Hardy Space Case

Proposition (cont.)

(iv) *(D. Farenick & W.Y. Lee, 1996) If $m \le N$ and $|a_{-m}| = |a_N| \neq 0$, then T_φ is hyponormal if and only if the following equation holds:*

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \bar{a}_{N-m+1} \\ \bar{a}_{N-m+2} \\ \vdots \\ \vdots \\ \bar{a}_N \end{pmatrix}. \quad \text{(hypon.)}$$

In this case, the rank of $[T_\varphi^*, T_\varphi]$ is $N - m$.

Trigonometric Symbols in the Hardy Space Case

Proposition (cont.)

(v) *(D. Farenick & W.Y. Lee, 1996)* T_φ *is normal if and only if* $m = N$, $|a_{-N}| = |a_N|$, *and* (hypon.) *holds with* $m = N$.

A Revealing Example

Let

$$\varphi \equiv \bar{z}^2 + 2z.$$

On the Hardy space $H^2(\mathbb{T})$, T_φ is not hyponormal, because $m = 2$, $N = 1$, and $m > N$.

However, on the Bergman space $A^2(\mathbb{D})$ T_φ is hyponormal, as we now prove. Consider a slight variation of the symbol, that is,

$$\varphi \equiv \bar{z}^2 + \alpha z. \quad (\alpha \in \mathbb{C})$$

.

Observe that

$$\left\langle [T_\varphi^*, T_\varphi] f, f \right\rangle = \left\langle |\alpha|^2 [T_{\bar{z}}, T_z] + [T_{z^2}, T_{\bar{z}^2}] f, f \right\rangle$$

so that T_φ is hyponormal if and only if

$$|\alpha|^2 \|zf\|^2 + \left\langle P(\bar{z}^2 f), \bar{z}^2 f \right\rangle \geq |\alpha|^2 \langle P(\bar{z} f), \bar{z} f \rangle + \left\| z^2 f \right\|^2$$

for all $f \in A^2(\mathbb{D})$.

A calculation, using the result on the next page, shows that this happens precisely when $|\alpha| \geq 2$. As a result, $T_{\bar{z}^2 + 2z}$ is hyponormal.

Key Difference Between Hardy and Bergman Cases

Lemma

For $u, v \geq 0$, *we have*

$$P(\bar{z}^u z^v) = \begin{cases} 0 & v < u \\ \frac{(v-u+1)}{v+1} z^{v-u} & v \geq u \end{cases}.$$

Proof.

$$\begin{aligned} P(\bar{z}^u z^v) &= \sum_{j=0}^{\infty} \left\langle \bar{z}^u z^v, \frac{z^j}{\|z^j\|} \right\rangle \frac{z^j}{\|z^j\|} \\ &= \sum_{j=0}^{\infty} \frac{\left\langle \bar{z}^u z^v, z^j \right\rangle z^j}{\|z^j\|^2} = \sum_{j=0}^{\infty} (j+1) \left\langle z^v, z^{u+j} \right\rangle z^j \\ &= \begin{cases} 0 & v < u \\ \frac{v-u+1}{v+1} z^{v-u} & v \geq u \end{cases}. \end{aligned}$$

$\square$

Corollary

For $v \geq u$ and $t \geq w$, we have

$$\begin{aligned}\left\langle P(\bar{z}^u z^v), P(\bar{z}^w z^t)\right\rangle &= \left\langle \frac{v-u+1}{v+1} z^{v-u}, \frac{t-w+1}{t+1} z^{t-w} \right\rangle \\ &= \frac{(t-w+1)}{(v+1)(t+1)} \delta_{u+t,v+w}.\end{aligned}$$

Some Known Results

(H. Sadraoui, PhD Thesis, Purdue Univ., 1992)

If $\varphi \equiv \bar{g} + f$, the following are equivalent:

(i) T_φ is hyponormal on $A^2(\mathbb{D})$;

(ii) $H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}$;

(iii) $H_{\bar{g}} = C H_{\bar{f}}$, where C is a contraction on $A^2(\mathbb{D})$.

(I.S. Hwang, J. Korean Math. Soc., 2005)

Let $\varphi \equiv a_{-m}\bar{z}^m + a_{-N}\bar{z}^N + a_m z^m + a_N z^N$ $(0 < m < N)$ satisfying $a_m \bar{a}_N = \bar{a}_{-m} a_{-N}$, then T_φ is hyponormal if and only if

$$\begin{aligned} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) &\geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) \ \ (\text{if } |a_{-N}| \leq |a_N|) \\ N^2(|a_{-N}|^2 - |a_N|^2) &\leq m^2(|a_m|^2 - |a_{-m}|^2) \ \ (\text{if } |a_N| \leq |a_{-N}|). \end{aligned}$$

The last condition is not sufficient.

(I.S. Hwang, J. Korean Math. Soc., 2008) Let $\varphi \equiv 4\bar{z}^3 + 2\bar{z}^2 + \bar{z} + z + 2z^2 + \beta z^3$ $(|\beta| = 4)$. Then T_φ is hyponormal if and only if $\beta = 4$.

(I.S. Hwang, J. Korean Math. Soc., 2008) Let

$$\varphi \equiv 8\bar{z}^3 + \bar{z}^2 + \beta\bar{z} + \gamma z + 7z^2 + 2z^3 \quad (|\beta| = |\gamma|).$$

Then T_φ is not hyponormal.

(P. Ahern & Ž Čučković, Pacific J. Math., 1996) Let $\varphi \equiv \bar{g} + f \in L^\infty(\mathbb{D})$, and assume that T_φ is hyponormal. Then

$$T_\varphi u \geq u,$$

where $u := |f|^2 - |g|^2$.

(P. Ahern & Ž Čučković, Pacific J. Math., 1996) Let $\varphi \equiv \bar{g} + f \in L^{infty}(\mathbb{D})$, and assume that T_φ is hyponormal. Then

$$\bar{lim}_{z \to \zeta} (|f'(z)|^2 - |g'(z)|^2) \geq 0$$

for all $\zeta \in \mathbb{T}$. In particular, if f' and g' are continuous at ζ then $|f'(\zeta)| \geq |g'(\zeta)|$.

(I.S. Hwang & J. Lee, Math. Ineq. Appl., 2012) studied hyponormality on weighted Bergman spaces.

(Y. Lu & C. Liu, J. Korean Math. Soc., 2009) considered radial symbols.

(Y. Lu & Y. Shi, IEOT, 2009) studied the weighted Bergman space case.

Hyponormality of Toeplitz Operators on the Bergman Space

The self-commutator of T_φ is

$$C := [T_\varphi^*, T_\varphi]$$

We seek necessary and sufficient conditions on the symbol φ to ensure that $C \geq 0$.

The following result gives a flavor of the type of calculations we face when trying to decipher the hyponormality of a Toeplitz operator acting on the Bergman space. Although the calculation therein will be superseded by the calculations in the following section, it serves both as a preliminary example and as motivation for the organization of our work.

Proposition

Assume $k, \ell \geq \max\{a, b\}$. *Then*

$$
\begin{aligned}
& \left\langle [T_{\bar{z}^a}, T_{z^b}] (z^k + cz^\ell), z^k + cz^\ell \right\rangle \\
= \quad & a^2 \left[\frac{1}{(k+1)^2(k+1+a)} + c^2 \cdot \frac{1}{(\ell+1)^2(\ell+1+a)} \right] \delta_{a,b} \\
& + ac[\frac{k-\ell+a}{(a+k+1)(k+1)(\ell+1)} \delta_{a+k,b+\ell} \\
& + \frac{\ell-k+a}{(a+\ell+1)(k+1)(\ell+1)} \delta_{a+\ell,b+k}]
\end{aligned}
$$

Corollary

Assume $a = b$, $k, \ell \geq a$ and $k \neq \ell$. Then

$$\begin{aligned} & \left\langle [T_{\bar{z}^a}, T_{z^a}] (z^k + cz^\ell), z^k + cz^\ell \right\rangle \\ = \quad & a^2 \left[\frac{1}{(k+1)^2(k+1+a)} + c^2 \cdot \frac{1}{(\ell+1)^2(\ell+1+a)} \right] \end{aligned}$$

Self-Commutators

We focus on the action of the self-commutator C of certain Toeplitz operators T_φ on suitable vectors f in the space $A^2(\mathbb{D})$. The symbol φ and the vector f are of the form

$$\varphi := \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q \ \ (n > m;\ p < q)$$

and

$$f := z^k + cz^\ell + dz^r \ \ (k < \ell < r),$$

respectively. Our ultimate goal is to study the asymptotic behavior of this action as k goes to infinity. Thus, we consider the expression $\langle Cf, f\rangle$, given by

$$\left\langle [(T_{\alpha z^n+\beta z^m+\gamma \overline{z}^p+\delta \overline{z}^q})^*, T_{\alpha z^n+\beta z^m+\gamma \overline{z}^p+\delta \overline{z}^q}](z^k + cz^\ell + dz^r), z^k + cz^\ell + dz^r \right\rangle,$$

for large values of k (and consequently large values of ℓ and r.) It is straightforward to see that $\langle Cf, f\rangle$ is a quadratic form in c and d, that is,

$$\langle Cf, f\rangle \equiv A_{00} + 2\,\mathrm{Re}(A_{10}c) + 2\,\mathrm{Re}(A_{01}d) + A_{20}c\bar{c} + 2\,\mathrm{Re}(A_{11}c\bar{d}) + A_{02}d\bar{d}. \tag{1}$$

Alternatively, the matricial form of (1) is

$$\left\langle \begin{pmatrix} A_{00} & A_{10} & A_{01} \\ \overline{A}_{10} & A_{20} & \overline{A}_{11} \\ \overline{A}_{01} & A_{11} & A_{02} \end{pmatrix} \begin{pmatrix} 1 \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} \right\rangle . \tag{2}$$

We now observe that the coefficient A_{00} arises from the action of C on the monomial z^k, that is,

$$A_{00} = \left\langle Cz^k, z^k \right\rangle \equiv \left\langle [(T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q})^*, T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q}] z^k, z^k \right\rangle .$$

Similarly,

$$A_{10} = \left\langle Cz^\ell, z^k \right\rangle ,$$

$$A_{01} = \left\langle Cz^r, z^k \right\rangle ,$$

$$A_{20} = \left\langle Cz^\ell, z^\ell \right\rangle ,$$

$$A_{11} = \left\langle Cz^r, z^\ell \right\rangle ,$$

$$A_{02} = \left\langle Cz^r, z^r \right\rangle .$$

To calculate A_{00} explicitly, we first recall that the algebra of Toeplitz operators with analytic symbols is commutative, and therefore T_{z^n} commutes with T_{z^m}, T_{z^p} and T_{z^q}.

We also recall that two monomials z^u and z^v are orthogonal whenever $u \neq v$. As a result, the only nonzero contributions to A_{00} must come from the inner products $\left\langle [T_{z^n}^*, T_{z^n}]z^k, z^k \right\rangle$, $\left\langle [T_{z^m}^*, T_{z^m}]z^k, z^k \right\rangle$, $\left\langle [T_{z^p}^*, T_{z^p}]z^k, z^k \right\rangle$ and $\left\langle [T_{z^q}^*, T_{z^q}]z^k, z^k \right\rangle$.

Applying Corollary 7 we see that

$$A_{00} = \frac{1}{(k+1)^2} \left(\frac{|\alpha|^2 n^2}{k+n+1} + \frac{|\beta|^2 m^2}{k+m+1} - \frac{|\gamma|^2 p^2}{k+p+1} - \frac{|\delta|^2 q^2}{k+q+1} \right).$$

Similarly,

$$\begin{aligned} A_{10} &= \overline{\alpha}\beta\left(\frac{1}{\ell+m+1}-\frac{k-m+1}{(k+1)(\ell+1)}\right)\delta_{n+k,m+\ell} \\ &+\alpha\overline{\beta}\left(\frac{1}{\ell+n+1}-\frac{k-n+1}{(k+1)(\ell+1)}\right)\delta_{m+k,n+\ell} \\ &-\overline{\gamma}\delta\left(\frac{1}{\ell+q+1}-\frac{k-q+1}{(k+1)(\ell+1)}\right)\delta_{q+k,p+\ell} \\ &-\gamma\overline{\delta}\left(\frac{1}{\ell+p+1}-\frac{k-p+1}{(k+1)(\ell+1)}\right)\delta_{p+k,q+\ell}. \end{aligned} \tag{3}$$

Now recall that $m < n$ and $k < \ell$, so that $m+k < n+\ell$, and therefore $\delta_{m+k,n+\ell}$=0. Also, $p < q$ implies $p+k < q+\ell$, so that $\delta_{p+k,q+\ell} = 0$. As a consequence,

$$\begin{aligned} A_{10} &= \overline{\alpha}\beta\left(\frac{1}{\ell+m+1}-\frac{k-m+1}{(k+1)(\ell+1)}\right)\delta_{n+k,m+\ell} \\ &-\overline{\gamma}\delta\left(\frac{1}{\ell+q+1}-\frac{k-q+1}{(k+1)(\ell+1)}\right)\delta_{q+k,p+\ell}. \end{aligned} \tag{4}$$

In a completely analogous way, we obtain

$$\begin{aligned} A_{01} &= \overline{\alpha}\beta\left(\frac{1}{r+m+1} - \frac{k-m+1}{(k+1)(r+1)}\right)\delta_{n+k,m+r} \\ &\quad -\overline{\gamma}\delta\left(\frac{1}{r+q+1} - \frac{k-q+1}{(k+1)(r+1)}\right)\delta_{q+k,p+r}, \qquad (5) \end{aligned}$$

$$\begin{aligned} A_{11} &= \alpha\overline{\beta}\left(\frac{1}{\ell+n+1} - \frac{r-n+1}{(r+1)(\ell+1)}\right)\delta_{m+r,n+\ell} \\ &\quad -\gamma\overline{\delta}\left(\frac{1}{\ell+p+1} - \frac{r-p+1}{(r+1)(\ell+1)}\right)\delta_{p+r,q+\ell}, \qquad (6) \end{aligned}$$

$$A_{20} = \frac{1}{(\ell+1)^2}\left(\frac{|\alpha|^2 n^2}{\ell+n+1} + \frac{|\beta|^2 m^2}{\ell+m+1} - \frac{|\gamma|^2 p^2}{\ell+p+1} - \frac{|\delta|^2 q^2}{\ell+q+1}\right),$$

and

$$A_{02} = \frac{1}{(r+1)^2}\left(\frac{|\alpha|^2 n^2}{r+n+1} + \frac{|\beta|^2 m^2}{r+m+1} - \frac{|\gamma|^2 p^2}{r+p+1} - \frac{|\delta|^2 q^2}{r+q+1}\right).$$

Recall again that $k < \ell < r$ and assume, for the sake of simplicity, that $\ell := k + n - m$ and $r := \ell + q - p$. It follows that $n + k = m + \ell < m + r$ and $q + k < q + \ell = p + r$. Therefore, both Kronecker deltas appearing in A_{01} are zero, and thus $A_{01} = 0$. Moreover,

$$\begin{aligned} A_{10} &= \overline{\alpha}\beta\left(\frac{1}{\ell + m + 1} - \frac{k - m + 1}{(k+1)(\ell+1)}\right) \\ &\quad -\overline{\gamma}\delta\left(\frac{1}{\ell + q + 1} - \frac{k - q + 1}{(k+1)(\ell+1)}\right)\delta_{q+k,p+\ell}. \end{aligned}$$

Similarly,

$$\begin{aligned} A_{11} &= \alpha\overline{\beta}\left(\frac{1}{\ell + n + 1} - \frac{r - n + 1}{(r+1)(\ell+1)}\right)\delta_{m+r,n+\ell} \\ &\quad -\gamma\overline{\delta}\left(\frac{1}{\ell + p + 1} - \frac{r - p + 1}{(r+1)(\ell+1)}\right). \end{aligned}$$

The 3×3 matrix associated with C becomes

$$\begin{pmatrix} A_{00} & A_{10} & 0 \\ \overline{A}_{10} & A_{20} & \overline{A}_{11} \\ 0 & A_{11} & A_{02} \end{pmatrix}.$$

Surprisingly, A_{00}, A_{02} and A_{20} all have the same limit as $k \to \infty$. Also, A_{10} and $\overline{A}_{11}$ have the same limit. It follows that the asymptotic behavior is associated with a tri-diagonal matrix of the form

$$\begin{pmatrix} a & \rho & 0 \\ \bar{\rho} & a & \rho \\ 0 & \bar{\rho} & a \end{pmatrix}.$$

Here

$$a := |\alpha|^2 m^2 + |\beta|^2 n^2 - |\gamma|^2 p^2 - |\delta|^2 q^2$$

and

$$\rho := \bar{\alpha}\beta mn - \bar{\gamma}\delta pq$$

Now, if instead of using a vector of the form

$$f := z^k + cz^\ell + dz^r \ \ (k < \ell < r),$$

we use a longer vector,

$$f := z_1^k + c_2 z^{k_2} + \cdots + c_m z^{k_m} \ \ (k_1 < k_2 < \cdots < k_m),$$

then the associated asymptotic matrix would still be tridiagonal, with a in the diagonal and ρ in the superdiagonal.

Lemma

The spectrum of the infinite tridiagonal matrix

$$M := \begin{pmatrix} a & \rho & 0 & \cdots \\ \bar{\rho} & a & \rho & \cdots \\ 0 & \bar{\rho} & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

is $[a - 2|\rho|, a + 2|\rho|]$.

As a consequence, if M is positive (as an operator on $\ell^2(\mathbb{Z})$), then

$$a \geq 2\,|\rho|\,.$$

Theorem

Assume that T_φ is hyponormal on $A^2(\mathbb{D})$, with

$$\varphi := \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q \;\; (n > m;\; p < q).$$

Then

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2\,|\bar{\alpha}\beta mn - \bar{\gamma}\delta pq|.$$

Specific Case

When $p = m$ and $q = n$ in

$$\varphi := \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q \ \ (n > m;\ p < q).$$

the inequality

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 \left| \bar{\alpha}\beta mn - \bar{\gamma}\delta pq \right|.$$

reduces to

$$n^2(|\alpha|^2 - |\delta|^2) + m^2(|\beta|^2 - |\gamma|^2) \geq 2mn \left| \bar{\alpha}\beta - \bar{\gamma}\delta \right|.$$

This not only generalizes previous estimates, but also sharpens them, since previous results did not include the factor 2 in the right-hand side.