A New Necessary Condition for the Hyponormality of Toeplitz Operators on the Bergman Space

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Abstract

It is a well known result of C. Cowen that, for a symbol

$$\varphi \in L^\infty, \varphi \equiv \overline{f} + g \ (f, g \in H^2),$$

the Toeplitz operator $T_\varphi$ acting on the Hardy space of the unit circle is hyponormal if and only if

$$f = c + T_{\overline{h}} g,$$

for some $c \in \mathbb{C}$, $h \in H^\infty$, $\|h\|_\infty \leq 1$. In this talk we will consider possible versions of this result in the Bergman space case.
Concretely, we consider Toeplitz operators on the Bergman space of the unit disk, with symbols of the form

\[ \varphi \equiv \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q, \]

where \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) and \( m, n, p, q \in \mathbb{Z}_+ \), \( m < n \) and \( p < q \). By letting \( T_\varphi \) act on vectors of the form

\[ z^k + cz^\ell + dz^r \quad (k < \ell < r), \]

we study the asymptotic behavior of a suitable matrix of inner products, as \( k \to \infty \). As a consequence, we obtain a rather sharp inequality involving the above mentioned data:

\[ |\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 |\bar{\alpha}\beta mn - \bar{\gamma}\delta pq|. \]

This result is intended to be a precursor of basic necessary conditions for joint hyponormality of commuting tuples of Toeplitz operators acting on Bergman spaces in several complex variables.
$T \in \mathcal{L}(\mathcal{H})$: algebra of bounded operators on a Hilbert space $\mathcal{H}$

- **normal** if $T^* T = TT^*$
- **quasinormal** if $T$ commutes with $T^* T$
- **subnormal** if $T = N|_\mathcal{H}$, where $N$ is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- **hyponormal** if $T^* T \geq TT^*$
- **2-hyponormal** if $(T, T^2)$ is (jointly) hyponormal ($k \geq 1$)

$$
\begin{pmatrix}
[T^*, T] & [T^*^2, T] \\
[T^*, T^2] & [T^*^2, T^2]
\end{pmatrix} \geq 0
$$

**Normality Chain**:

normal $\Rightarrow$ quasinormal $\Rightarrow$ subnormal $\Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal.
\( L^\infty \equiv L^\infty(\mathbb{T}); \ H^\infty \equiv H^\infty(\mathbb{T}); \ L^2 \equiv L^2(\mathbb{T}); \ H^2 \equiv H^2(\mathbb{T}), \)

\( P : L^2 \to H^2 \) orthogonal projection

For \( \varphi \in L^\infty \), the Toeplitz operator with symbol \( \varphi \) is

\( T_\varphi : H^2 \to H^2 \), given by

\[
T_\varphi f := P(\varphi f) \quad (f \in H^2)
\]

\( T_\varphi \) is said to be \textit{analytic} if \( \varphi \in H^\infty \)

Halmos’s Problem 5 (1970):
Is every subnormal Toeplitz operator either normal or analytic?

C. Cowen and J. Long (1984): \textit{No}
C. Cowen (1988)

\[ \varphi \in L^\infty, \varphi = \overline{f} + g \ (f, g \in H^2) \]

\( T_{\varphi} \) is hyponormal \( \iff \ f = c + T_{\overline{h}}g, \)

for some \( c \in \mathbb{C}, \ h \in H^\infty, \ \|h\|_\infty \leq 1. \)

Nakazi-Takahashi (1993)

For \( \varphi \in L^\infty, \) let

\[ \mathcal{E}(\varphi) := \{ k \in H^\infty : \|k\|_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty \}. \]

Then

\( T_{\varphi} \) is hyponormal \( \iff \mathcal{E}(\varphi) \neq \emptyset. \)
Natural Questions:

1) When is $T_\varphi$ subnormal?
At present, there's no known characterization of subnormality in terms of the symbol $\varphi$.

2) Characterize 2-hyponormality for Toeplitz operators

Sample Result:

Theorem

(RC and WY Lee, 2001) Every 2-hyponormal trigonometric Toeplitz operator is subnormal.
In 2001, RC and W.Y. Lee completely characterized the hyponormality of Toeplitz pairs $T \equiv (T_\phi, T_\psi)$, when both symbols $\phi$ and $\psi$ are trigonometric polynomials.

**Key building block:** For $\phi$ and $\psi$ trigonometric polynomials, the hyponormality of $T \equiv (T_\phi, T_\psi)$ forces the co-analytic parts of $\phi$ and $\psi$ to necessarily coincide up to a constant multiple; equivalently,

$$[ECAP] \quad \phi - \beta \psi \in \mathcal{H}^2 \text{ for some } \beta \in \mathbb{C}.$$ 

**ECAP:** Equality (up to a constant multiple) of Co-Analytic Parts
Hyponormal Toeplitz Operators on the Bergman Space

\[ L^\infty \equiv L^\infty(D); \ H^\infty \equiv H^\infty(D); \ L^2 \equiv L^2(D); \ A^2 \equiv A^2(D), \]

\[ P : L^2 \to A^2 \text{ orthogonal projection} \]

For \( \varphi \in L^\infty \), the Toeplitz operator on the Bergman space with symbol \( \varphi \) is

\[ T_\varphi : A^2(D) \to A^2(D), \]

given by

\[ T_\varphi f := P(\varphi f) \quad (f \in A^2). \]

\( T_\varphi \) is said to be \textit{analytic} if \( \varphi \in H^\infty \).
Proposition

Let $\varphi$ be a trigonometric polynomial of the form

$$
\varphi(z) = \sum_{n=-m}^{N} a_n z^n.
$$

(i) (D. Farenick & W.Y. Lee, 1996) If $T_\varphi$ is a hyponormal operator then $m \leq N$ and $|a_{-m}| \leq |a_N|$.

(ii) (D. Farenick & W.Y. Lee, 1996) If $T_\varphi$ is a hyponormal operator then $N - m \leq \text{rank} [T_\varphi^*, T_\varphi] \leq N$. 
Proposition (cont.)

(iii) \( (RC & W.Y. \text{ Lee, 2001}) \) The hyponormality of \( T_\varphi \) is independent of the particular values of the Fourier coefficients \( a_0, a_1, \ldots, a_{N-m} \) of \( \varphi \). Moreover, for \( T_\varphi \) hyponormal, the rank of the self-commutator of \( T_\varphi \) is independent of those coefficients.
Proposition (cont.)

(iv) \textit{(D. Farenick & W.Y. Lee, 1996)} If \( m \leq N \) and \(|a_{-m}| = |a_N| \neq 0\),
then \( T_\varphi \) is hyponormal if and only if the following equation holds:

\[
\begin{pmatrix}
a_{-1} \\
a_{-2} \\
\vdots \\
\vdots \\
a_{-m}
\end{pmatrix} = a_{-m} \begin{pmatrix}
\bar{a}_{N-m+1} \\
\bar{a}_{N-m+2} \\
\vdots \\
\vdots \\
\bar{a}_N
\end{pmatrix}. \quad \text{(hypon.)}
\]

In this case, the rank of \([T_\varphi^*, T_\varphi]\) is \( N - m \).
Proposition (cont.)

(v) *(D. Farenick & W.Y. Lee, 1996)* $T_\varphi$ is normal if and only if

$m = N$, $|a_{-N}| = |a_N|$, and *(hypon.)* holds with $m = N$. 
A Revealing Example

Let

\[ \varphi \equiv \bar{z}^2 + 2z. \]

On the Hardy space \( H^2(\mathbb{T}) \), \( T_\varphi \) is not hyponormal, because \( m = 2 \), \( N = 1 \), and \( m > N \).

However, on the Bergman space \( A^2(\mathbb{D}) \) \( T_\varphi \) is hyponormal, as we now prove. Consider a slight variation of the symbol, that is,

\[ \varphi \equiv \bar{z}^2 + \alpha z. \quad (\alpha \in \mathbb{C}) \]

Observe that

\[ \langle [T_\varphi^*, T_\varphi]f, f \rangle = \langle |\alpha|^2 [T\bar{z}, Tz] + [Tz^2, T\bar{z}^2]f, f \rangle \]
so that $T_\varphi$ is hyponormal if and only if

$$
|\alpha|^2 \|zf\|^2 + \langle P(\bar{z}^2 f), \bar{z}^2 f \rangle \geq |\alpha|^2 \langle P(\bar{zf}), \bar{zf} \rangle + \|z^2 f\|^2
$$

for all $f \in A^2(\mathbb{D})$.

A calculation, using the result on the next page, shows that this happens precisely when $|\alpha| \geq 2$. As a result, $T_{\bar{z}^2+2z}$ is hyponormal.
Key Difference Between Hardy and Bergman Cases

**Lemma**

For \( u, v \geq 0 \), we have

\[
P(\bar{z}^u z^v) = \begin{cases} 
0 & \text{if } v < u \\
\frac{(v-u+1)}{v+1} z^{v-u} & \text{if } v \geq u
\end{cases}.
\]

**Proof.**

\[
P(\bar{z}^u z^v) = \sum_{j=0}^{\infty} \frac{\langle \bar{z}^u z^v, z^j \rangle}{\|z^j\|} \frac{z^j}{\|z^j\|} = \sum_{j=0}^{\infty} \frac{\langle \bar{z}^u z^v, z^j \rangle}{\|z^j\|^2} z^j = \sum_{j=0}^{\infty} (j+1) \langle z^v, z^{u+j} \rangle z^j
\]

\[
= \begin{cases} 
0 & \text{if } v < u \\
\frac{v-u+1}{v+1} z^{v-u} & \text{if } v \geq u
\end{cases}.
\]
Corollary

For \( v \geq u \) and \( t \geq w \), we have

\[
\langle P(\bar{z}^u z^v), P(\bar{z}^w z^t) \rangle = \frac{v - u + 1}{v + 1} z^{v-u} \frac{t - w + 1}{t + 1} z^{t-w}
\]

\[
= \frac{(t - w + 1)}{(v + 1)(t + 1)} \delta_{u+t, v+w}.
\]
Some Known Results

If \( \varphi \equiv \bar{g} + f \), the following are equivalent:

(i) \( T_\varphi \) is hyponormal on \( A^2(\mathbb{D}) \);

(ii) \( H_\bar{g}^* H_\bar{g} \leq H_\bar{f}^* H_\bar{f} \);

(iii) \( H_\bar{g} = C H_\bar{f} \), where \( C \) is a contraction on \( A^2(\mathbb{D}) \).

Let \( \varphi \equiv a_{-m} \bar{z}^m + a_{-N} \bar{z}^N + a_m z^m + a_N z^N \) \((0 < m < N)\) satisfying
\( a_m \bar{a}_N = a_{-m} a_{-N} \), then \( T_\varphi \) is hyponormal if and only if

\[
\frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) \quad \text{(if } |a_{-N}| \leq |a_N|)\]

\[
N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) \quad \text{(if } |a_N| \leq |a_{-N}|)\].

The last condition is not sufficient.
Let \( \varphi \equiv 4\bar{z}^3 + 2\bar{z}^2 + \bar{z} + z + 2z^2 + \beta z^3 \ (|\beta| = 4) \). Then \( T_\varphi \) is hyponormal if and only if \( \beta = 4 \).

Then \( T_\varphi \) is not hyponormal.

Let \( \varphi \equiv \bar{g} + f \in L^\infty(\mathbb{D}) \), and assume that \( T_\varphi \) is hyponormal. Then

\[
T_\varphi u \geq u,
\]

where \( u := |f|^2 - |g|^2 \).
Let $\varphi \equiv \bar{g} + f \in L^{\infty}(\mathbb{D})$, and assume that $T_\varphi$ is hyponormal. Then

$$\lim_{z \to \zeta} (|f'(z)|^2 - |g'(z)|^2) \geq 0$$

for all $\zeta \in \mathbb{T}$. In particular, if $f'$ and $g'$ are continuous at $\zeta$ then $|f'(\zeta)| \geq |g'(\zeta)|$.


(Y. Lu & Y. Shi, IEOT, 2009) studied the weighted Bergman space case.
The self-commutator of $T_\varphi$ is

$$C := [T_\varphi^*, T_\varphi]$$

We seek necessary and sufficient conditions on the symbol $\varphi$ to ensure that $C \geq 0$.

The following result gives a flavor of the type of calculations we face when trying to decipher the hyponormality of a Toeplitz operator acting on the Bergman space. Although the calculation therein will be superseded by the calculations in the following section, it serves both as a preliminary example and as motivation for the organization of our work.
Proposition

Assume $k, \ell \geq \max\{a, b\}$. Then

$$
\langle [T_{\bar{z}}^a, T_{z}^b](z^k + c z^\ell), z^k + c z^\ell \rangle
$$

$$
= a^2 \left[ \frac{1}{(k + 1)^2(k + 1 + a)} + c^2 \cdot \frac{1}{(\ell + 1)^2(\ell + 1 + a)} \right] \delta_{a,b}
$$

$$
+ ac \left[ \frac{k - \ell + a}{(a + k + 1)(k + 1)(\ell + 1)} \delta_{a+k,b+\ell}
\right.
$$

$$
+ \left. \frac{\ell - k + a}{(a + \ell + 1)(k + 1)(\ell + 1)} \delta_{a+\ell,b+k} \right]
$$
Corollary

Assume $a = b$, $k, \ell \geq a$ and $k \neq \ell$. Then

$$\langle [T_{\bar{z}^a}, T_{z^a}] (z^k + cz^\ell), z^k + cz^\ell \rangle = a^2 \left[ \frac{1}{(k + 1)^2(k + 1 + a)} + c^2 \cdot \frac{1}{(\ell + 1)^2(\ell + 1 + a)} \right]$$
We focus on the action of the self-commutator $C$ of certain Toeplitz operators $T_\varphi$ on suitable vectors $f$ in the space $A^2(\mathbb{D})$. The symbol $\varphi$ and the vector $f$ are of the form

$$\varphi := \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q \quad (n > m; \ p < q)$$

and

$$f := z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

respectively. Our ultimate goal is to study the asymptotic behavior of this action as $k$ goes to infinity. Thus, we consider the expression $\langle Cf, f \rangle$, given by

$$\langle [(T_\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q)^*, \ T_\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q](z^k + cz^\ell + dz^r), z^k + cz^\ell + dz^r \rangle,$$

for large values of $k$ (and consequently large values of $\ell$ and $r$.) It is straightforward to see that $\langle Cf, f \rangle$ is a quadratic form in $c$ and $d$, that is,
\begin{align*}
\langle Cf, f \rangle & \equiv A_{00} + 2 \Re(A_{10} c) + 2 \Re(A_{01} d) + A_{20} \overline{c} \overline{c} + 2 \Re(A_{11} \overline{c} \overline{d}) + A_{02} \overline{d} \overline{d}. \\
\text{Alternatively, the matricial form of (1) is} \\
\langle \begin{pmatrix} A_{00} & A_{10} & A_{01} \\ \overline{A}_{10} & A_{20} & \overline{A}_{11} \\ \overline{A}_{01} & A_{11} & A_{02} \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} \rangle. 
\end{align*}
We now observe that the coefficient $A_{00}$ arises from the action of $C$ on the monomial $z^k$, that is,

$$A_{00} = \langle Cz^k, z^k \rangle \equiv \langle [(T_{\alpha}z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q)^*, T_{\alpha}z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q]z^k, z^k \rangle.$$ 

Similarly,

$$A_{10} = \langle Cz^\ell, z^k \rangle,$$

$$A_{01} = \langle Cz^r, z^k \rangle,$$

$$A_{20} = \langle Cz^\ell, z^\ell \rangle,$$

$$A_{11} = \langle Cz^r, z^\ell \rangle,$$

$$A_{02} = \langle Cz^r, z^r \rangle.$$
To calculate $A_{00}$ explicitly, we first recall that the algebra of Toeplitz operators with analytic symbols is commutative, and therefore $T_{z^n}$ commutes with $T_{z^m}$, $T_{z^p}$ and $T_{z^q}$.

We also recall that two monomials $z^u$ and $z^v$ are orthogonal whenever $u \neq v$. As a result, the only nonzero contributions to $A_{00}$ must come from the inner products $\langle [T_{z^n}^*, T_{z^n}]z^k, z^k \rangle$, $\langle [T_{z^m}^*, T_{z^m}]z^k, z^k \rangle$, $\langle [T_{z^p}^*, T_{z^p}]z^k, z^k \rangle$ and $\langle [T_{z^q}^*, T_{z^q}]z^k, z^k \rangle$.

Applying Corollary 7 we see that

$$A_{00} = \frac{1}{(k + 1)^2} \left( \frac{|\alpha|^2 n^2}{k + n + 1} + \frac{|\beta|^2 m^2}{k + m + 1} - \frac{|\gamma|^2 p^2}{k + p + 1} - \frac{|\delta|^2 q^2}{k + q + 1} \right).$$
Similarly,

$$A_{10} = \bar{\alpha} \beta \frac{1}{\ell + m + 1} - \frac{k - m + 1}{(k + 1)(\ell + 1)} \delta_{n+k,m+\ell}$$

$$+ \alpha \bar{\beta} \left( \frac{1}{\ell + n + 1} - \frac{k - n + 1}{(k + 1)(\ell + 1)} \right) \delta_{m+k,n+\ell}$$

$$- \bar{\gamma} \delta \left( \frac{1}{\ell + q + 1} - \frac{k - q + 1}{(k + 1)(\ell + 1)} \right) \delta_{q+k,p+\ell}$$

$$- \gamma \bar{\delta} \left( \frac{1}{\ell + p + 1} - \frac{k - p + 1}{(k + 1)(\ell + 1)} \right) \delta_{p+k,q+\ell}. \quad (3)$$

Now recall that $m < n$ and $k < \ell$, so that $m + k < n + \ell$, and therefore $\delta_{m+k,n+\ell}=0$. Also, $p < q$ implies $p + k < q + \ell$, so that $\delta_{p+k,q+\ell} = 0$. As a consequence,

$$A_{10} = \bar{\alpha} \beta \frac{1}{\ell + m + 1} - \frac{k - m + 1}{(k + 1)(\ell + 1)} \delta_{n+k,m+\ell}$$

$$- \bar{\gamma} \delta \left( \frac{1}{\ell + q + 1} - \frac{k - q + 1}{(k + 1)(\ell + 1)} \right) \delta_{q+k,p+\ell}. \quad (4)$$
In a completely analogous way, we obtain

\[
A_{01} = \alpha\beta \left( \frac{1}{r + m + 1} - \frac{k - m + 1}{(k + 1)(r + 1)} \right) \delta_{n+k,m+r} \\
- \bar{\gamma}\delta \left( \frac{1}{r + q + 1} - \frac{k - q + 1}{(k + 1)(r + 1)} \right) \delta_{q+k,p+r},
\]

(5)

\[
A_{11} = \alpha\bar{\beta} \left( \frac{1}{\ell + n + 1} - \frac{r - n + 1}{(r + 1)(\ell + 1)} \right) \delta_{m+r,n+\ell} \\
- \gamma\bar{\delta} \left( \frac{1}{\ell + p + 1} - \frac{r - p + 1}{(r + 1)(\ell + 1)} \right) \delta_{p+r,q+\ell},
\]

(6)

\[
A_{20} = \frac{1}{(\ell + 1)^2} \left( \frac{|\alpha|^2 n^2}{\ell + n + 1} + \frac{|\beta|^2 m^2}{\ell + m + 1} - \frac{|\gamma|^2 p^2}{\ell + p + 1} - \frac{|\delta|^2 q^2}{\ell + q + 1} \right),
\]

and

\[
A_{02} = \frac{1}{(r + 1)^2} \left( \frac{|\alpha|^2 n^2}{r + n + 1} + \frac{|\beta|^2 m^2}{r + m + 1} - \frac{|\gamma|^2 p^2}{r + p + 1} - \frac{|\delta|^2 q^2}{r + q + 1} \right).
\]
Recall again that $k < \ell < r$ and assume, for the sake of simplicity, that $\ell := k + n - m$ and $r := \ell + q - p$. It follows that $n + k = m + \ell < m + r$ and $q + k < q + \ell = p + r$. Therefore, both Kronecker deltas appearing in $A_{01}$ are zero, and thus $A_{01} = 0$. Moreover,

$$A_{10} = \bar{\alpha} \beta \left( \frac{1}{\ell + m + 1} - \frac{k - m + 1}{(k + 1)(\ell + 1)} \right) - \bar{\gamma} \delta \left( \frac{1}{\ell + q + 1} - \frac{k - q + 1}{(k + 1)(\ell + 1)} \right) \delta_{q+k,p+\ell}. $$

Similarly,

$$A_{11} = \alpha \bar{\beta} \left( \frac{1}{\ell + n + 1} - \frac{r - n + 1}{(r + 1)(\ell + 1)} \right) \delta_{m+r,n+\ell} - \gamma \bar{\delta} \left( \frac{1}{\ell + p + 1} - \frac{r - p + 1}{(r + 1)(\ell + 1)} \right).$$
The $3 \times 3$ matrix associated with $C$ becomes

$$
\begin{pmatrix}
A_{00} & A_{10} & 0 \\
\bar{A}_{10} & A_{20} & \bar{A}_{11} \\
0 & A_{11} & A_{02}
\end{pmatrix}.
$$

Surprisingly, $A_{00}$, $A_{02}$ and $A_{20}$ all have the same limit as $k \to \infty$. Also, $A_{10}$ and $\bar{A}_{11}$ have the same limit. It follows that the asymptotic behavior is associated with a tri-diagonal matrix of the form
\[
\begin{pmatrix}
  a & \rho & 0 \\
  \bar{\rho} & a & \rho \\
  0 & \bar{\rho} & a
\end{pmatrix}.
\]

Here
\[
a := |\alpha|^2 m^2 + |\beta|^2 n^2 - |\gamma|^2 p^2 - |\delta|^2 q^2
\]
and
\[
\rho := \bar{\alpha}\beta mn - \bar{\gamma}\delta pq
\]
Now, if instead of using a vector of the form

\[ f := z^k + cz^\ell + dz^r \ (k < \ell < r), \]

we use a longer vector,

\[ f := z_1^k + c_2 z_2^{k_2} + \cdots + c_m z_m^{k_m} \ (k_1 < k_2 < \cdots < k_m), \]

then the associated asymptotic matrix would still be tridiagonal, with \( a \) in the diagonal and \( \rho \) in the superdiagonal.
Lemma

The spectrum of the infinite tridiagonal matrix

\[ M := \begin{pmatrix}
  a & \rho & 0 & \cdots \\
  \bar{\rho} & a & \rho & \cdots \\
  0 & \bar{\rho} & a & \cdots \\
  \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}. \]

is \([a - 2|\rho|, a + 2|\rho|]\).
As a consequence, if $M$ is positive (as an operator on $\ell^2(\mathbb{Z})$), then

$$a \geq 2|\rho|.$$ 

**Theorem**

Assume that $T_\varphi$ is hyponormal on $A^2(\mathbb{D})$, with

$$\varphi := \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q \quad (n > m; \ p < q).$$

Then

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2|\overline{\alpha} \beta mn - \overline{\gamma} \delta pq|.$$
Specific Case

When \( p = m \) and \( q = n \) in

\[
\varphi := \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q \quad (n > m; \ p < q).
\]

the inequality

\[
|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2|\bar{\alpha}\beta mn - \bar{\gamma}\delta pq|.
\]

reduces to

\[
n^2(|\alpha|^2 - |\delta|^2) + m^2(|\beta|^2 - |\gamma|^2) \geq 2mn|\bar{\alpha}\beta - \bar{\gamma}\delta|.
\]

This not only generalizes previous estimates, but also sharpens them, since previous results did not include the factor 2 in the right-hand side.