Non-extremal sextic moment problems
(joint work with Seonguk Yoo)

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IWOTA 2015
Tbilisi, July 7, 2015

The Truncated Complex Moment Problem

- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \dots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 2n);$$

  μ is called a **rep. meas.** for γ.

  In earlier joint work with L. Fialkow,

- We have introduced an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0,\infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a,b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**. For simplicity, in this talk we focus on **one complex variable or two real variables**, although several results have multivariable versions.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We have fully resolved, among others, the cases

$$\bar{Z} = \alpha 1 + \beta Z$$

and

$$Z^k = p_{k-1}(Z, \bar{Z}) \ \ (1 \leq k \leq [\frac{n}{2}] + 1; \deg p_{k-1} \leq k-1).$$

- We obtain applications to quadrature problems in numerical analysis.

Basic Positivity Condition

$\mathcal{P}_n$: polynomials p in z and $\bar{z}$, $\deg p \leq n$

Given $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$,

$$0 \leq \int |\ p(z, \bar{z})\ |^2 \, d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell, j+k}.$$

- To understand this **“matricial” positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots$$

Define $M[i,j]$ as in

$$M[3,2] := \begin{pmatrix} \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} \end{pmatrix}$$

Then

$$(\textbf{“matricial” positivity}) \quad \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell}\gamma_{i+\ell,j+k} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0,0] & M[0,1] & ... & M[0,n] \\ M[1,0] & M[1,1] & ... & M[1,n] \\ ... & ... & ... & \dots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix} \geq 0.$$

For example,

- $$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

- $$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Similarly, one can build $M(\infty)$.

In the real case, $M(n)_{ij} := \gamma_{i+j}, \ i,j \in \mathbb{Z}_+^2$.

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{06}, \ldots, \gamma_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. But this can also be done for $n = 2$.

Positivity of Block Matrices

Theorem

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases}.$$

Moreover, $\operatorname{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A \Leftrightarrow C = W^*AW$.

Corollary

Assume $\text{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank}\, A$. *Then*

$$A \geq 0 \Leftrightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

We say that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is a flat extension of A. *Observe that*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix}.$$

Corollary

Assume that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Then

$$\begin{aligned} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} &= \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - W^*AW \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix}^* \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}^* \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix} \end{aligned}$$

(sum-of-squares representation).

Functional Calculus

For $p \in \mathcal{P}_n$, $p(z,\bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$, let $\widehat{p}$ denote the vector of coefficients and define

$$p(Z,\bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j \equiv M(n)\widehat{p}.$$

If there exists a rep. meas. μ, then

$$p(Z,\bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(RG) If $p, q, pq \in \mathcal{P}_n$, **and** $p(Z,\bar{Z}) = 0$,

then $(pq)(Z,\bar{Z}) = 0.$

Singular TMP

- Given a finite family of moments, build moment matrix.
- Identify all column relations $p(Z, \bar{Z}) = 0$, i.e., $M(n)\widehat{p} = 0$.
- Build algebraic variety

$$\mathcal{V} := \bigcap_{p \in \mathcal{P}_n,\ \widehat{p} \in \ker M(n)} \mathcal{Z}_p.$$

- Always true: in the presence of a measure,

$$\text{supp } \mu \subseteq \mathcal{V}.$$

Therefore,

$$r := \text{rank}\, M(n) \le \text{card supp } \mu \le v := \text{card}\, \mathcal{V}(\gamma).$$

Thus, if the variety is finite there's a natural candidate for supp μ, i.e., supp $\mu = \mathcal{V}(\gamma)$

(**However, it is possible for the inclusion** supp $\mu \subseteq \mathcal{V}$ **to be proper**.)

Real TMP

Let $\beta \equiv \beta^{(2n)} = \{\beta_{ij}\}_{i,j\geq 0, i+j\leq 2n}$ denote a 2-dimensional real multisequence, and let K (closed) $\subseteq \mathbb{R}^2$. The truncated K-moment problem asks for necessary and sufficient conditions on β to guarantee the existence of a positive Borel measure μ supported in K such that

$$\beta_{ij} = \int x^i y^j \, d\mu \qquad (i, \geq 0, i+j \leq 2n);$$

μ is called a **rep. meas.** for β.

Associated with β is a moment matrix $M \equiv M(n)$, defined by

$$M(n)_{ij} := \beta_{i+j} \qquad (i, j \geq 0, i+j \leq 2n).$$

The Real TMP and the Complex TMP are entirely equivalent, via the degree-one transformation $z \equiv x + iy$.

- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
- Recursively determinate moment matrices
- Strategy: Build positive extension, repeat, and eventually the TMP is extremal

$$\text{rank } M(n) \leq \text{rank } M(n+1) \leq \text{card } \mathcal{V}(M(n+1)) \leq \text{card}\, \mathcal{V}(M(n))$$

- General case.

Recursively Determinate Moment Matrices

A bivariate moment matrix $M(n)$ is *recursively determinate* if there are column dependence relations of the form

$$X^k = p(X, Y) \quad (p \in \mathcal{P}_{k-1},\ k \leq n) \tag{3.1}$$

and

$$Y^\ell = q(X, Y) \quad (q \in \mathcal{P}_\ell,\ q \text{ has no } y^\ell \text{ term},\ \ell \leq n), \tag{3.2}$$

or with similar relations with the roles of p and q reversed.

First Existence Criterion

Theorem

(RC-L. Fialkow, 1998) Let γ be a truncated moment sequence. TFAE:

(i) γ has a rep. meas.;

(ii) γ has a rep. meas. with moments of all orders;

(iii) γ has a compactly supported rep. meas.;

(iv) γ has a finitely atomic rep. meas. (with at most $(n+2)(2n+3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$, $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n+k+1)$ (here $k \leq 2n^2 + 6n + 6$) .

Case of Flat Data

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

γ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

Theorem

(RC-L. Fialkow, 1996) If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

Theorem

(RC-L. Fialkow, 1996) The truncated moment sequence γ has a rank $M(n)$*-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

The finitely atomic measure μ can be found concretely.

The Quartic Moment Problem

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A\,1$ (Dirac measure)
- $\bar{Z} = A\,1 + B\,Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A\,1 + B\,Z + C\,\bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$\begin{aligned} D &= 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \text{ and } C = \bar{B} \\ &\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2 \\ &\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}. \end{aligned}$$

Using the Flat Data Theorem, one can reduce the study to cases corresponding to the following five real conics:

(a)	$\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$	parabola; $y = x^2$
(b)	$\bar{W}^2 = -4i1 + W^2$	hyperbola; $yx = 1$
(c)	$\bar{W}^2 = W^2$	pair of intersect. lines; $yx = 0$
(d)	$\bar{W}W = 1$	unit circle; $x^2 + y^2 = 1$
(e)	$\bar{W}^2 + 2\bar{W}W + W^2 = 4$	pair of parallel lines; $x^2 = 1$.

Theorem QUARTIC

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ *be given, and assume* $M(2) \geq 0$ *and* $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ *is a basis for* $\mathcal{C}_{M(2)}$. *Then* $\gamma^{(4)}$ *admits a rep. meas.* μ. *Moreover, it is possible to find* μ *with* card supp $\mu =$ rank $M(2)$, *except in some cases when* $\mathcal{V}(\gamma^{(4)})$ *is a pair of intersecting lines, in which cases there exist* μ *with* card supp $\mu \leq 6$.

Corollary

Assume that $M(2) \geq 0$, $M(2)$ *singular, and that* rank $M(2) \leq$ card $\mathcal{V}(\gamma^{(4)})$. *Then* $M(2)$ *admits a representing measure.*

The Case of Invertible $M(2)$

(L. Fialkow and J. Nie, 2010) Consider a quartic moment problem with invertible $M(2)$. Then there exists a representing measure.
The proof is abstract, using convex analysis.

(RC and S. Yoo; Proc. AMS, 2015) Concrete construction of a representing measure, when $M(2)$ is invertible. Moreover, there exists a 6-atomic representing measure, that is, $M(2)$ admits a flat extension $M(3)$.

The Case of Recursively Determinate Matrices

Recall that

$$X^k = p(X, Y) \quad (p \in \mathcal{P}_{k-1},\ k \leq n) \tag{5.1}$$

and

$$Y^\ell = q(X, Y) \quad (q \in \mathcal{P}_\ell,\ q \text{ has no } y^\ell \text{ term},\ \ell \leq n), \tag{5.2}$$

Theorem

(RC and L. Fialkow, 2013) If $M(n)$ is positive, with column relations generated entirely by X^k and Y^ℓ via recursiveness and linearity, then $M(n)$ admits a unique RG extension $M(n+1)$, i.e., Ran $B(n+1) \subseteq$ Ran $M(n)$, and $M(n+1)$ is recursively generated.

Basic necessary conditions for the existence of a representing measure

$$\text{(Positivity)}\ M(n) \geq 0 \tag{5.3}$$

$$\text{(Consistency)}\ p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0 \tag{5.4}$$

(where Λ is the Riesz functional associated to $M(n)$)

$$\text{(Variety Condition)}\ r \leq v, \text{ i.e., } \text{rank } M(n) \leq \text{card } \mathcal{V}. \tag{5.5}$$

Consistency implies

$$\text{(Recursiveness)}\ p, q, pq \in \mathcal{P}_n,\ \widehat{p} \in \ker M(n) \Longrightarrow \widehat{pq} \in \ker M(n). \tag{5.6}$$

(ideal-like property)

Theorem (Extremal TMP)

(RC, L. Fialkow and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ **extremal**, *i.e.,* $r = v$, *the following are equivalent:*

(i) β *has a representing measure;*

(ii) β *has a unique representing measure, which is* rank $M(n)$*-atomic (minimal);*

(iii) $M(n) \geq 0$ *and* β *is consistent.*

Cubic Column Relations

Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$, and that $Z^3 = p_2(Z, \bar{Z})$, with deg $p_2 \leq 2$.

Recall

Theorem A

(RC-L. Fialkow) If $M(n)$ admits a column relation of the form $Z^k = p_{k-1}(Z, \bar{Z})$ $(1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and deg $p_{k-1} \leq k-1)$, then $M(n)$ admits a flat extension $M(n+1)$, and therefore a representing measure.

Now, if $k = 3$, Theorem A can be used only if $n \geq 4$. Thus, one strategy is to somehow extend $M(3)$ to $M(4)$ and preserve the column relation $Z^3 = p_2(Z, \bar{Z})$. This requires adding new moments of degree 4 and checking that the C block in the extension satisfies the Toeplitz condition, something highly nontrivial.

Here's a different approach:

We'd like to study the case of harmonic poly's: $q(z,\bar{z}) := f(z) - \overline{g(z)}$, with deg $q = 3$.

Recall that rank $M(n) \leq$ card $\mathcal{Z}(q)$

so of special interest is the case when card $\mathcal{Z}(q) \geq 7$, since otherwise the TMP admits a flat extension, or has no representing measure. In the case when $g(z) \equiv z$, we have

Lemma

(Wilmshurst '98, Sarason-Crofoot, '99, Khavinson-Swiatek, '03)

$$\text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7.$$

Bézout's Theorem predicts card $\mathcal{Z}(f(z) - \bar{z}) \leq 9$

We thus consider the harmonic polynomial $q_7(z, \bar{z}) := z^3 - itz - u\bar{z}$.

Proposition

(RC-S. Yoo, '09) card $\mathcal{Z}(q_7) = 7$. *In fact, for* $0 < |u| < t < 2|u|$,

$$\mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r + ir, -r - ir\},$$

where $p, q, r > 0$, $p^2 + q^2 = u$ *and* $r^2 = \frac{t-u}{2}$.

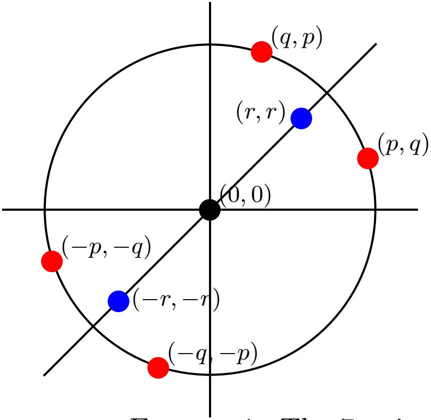

FIGURE 1. The 7-point set $\mathcal{Z}(q_7)$, where $r = \sqrt{\frac{t-u}{2}}$, $p = \frac{1}{2}(2u + \sqrt{4u^2 - t^2})$ and $p^2 + q^2 = u$

Notation

Define

$$\begin{aligned} q_{LC}(z,\bar{z}) &:= \bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z} \\ &= i(z - i\bar{z})(\bar{z}z - u). \end{aligned}$$

Observe that the zero set of q_{LC} is the union of a line and a circle, and that $\mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC})$.

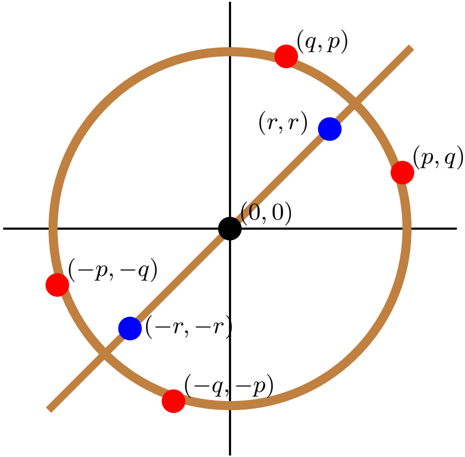

FIGURE 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$

Theorem (RC & S. Yoo, J. Funct. Anal., 2014)

Let $M(3) \geq 0$, with $M(2) > 0$ and $q_7(Z, \bar{Z}) = 0$. There exists a representing measure for $M(3)$ if and only if

$$\begin{cases} \Lambda(q_{LC}) &= 0 \\ \Lambda(zq_{LC}) &= 0. \end{cases} \tag{5.7}$$

where $\Lambda \equiv \Lambda_\beta$ is the Riesz functional. Equivalently,

$$\begin{cases} Re\ \gamma_{12} - Im\ \gamma_{12} = u(Re\ \gamma_{01} - Im\ \gamma_{01}) &= 0 \\ \gamma_{22} = (t+u)\gamma_{11} - 2u\ Im\ \gamma_{02} &= 0. \end{cases}$$

Equivalently,

$$q_{LC}(Z, \bar{Z}) = 0 \tag{5.8}$$

Proof uses Consistency Property.

A New Tool: Rank Reduction

Given a point $(a, b) \in \mathbb{R}^2$ we let $\mathbf{v} \equiv \mathbf{v}_{(a,b)}$ denote the row vector

$$(1, a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3)$$

We also let $\delta_{(a,b)}$ denote the point mass at (a, b). It is easy to see that the moment matrix associated with $\delta_{(a,b)}$ is $\mathbf{vv}^T$, that is, the matrix whose entries are $M(3)_{ij} = a^i b^j$. For this moment matrix, $r = 1$ and $\mathcal{V} = \{(a, b)\}$.

Theorem (RC S. Yoo, J. Funct. Anal., 2015)

Assume $M(3) \geq 0$, $M(2) > 0$, rank $M(3) = 7$ and card $\mathcal{V} \geq 8$. Assume also that $M(3)$ satisfies the Consistency Property. Then $M(3)$ admits a flat extension $M(4)$; that is, there exists a representing measure μ with card supp $\mu = 7$.

Sketch of Proof. WLOG, assume

$$\mathcal{V} = \{(x_1, y_1), \ldots, (x_8, y_8)\}.$$

Also assume that in $M(3)$ the first seven columns are linearly independent.

Now form the Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & x_1^3 & x_1^2 y_1 & x_1 y_1^2 & y_1^3 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & x_2^3 & x_2^2 y_2 & x_2 y_2^2 & y_2^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_8 & y_8 & x_8^2 & x_8 y_8 & y_8^2 & x_8^3 & x_8^2 y_8 & x_8 y_8^2 & y_8^3 \end{pmatrix}.$$

This is an 8×10 matrix, with rank 7. It follows that exactly seven rows are linearly independent, so one of them must be a linear combination of the other seven, say

$$R_j = \sum_{i \neq j} \lambda_i R_i.$$

The row R_j must be associated with a point $(x_j, y_j) \in \mathcal{V}$. To single out this point, we will denote it by (a, b). Now let

$$\mathcal{V}' := \mathcal{V} \setminus \{(a, b)\}.$$

Claim. No conic goes through $\mathcal{V}'$. For, let

$$C(x, y) \equiv C_{00} + C_{10}x + C_{01}y + C_{20}x^2 + C_{11}xy + C_{02}y^2$$

be such a conic, that is, if $\widehat{C}$ denotes the vector of coefficients of C regarded as a 10-vector, then

$$R_i \widehat{C} = C(x_i, y_i) = 0,$$

and therefore

$$\sum_{i \neq j} \lambda_i R_i \widehat{C} = 0;$$

that is, $C(a,b) \equiv R_j \widehat{C} = 0$, which implies that C also vanishes on (a,b), and a fortiori C vanishes on the entire algebraic variety $\mathcal{V}$. Then, by the Consistency Property, $C(X,Y) = 0$, that is, the moment matrix $M(3)$ admits a quadratic column relation, a contradiction to the fact that $M(2) > 0$.

We now define

$$\widetilde{M(3)} := M(3) - \rho \mathbf{v}\mathbf{v}^T,$$

where $\mathbf{v}$ is the row vector associated with the point (a,b).

We wish to prove that rank $\widetilde{M(3)} = 6$ for some positive value of ρ. If we do this, then $\widetilde{M(3)}$ will be a flat extension of $\widetilde{M(2)}$, and we will have a 6-atomic measure for $\widetilde{M(3)}$, and therefore a 7-atomic measure for $M(3)$, since $M(3) = \widetilde{M(3)} + \rho \mathbf{v}\mathbf{v}^T$. Moreover, one can show that rank $\widetilde{M(2)} = 6$. Let λ denote the nonzero eigenvalue of $\mathbf{v}\mathbf{v}^T$, and let $\mathcal{B}$ be the basis of the column space of $M(3)$. Then

$$\det \widetilde{M(3)}_{\mathcal{B}} = \det M(3)_{\mathcal{B}} - \rho\lambda \det(M(3)_{\mathcal{B}}|_{\{2,3,4,5,6,7\}}).$$

Thus, with

$$\rho := \frac{\det M(3)_{\mathcal{B}}}{\lambda \det(M(3)_{\mathcal{B}}|_{\{2,3,4,5,6,7\}})},$$

we successfully reduce the rank.

Problem

Assume $M(3) \geq 0$, $M(2) > 0$, rank $M(3) = 8$, card $\mathcal{V} = 9$. Under what conditions does the moment sequence admit a representing measure?

Remark

If $\mathcal{M}(n)$ has an r-atomic measure $\mu \equiv \sum_{i=1}^{r} \rho_i \delta_{(x_i, y_i)}$, then we may write $\mathcal{M}(n)$ as

$$\mathcal{M}(n) = \sum_{i=1}^{r} \rho_i \mathbf{v}_i \mathbf{v}_i^*,$$

where the densities ρ_i are positive, the column vector $\mathbf{v}_i$ is given by $(1, x_i, y_i, \ldots, x_i^n, x_i^{n-1} y_i, \ldots, x_i y_i^{n-1}, y_i^n)^T$, and the point (x_i, y_i) is in the algebraic variety $\mathcal{V}$ for all $i = 1, \ldots, r$.

Classification of sextic MP

r_3	v_3	$v_3 - r_3$	Max Ext		Solution Presented in
7	7	0	$\mathcal{M}(4)$	extremal	RC-S. Yoo; JFA(2014), 2015
7	8	1	$\mathcal{M}(5)$		RC-S. Yoo; JFA(2015)
7	9	2	$\mathcal{M}(6)$		RC-S. Yoo; JFA(2015)
7	∞	N/A	N/A		RC-S. Yoo; JFA(2015)
8	8	0	$\mathcal{M}(4)$	extremal	RC-S. Yoo; 2015
8	9	1	$\mathcal{M}(5)$		RC-S. Yoo; JFA(2015)
8	∞	N/A	N/A		RC-S. Yoo; JFA(2015)
9	∞	N/A	N/A		L. Fialkow; TAMS(2011) (cases)
10	∞	N/A	N/A		unknown

Some Matricial Results

When we decompose a moment matrix as a sum $\mathcal{M}(n) = \widetilde{\mathcal{M}(n)} + P$, the goal is to both reduce the rank (i.e., rank $\widetilde{\mathcal{M}(n)} <$ rank $\mathcal{M}(n)$) and obtain a moment matrix $\widetilde{\mathcal{M}(n)}$ for which we can use previous known results to solve TMP. In some cases, we can even make $\widetilde{\mathcal{M}(n)}$ flat.

Lemma

Let A and B be finite matrices. Then

$$\text{rank } (A + B) \leq \text{rank } A + \text{rank } B$$

We must also ensure $\widetilde{\mathcal{M}(n)} \geq 0$; that is, the minimum eigenvalue is nonnegative. For a self-adjoint $n \times n$ matrix A, we list the eigenvalues as

$$\lambda_1(A) \leq \cdots \leq \lambda_n(A).$$

Eigenvalue Inequalities

Theorem

(Horn-Johnson, 1990) Let $A \in M_n$ be Hermitian and let $z \in \mathbb{C}^n$ be a given vector. If the eigenvalues of A and $A \pm zz^*$ are arranged in increasing order as above, we have for $k = 1, 2, \ldots, n-2$,

(i) $\lambda_k(A \pm zz^*) \leq \lambda_{k+1}(A) \leq \lambda_{k+2}(A \pm zz^*)$,

(ii) $\lambda_k(A) \leq \lambda_{k+1}(A \pm zz^*) \leq \lambda_{k+2}(A)$.

One easily checks that, for $\alpha \in \mathbb{R}$

$$\det(A - \alpha\, I_1) = \det(A) - \alpha \det\left(A_{\{2,3,\ldots,m\}}\right). \tag{5.9}$$

On the other hand, the Spectral Theorem guarantees that any rank-one Hermitian matrix is unitarily equivalent to a scalar multiple of $I_1(m)$.
If P is rank-one, there exists a unitary operator U such that $U^* PU = \lambda\, I_1$, where λ is the only nonzero eigenvalue of P. We now generalize this as follows.

Proposition

Let A be an arbitrary square matrix of size m, and let P, U and λ be as above. Then

$$\det(A - \rho\, P) = \det(A) - \rho\, \lambda \det\left((U^* AU)_{\{2,3,\cdots,m\}}\right) \quad \textit{for all } \rho \in \mathbb{R}.$$

$\mathcal{M}(3)$ WITH $r = 8$ AND $v = 9$

To date, most concrete solutions of sextic moment problems include numerical conditions on one or more of the moments; it is generally intricate to express these numerical conditions as specific properties of the moment matrix.

Moreover, when we solve a recursively determinate sextic moment problem ($r = 8$ and $v \geq 8$), we need to maintain recursiveness to build the extension $\mathcal{M}(4)$, and we must also verify the positivity of $\mathcal{M}(4)$.

This leads naturally to an algorithmic approach to TMP.

PROBLEM

Suppose $\mathcal{M}(3) \geq 0$ is of rank 8, consistent, with $\mathcal{M}(2) > 0$, and with $v = 9$. Let $\mathcal{V} \equiv \{(x_i, y_i)\}_{i=1}^{9}$ be the algebraic variety of $\mathcal{M}(3)$. Under what conditions does the moment sequence admit a representing measure?

Algorithm.

Step 1. Build the generalized Vandermonde matrix of $\mathcal{V}$, namely,

$$\mathcal{E} := \begin{pmatrix} 1 & x_i & y_i & x_i^2 & x_i y_i & y_i^2 & x_i^3 & x_i^2 y_i & x_i y_i^2 & y_i^3 \end{pmatrix}_{i=1}^{9}.$$

Since $\mathcal{E}$ has 8 linearly independent rows, we can pick a point $(a,b) \in \mathcal{V}$ such that the row $R_{(a,b)}$ associated with (a,b) is linearly dependent of the other 8 rows.

Step 2. Let $\mathcal{B}$ be the basis for the column space of $\mathcal{E}$ and let $\mathcal{E}_{\mathcal{B}}$ denote the resulting matrix after removing the two dependent columns and the row $R_{(a,b)}$ from $\mathcal{E}$. We prove that it is invertible, using the Cayley-Bacharach Theorem, stating that every cubic passing through any eight of the nine points also passes through the ninth point.

Step 3. Once we know that $\mathcal{E}_{\mathcal{B}}$ is invertible, we choose another point $(c,d) \in \mathcal{V}$ $((c,d) \neq (a,b))$ and eliminate the row $R_{(c,d)}$ associated with (c,d) from $\mathcal{E}_{\mathcal{B}}$; we denote this matrix as $\mathcal{E}'_{\mathcal{B}}$. Note that $\mathcal{E}'_{\mathcal{B}}$ has rank 7 and this fact implies that there is a new cubic polynomial $r(x,y)$ vanishing on $\widehat{\mathcal{V}} := \mathcal{V} - \{(a,b),(c,d)\}$, besides $p(x,y)$ and $q(x,y)$.

Step 4. We will use a rank-one decomposition of $\mathcal{M}(3)$, and try to understand the structure of the decomposition in case a representing measure exists. Suppose $\mathcal{M}(3)$ has a representing measure. Then the variety condition forces a measure to be 8- or 9-atomic. Let $\mathbf{v}(x,y) := \left(1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\right)^T$. We write

$$\mathcal{M}(3) = \widetilde{\mathcal{M}(3)} + m_1 \mathbf{v}(a,b)\mathbf{v}(a,b)^T + m_2 \mathbf{v}(c,d)\mathbf{v}(c,d)^T,$$

where m_1 and m_2 are nonnegative (not simultaneously zero) for $(a,b),(c,d) \in \mathcal{V}$. The moment matrices $\mathbf{v}(a,b)\mathbf{v}(a,b)^T$ and $\mathbf{v}(c,d)\mathbf{v}(c,d)^T$, have representing measures $\delta_{(a,b)}$ and $\delta_{(c,d)}$, resp. Thus, in the presence of a measure, we should be able to find a moment matrix $\widetilde{\mathcal{M}(3)}$ with a 6- or 7-atomic measure (since rank $\widetilde{\mathcal{M}(3)} = 6$ or 7). Denote such measure by $\tilde{\mu}$, with supp $\tilde{\mu} \subseteq \widehat{\mathcal{V}}$. Since $\widehat{\mathcal{V}} \subseteq \mathcal{Z}(r)$, it follows that supp $\tilde{\mu} \subseteq \mathcal{Z}(r)$, and therefore $r(X,Y) = \mathbf{0}$. Thus, we must find $m_1, m_2 \geq 0$ such that $r(X,Y) = \mathbf{0}$.

Step 5. In order to find such m_1 and m_2, we need to solve a linear system of 10 equations with the two unknowns m_1 and m_2. If no nonnegative solutions exist, $\mathcal{M}(3)$ does not have a representing measure. In the case when a solution does exist, we must check whether $\widetilde{\mathcal{M}(3)} \geq 0$ with the fixed m_1 and m_2 (equivalently, $\Lambda_{\widetilde{\mathcal{M}(3)}}(x^i y^j r) = 0$ for $0 \leq i + j \leq 3$).

Step 6. After checking positive semidefiniteness, we still have the two possible cases based on the values of rank $\widetilde{\mathcal{M}(3)}$: If rank $\widetilde{\mathcal{M}(3)} = 6$, then $\widetilde{\mathcal{M}(3)}$ is a flat extension of $\widetilde{\mathcal{M}(2)}$; hence $\widetilde{\mathcal{M}(3)}$ has a 6-atomic measure, and so $\mathcal{M}(3)$ has an 8-atomic measure. Finally, to cover the case rank $\widetilde{\mathcal{M}(3)} = 7$, notice that card $\mathcal{V}(\widetilde{\mathcal{M}(3)}) = 7$; if the cardinality of the variety is 7, then $\widetilde{\mathcal{M}(3)}$ is extremal, so we use Division Algorithm techniques. If card $\mathcal{V}(\widetilde{\mathcal{M}(3)}) \geq 8$, then it follows that $\widetilde{\mathcal{M}(3)}$ admits a representing measure and so does $\mathcal{M}(3)$.

The construction of the Algorithm is therefore complete. □

$\mathcal{M}(3)$ with $r = 8$ and $v = \infty$

Theorem

(RC-L. Fialkow; Op, Th. Adv. Appl., 1998) Assume that $\mathcal{M}(n) \geq 0$ *satisfies (RG) and that* $Y = A1 + BX$ *for some* $A, B \in \mathbb{R}$*. Then* $\mathcal{M}(n)$ *admits a flat extension* $\mathcal{M}(n+1)$*.*

Problem

Let $\mathcal{V}$ *be the algebraic variety of* $\mathcal{M}(3)$*. Assume* $\mathcal{M}(3) \geq 0$*, of rank 8, consistent, with* $\mathcal{M}(2) > 0$*, and with* $v = \infty$*. Under what conditions, does the moment sequence admits a representing measure?*

Proposition

Let $\mathcal{M}(3)$ be a positive semidefinite, recursively generated moment matrix satisfying $XY = 0$. Then $\mathcal{M}(3)$ has a 7-atomic representing measure.

Proposition

Let $\mathcal{M}(3)$ be a positive semidefinite, recursively generated moment matrix, with column relation $X^2 = X$. Then $\mathcal{M}(3)$ admits a 7-atomic representing measure.

Algorithm

Write $p(x,y) = \ell_1(x,y)c_1(x,y)$ and $q(x,y) = \ell_2(x,y)c_2(x,y)$, where ℓ_i is a line and c_i is a conic for $i = 1, 2$.

Case 1. $c(x,y) \equiv c_1(x,y) = c_2(x,y)$

Let $(a_0, b_0) \in \mathcal{Z}(\ell_1) \cap \mathcal{Z}(\ell_2)$. Then (a_0, b_0) must be in the support of a representing measure; otherwise, consistency of $\mathcal{M}(3)$ forces to be a conic column relation in $\mathcal{M}(3)$. Notice that $\mathcal{M}(3)$ has a representing measure if and only if we may write $\mathcal{M}(3)$ as a sum of two moment matrices, that is, for some $\rho_0 > 0$,

$$\mathcal{M}(3) = \rho_0 \mathbf{v}\mathbf{v}^T + \mathcal{M}_c(3), \tag{5.10}$$

where $\mathbf{v} = \left(1, a_0, b_0, a_0^2, a_0 b_0, b_0^2, a_0^3, a_0^2 b_0, a_0 b_0^2, b_0^3\right)^T$ and $\mathcal{M}_c(3)$ is a moment matrix generated with atoms in the graph of $c(x,y) = 0$. It follows that $\mathcal{M}_c(3)$ has the quadratic column relation $c(X,Y) = \mathbf{0}$, and hence $\mathcal{M}_c(3)$ has at least 3 column relations.

Indeed, there is a positive ρ_0 such that rank $\mathcal{M}_c(3) = 7$. For, let $\mathcal{B}$ be the basis of the column space of $\mathcal{M}(3)$. All leading principal minors of $\mathcal{M}_c(3)_{\mathcal{B}}$ are linear in ρ; that is,

$$\det\left(\mathcal{M}_c(3)_{\mathcal{B}}\right) = \det\left(\mathcal{M}(3)_{\mathcal{B}}\right) - \rho\,\lambda \det\left((U^{*}\mathcal{M}(3)_{\mathcal{B}}U)_{\{2,3,\ldots,8\}}\right), \quad (5.11)$$

for some unitary matrix U, where λ is the only nonzero eigenvalue of $(\mathbf{v}\mathbf{v}^{T})_{\mathcal{B}}$. Positive definiteness of $\mathcal{M}(3)_{\mathcal{B}}$ implies that both $\det\left(\mathcal{M}(3)_{\mathcal{B}}\right)$ and $\lambda \det\left((U^{*}\mathcal{M}(3)_{\mathcal{B}}U)_{\{2,3,\ldots,8\}}\right)$ are positive.

We thus can take ρ_0 as $\det(\mathcal{M}(3)_{\mathcal{B}}) / \left(\lambda \det\left((U^*\mathcal{M}(3)_{\mathcal{B}}U)_{\{2,3,\ldots,8\}}\right)\right)$.
Next, if the eigenvalues of $\mathcal{M}(3)$ and $\mathcal{M}_c(3)$ are arranged in ascending order, then we can see that $0 < \lambda_3(\mathcal{M}(3)) \le \lambda_4(\widetilde{\mathcal{M}_c(3)})$. Since rank $\mathcal{M}_c(3) = 7$, it follows that $\mathcal{M}_c(3)$ has the eigenvalue zero with multiplicity 3, through which we can conclude that $\lambda_k(\mathcal{M}_c(3)) = 0$ for $k = 1, 2, 3$ and $\lambda_k(\mathcal{M}_c(3)) > 0$ for $k = 4, \ldots, 10$. In other words, $\mathcal{M}_c(3)$ is positive semidefinite.
In summary, with the specific ρ_0, $\mathcal{M}(3)$ has a representing measure if and only if $\mathcal{M}_c(3)$ has a representing measure. If c is a circle, parabola, or hyperbola, then we know $\mathcal{M}_c(3)$ admits a representing measure; if c is a pair of intersecting or parallel lines, then we apply the two specific situations we studied earlier: $XY = 0$ and $X^2 = X$.

Case 2. $\ell(x,y) \equiv \ell_1(x,y) = \ell_2(x,y)$

Let $(c_i, d_i) \in \mathcal{Z}(c_1) \cap \mathcal{Z}(c_2)$ for $i = 1, \ldots, m_2$ $(3 \leq m_2 \leq 4)$. Similarly, $\mathcal{M}(3)$ has a representing measure if and only if $\mathcal{M}(3)$ can be written as a sum of two moment matrices:

$$\mathcal{M}(3) = \mathcal{M}_\ell(3) + \left(\beta_{ij}^{(c)}\right) \equiv \mathcal{M}_\ell(3) + \mathcal{M}_c(3), \tag{5.12}$$

where $\beta_{ij}^{(c)} = \sum_{k=1}^{m_2} \rho_k^{(c)} c_k^i d_k^j$ for some positive $\rho_i^{(c)}$ $(1 \leq i \leq m_2)$ and $\mathcal{M}_\ell(3)$ is a moment matrix generated by atoms in the line ℓ. We next need to see that $\mathcal{M}_\ell(3)$ must have the column relation $\ell(X,Y) = \mathbf{0}$. Applying the relation to $\mathcal{M}_\ell(3) = \mathcal{M}(3) - \mathcal{M}_c(3)$, we have a linear system of 10 equations in the unknowns, $\rho_1^{(c)}, \ldots, \rho_{m_2}^{(c)}$ (at least 3 of them are positive).

If the system does not have a nonnegative solution set, then $\mathcal{M}(3)$ does not have a representing measure. If we can find a solution of the system, then since $\mathcal{M}_c(3)$ obviously has a representing measure, it follows that we just need to check if $\mathcal{M}_\ell(3) \geq 0$ and if $\mathcal{M}_\ell(3)$ satisfies (RG). If $\mathcal{M}_\ell(3)$ passes both tests, then it has a representing measure and consequently, so does $\mathcal{M}(3)$. □

Classification of sextic MP

r_3	v_3	$v_3 - r_3$	Max Ext		Solution Presented in
7	7	0	$\mathcal{M}(4)$	extremal	RC-S. Yoo; JFA(2014), 2015
7	8	1	$\mathcal{M}(5)$		RC-S. Yoo; JFA(2015)
7	9	2	$\mathcal{M}(6)$		RC-S. Yoo; JFA(2015)
7	∞	N/A	N/A		RC-S. Yoo; JFA(2015)
8	8	0	$\mathcal{M}(4)$	extremal	RC-S. Yoo; 2015
8	9	1	$\mathcal{M}(5)$		RC-S. Yoo; JFA(2015)
8	∞	N/A	N/A		RC-S. Yoo; JFA(2015)
9	∞	N/A	N/A		L. Fialkow; TAMS(2011) (cases)
10	∞	N/A	N/A		unknown

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