

TRUNCATED MOMENT PROBLEMS:
THE EXTREMAL CASE
(JOINT WORK WITH L. FIALKOW AND M. MÖLLER)

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- For a degree $2n$ real d -dimensional multisequence

$\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}_+^d, |i| \leq 2n}$ to have a *representing measure* μ , it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite, and for the algebraic variety associated to β , $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy $\text{rank } \mathcal{M}(n) \leq \text{card } \mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on \mathcal{V} , then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$.

- In joint work with Lawrence Fialkow and Michael Möller, we prove that for the *extremal* case ($\text{rank } \mathcal{M}(n) = \text{card } \mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, rank $\mathcal{M}(n)$ -atomic) representing measure.

- The new results build on our operator-theoretic approach to truncated moment problems, based on matrix positivity and extension, which, via a “functional calculus” for the columns of the associated moment matrix, allows us to obtain existence theorems in case the columns satisfy one of several natural constraints.

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- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \dots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane \mathbb{C} such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \leq i + j \leq 2n);$$

μ is called a **rep. meas.** for γ .

- In earlier joint work with L. Fialkow,
- We have introduced an approach based on matrix positivity and extension, combined with a new “functional calculus” for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\text{supp } \mu \subseteq \mathbb{R} \quad (\text{Hamburger TMP})$$

$$\text{supp } \mu \subseteq [0, \infty) \quad (\text{Stieltjes TMP})$$

$$\text{supp } \mu \subseteq [a, b] \quad (\text{Hausdorff TMP})$$

$$\text{supp } \mu \subseteq \mathbb{T} \quad (\text{Toeplitz TMP})$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**. For simplicity, in this talk we focus on one complex variable or two real variables, although several results have multivariable versions.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We have fully resolved, among others, the cases

$$\bar{Z} = \alpha 1 + \beta Z$$

and

$$Z^k = p_{k-1}(Z, \bar{Z}) \quad (1 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1; \deg p_{k-1} \leq k - 1).$$

- We obtain applications to quadrature problems in numerical analysis.
- We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on \mathbb{R}^d .

- Very recently, we have begun to use our methods to solve FULL moment problems, by first solving truncated MP's, and then applying J. Stochel's limiting argument.
- Our matrix extension approach works equally well to **localize the support** of a rep. meas.
- In the specific case of $K := \text{supp } \mu$, a semi-algebraic set determined by a finite collection of complex polynomials $\mathcal{P} = \{p_i(z, \bar{z})\}_{i=1}^m$, i.e.,

$$K = K_{\mathcal{P}} := \{z \in \mathbb{C} : p_i(z, \bar{z}) \geq 0, 1 \leq i \leq m\},$$

we obtain an existence criterion expressed in terms of positivity and extension properties of the moment matrix $M(n)(\gamma)$ associated to γ and of the localizing matrix M_{p_i} corresponding to each p_i .

THEOREM

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases} .$$

Moreover, $\text{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank } A \Leftrightarrow C = W^*AW$.

COROLLARY

Let $A \geq 0$ and assume $\text{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank } A$. Then

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

BASIC POSITIVITY CONDITION

\mathcal{P}_n : polynomials p in z and \bar{z} , $\deg p \leq n$

Given $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$,

$$\begin{aligned} 0 &\leq \int |p(z, \bar{z})|^2 d\mu(z, \bar{z}) \\ &= \sum_{ijkl} a_{ij} \bar{a}_{kl} \int \bar{z}^{i+l} z^{j+k} d\mu(z, \bar{z}) \\ &= \sum_{ijkl} a_{ij} \bar{a}_{kl} \gamma_{i+l, j+k}. \end{aligned}$$

- To understand this “**matricial**” **positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, z, \bar{z}, z^2, \bar{z}z, \bar{z}^2, \dots$$

Define $M[i, j]$ as in

$$M[3, 2] := \begin{pmatrix} \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} \end{pmatrix}$$

Then

(“matricial” positivity) $\sum_{ijkl} a_{ij} \bar{a}_{kl} \gamma_{i+l, j+k} \geq 0$

$$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0, 0] & M[0, 1] & \dots & M[0, n] \\ M[1, 0] & M[1, 1] & \dots & M[1, n] \\ \dots & \dots & \dots & \dots \\ M[n, 0] & M[n, 1] & \dots & M[n, n] \end{pmatrix} \geq 0.$$

For example,



$$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

- $M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$

In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{06}, \dots, \gamma_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. But this can also be done for $n = 2$.

MOMENT PROBLEMS AND NONNEGATIVE POLYNOMIALS (FULL MP CASE)

- $\mathcal{M} := \{\gamma \equiv \gamma^{(\infty)} : \gamma \text{ admits a rep. meas. } \mu\}$
- $\mathcal{B}_+ := \{\gamma \equiv \gamma^{(\infty)} : M(\infty)(\gamma) \geq 0\}$

Clearly, $\mathcal{M} \subseteq \mathcal{B}_+$

- (Berg, Christensen and Ressel) $\gamma \in \mathcal{B}_+$, γ bounded $\Rightarrow \gamma \in \mathcal{M}$
- (Berg and Maserick) $\gamma \in \mathcal{B}_+$, γ exponentially bounded $\Rightarrow \gamma \in \mathcal{M}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ flat $\Rightarrow \gamma \in \mathcal{M}$

- \mathcal{P}_+ : nonnegative poly's
 Σ^2 : sums of squares of poly's

Clearly, $\Sigma^2 \subseteq \mathcal{P}_+$

Duality

For C a cone in $\mathbb{R}^{\mathbb{Z}_+^n}$, we let

$$C^* := \{x \in \mathbb{R}^{\mathbb{Z}_+^n} : \text{supp}(x) \text{ is finite and } \langle p, x \rangle \geq 0 \text{ for all } p \in C\}.$$

- (Haviland) $\mathcal{P}_+^* = \mathcal{M}$

For, the Riesz functional $\Lambda_\gamma(p) := p(\gamma)$ maps $\mathcal{M} \rightarrow \mathcal{P}_+^*$ ($\gamma \mapsto \Lambda_\gamma$), and Haviland's Theorem says that this map is onto, that is, there exists μ r.m. for γ if and only if $\Lambda_\gamma \geq 0$ on \mathcal{P}_+ .

At present, we are attempting to formulate and prove a version of this result for TMP.

- $\mathcal{P}_+ = \mathcal{M}^*$ (straightforward once we have a r.m.)
- $\mathcal{B}_+ = (\Sigma^2)^*$ (straightforward)
- (Berg, Christensen and Jensen) $(\mathcal{B}_+)^* = \Sigma^2$

- $(n = 1) \mathcal{P}_+ = \Sigma^2 \Rightarrow \mathcal{P}_+^* = (\Sigma^2)^* \Rightarrow \mathcal{M} = \mathcal{B}_+$ (Hamburger)
- (Hilbert) Description of pairs (n, d) for which every poly of degree d in n indeterminates which is nonnegative on \mathbb{R}^n is a sum of squares (cf. Reznick 2000)

FUNCTIONAL CALCULUS

For $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$ define

$$p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j.$$

If there exists a rep. meas. μ , then

$$p(Z, \bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(RG) If $p, q, pq \in \mathcal{P}_n$, and $p(Z, \bar{Z}) = 0$,

then $(pq)(Z, \bar{Z}) = 0$.

THEOREM

(RC-LF, 1998) Let γ be a truncated moment sequence. TFAE:

(i) γ has a rep. meas.;

(ii) γ has a rep. meas. with moments of all orders;

(iii) γ has a compactly supported rep. meas.;

(iv) γ has a finitely atomic rep. meas. (with at most $(n+2)(2n+3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n+k+1)$ (here $k \leq 2n^2 + 6n + 6$).

CASE OF FLAT DATA

Recall: If μ is a rep. meas. for $M(n)$, then $\text{rank } M(n) \leq \text{card supp } \mu$.

$$\gamma \text{ is flat if } M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}.$$

THEOREM

(RC-LF, 1996) *If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.*

THEOREM

(RC-LF, 1996) *The truncated moment sequence γ has a rank $M(n)$ -atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

To find μ concretely, let $r := \text{rank } M(n)$ and look for the relation

$$Z^r = c_0 1 + c_1 Z + \dots + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + \dots + c_{r-1} z^{r-1})$$

and solve the [Vandermonde](#) equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

Consider the full MP

$$\int \bar{z}^i z^j d\mu = \gamma_{ij} \quad (i, j \geq 0),$$

where $\text{supp } \mu \subseteq K$, for K a closed subset of \mathbb{C} .

The **Riesz functional** is given by

$$\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij} \quad (i, j \geq 0).$$

- Riesz-Haviland: There exists μ with $\text{supp } \mu \subseteq K \Leftrightarrow \Lambda_\gamma(p) \geq 0$ for all p such that $p|_K \geq 0$.

If q is a polynomial in z and \bar{z} , and

$$K \equiv K_q := \{z \in \mathbb{C} : q(z, \bar{z}) \geq 0\},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for μ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p} d\mu \geq 0 \quad (\text{all } p).$$

- K. Schmüdgen (1991): If K_q is compact, $L_\gamma(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all p , then there exists μ with $\text{supp } \mu \subseteq K_q$.
- We shall establish a version of this result for truncated MP's.

First, recall that $p(Z, \bar{Z}) = 0$ implies $\text{supp } \mu \subseteq \mathcal{Z}(p)$. We define the algebraic variety of γ as

$$\mathcal{V}(\gamma) := \bigcap_{\substack{p \in \mathcal{P}_n \\ p(Z, \bar{Z})=0}} \mathcal{Z}(p),$$

and observe that $\text{rank } M(n) \leq \text{card } \text{supp } \mu \leq \text{card } \mathcal{V}(\gamma)$, from which it follows that

$\text{card } \mathcal{V}(\gamma) < \text{rank } M(n) \Rightarrow$ there is no rep. meas. μ .

LOCALIZATION OF SUPPORT: MAIN THEOREM

THEOREM

(RC-LF, 2000) Let $M(n) \geq 0$ and suppose $\deg(q) = 2k$ or $2k - 1$ for some $k \leq n$. Then $\exists \mu$ with rank $M(n)$ atoms and $\text{supp } \mu \subseteq K_q$ if and only if \exists a flat extension $M(n+1)$ for which $M_q(n+k) \geq 0$. In this case, $\exists \mu$ with exactly $\text{rank } M(n) - \text{rank } M_q(n+k)$ atoms in $\mathcal{Z}(q)$.

REMARK

M. Laurent (2005) has recently found an alternative proof, using ideas from real algebraic geometry.

THE ALGEBRAIC VARIETY OF A TMP

Recall that if $\gamma^{(2n)}$ admits a rep. meas., then

$$\begin{aligned} M(n) &\equiv M(n)(\gamma) \geq 0 \\ M(n) &\text{ is } RG && (10.1) \\ \text{rank } M(n) &\leq \text{card } \mathcal{V}(\gamma). \end{aligned}$$

QUESTION

Assume $M(n)$ satisfies (10.1), and $M(n)$ is singular. Does γ admit a rep. meas.?

QUESTION

For which $p \in \mathcal{P}_n$ do (10.1) and $p(Z, \bar{Z}) = 0$ imply that γ has a rep. meas.?

(Fialkow) Consider $p(z, \bar{z}) \equiv z^k - q(z)$, with $\deg q < k$. If k is minimal, if γ satisfies (10.1) and if $p(Z, \bar{Z}) = 0$, then $B := \{1, Z, Z^2, \dots, Z^{k-1}\}$ is lin. indep. Moreover, $k \geq \text{card } \mathcal{V}(\gamma) \geq \text{rank } M(n) \geq k$, so B is indeed a basis for $\mathcal{C}_{M(n)}$. It follows that $M(n)$ is flat, and it therefore admits a k -atomic rep. meas.

THE QUARTIC MOMENT PROBLEM

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A 1$ (Dirac measure)
- $\bar{Z} = A 1 + B Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A 1 + B Z + C \bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A 1 + B Z + C \bar{Z} + D Z^2$

$$D = 0 \Rightarrow \bar{Z}Z = A 1 + B Z + \bar{B} \bar{Z} \text{ and } C = \bar{B}$$

$$\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2$$

$$\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}.$$

The functional calculus we have constructed is such that $p(Z, \bar{Z}) = 0$ implies $\text{supp } \mu \subseteq \mathcal{Z}(p)$.

When $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$, the associated algebraic variety is the zero set of a real quadratic equation in $x := \text{Re}[z]$ and $y := \text{Im}[z]$.

Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

- (a) $\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$ parabola; $y = x^2$
- (b) $\bar{W}^2 = -4i1 + W^2$ hyperbola; $yx = 1$
- (c) $\bar{W}^2 = W^2$ pair of intersect. lines; $yx = 0$
- (d) $\bar{W}W = 1$ unit circle; $x^2 + y^2 = 1$.

THEOREM QUARTIC

(RC-LF, 2005) Let $\gamma^{(4)}$ be given, and assume $M(2) \geq 0$ and $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. Then $\gamma^{(4)}$ admits a rep. meas. μ . Moreover, it is possible to find μ with $\text{card supp } \mu = \text{rank } M(2)$, except in some cases when $\mathcal{V}(\gamma^{(4)})$ is a *pair of intersecting lines*, in which cases there exist μ with $\text{card supp } \mu \leq 6$.

Consider now the following property for a polynomial $P \in \mathbb{R}_n[x, y]$:

$\beta \equiv \beta^{(2n)}$ has a rep. meas. supported in $\mathcal{Z}(P)$ if and only if (A'_n)

$\mathcal{M}(n)(\beta)$ is positive semi-definite, RG ,

$P(X, Y) = 0$ in $\mathcal{C}_{\mathcal{M}(n)}$, and $\text{rank } \mathcal{M}(n) \leq \text{card } \mathcal{V}(\mathcal{M}(n))$.

Polynomials which satisfy (A'_n) form an attractive class, because if P satisfies (A'_n) , then the degree- $2n$ moment problem on $P(x, y) = 0$ can be solved by concrete tests involving only elementary linear algebra and the calculation of roots of polynomials.

THEOREM

(RC-LF, 2005) If $\deg P \leq 2$, then P satisfies (A'_n) for every $n \geq \deg P$.

Despite this theorem, there are differences between the parabolic and elliptic moment problems, and the hyperbolic problem. In the former cases, the conditions of (A'_n) always imply the existence of a rank $\mathcal{M}(n)$ -atomic rep. meas., corresponding to a flat extension of $\mathcal{M}(n)$; for this reason, positive Borel measures supported on these curves always admit *Gaussian* cubature rules, i.e., rank $\mathcal{M}(n)$ -atomic cubature rules of degree $2n$ (Fialkow and Petrovic, 2003). By contrast, in the hyperbolic case, minimal rep. meas. μ sometimes entail $\text{card supp } \mu > \text{rank } \mathcal{M}(n)$ (and Gaussian cubature rules may fail to exist).

- The above Theorem is motivated in part by results of J. Stochel, who solved the full moment problem on planar curves of degree at most 2. Paraphrasing Stochel's work (i.e., translating from the language of *moment sequences* into the language of moment matrices), we consider the following property of a polynomial P :

$$\beta^{(\infty)} \text{ has a rep. meas. supported in } P(x, y) = 0 \quad (A)$$

if and only if $\mathcal{M}(\infty)(\beta) \geq 0$ and $P(X, Y) = 0$ in $\mathcal{C}_{\mathcal{M}(\infty)}$.

THEOREM

(Stochel, 1992) If $\deg P \leq 2$, then P satisfies (A).

- Stochel also proved that there exist polynomials of degree 3 that do not satisfy (A).
- The link between TMP and FMP is provided by another result of Stochel (2001):

THEOREM

$\beta^{(\infty)}$ has a rep. meas. supported in a closed set $K \subseteq \mathbb{R}^2$ if and only if, for each n , $\beta^{(2n)}$ has a rep. meas. supported in K .

Recall

Riesz-Haviland: There exists μ with $\text{supp } \mu \subseteq K \Leftrightarrow \Lambda_\gamma(p) \geq 0$ for all p such that $p|_K \geq 0$.

For TMP, the natural analogue won't work; for example, if $d = 1$, $K = \mathbb{R}$, and

$$\mathcal{M}(2) := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0,$$

then Λ_β is \mathbb{R} -positive, but no r.m. exists.

Possible Analogue: $\beta^{(2n)}$ admits a K -r.m. if and only if Λ_β admits a K -positive extension $\Lambda : \mathcal{P}_{2n+2} \rightarrow \mathbb{R}$.

Another useful and relevant notion is that of \mathcal{V} -positivity.

$$\beta_i = \int_{\mathbb{R}^d} x^i d\mu, \quad |i| \leq 2n; \quad (12.1)$$

$\mathcal{P} \equiv \mathbb{R}^d[x] = \mathbb{R}[x_1, \dots, x_d]$: space of real valued d -variable polynomials

$\mathcal{P}_k \equiv \mathbb{R}_k^d[x]$: the subspace of \mathcal{P} consisting of polynomials p with $\deg p \leq k$ ($k \geq 1$).

$\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \rightarrow \mathbb{R}$ (Riesz functional): if $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$, then

$$\Lambda(p) := \sum_{|i| \leq 2n} a_i \beta_i$$

- In the presence of a representing measure μ , we have $\Lambda(p) = \int p d\mu$.

\hat{p} : coefficient vector (a_i) of p .

$\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$: moment matrix, with rows and columns X^i indexed by the monomials of \mathcal{P}_n in degree-lexicographic order

$d = n = 2$: the columns of $\mathcal{M}(2)$ are denoted as $1, X_1, X_2, X_1^2, X_2 X_1, X_2^2$

- $\mathcal{M}(n)$ is a real symmetric matrix characterized by

$$\langle \mathcal{M}(n)\hat{p}, \hat{q} \rangle = \Lambda(pq) \quad (p, q \in \mathcal{P}_n). \quad (12.2)$$

- If μ is a representing measure for β , then

$$\langle \mathcal{M}(n)\hat{p}, \hat{p} \rangle = \Lambda(p^2) = \int p^2 d\mu \geq 0; \text{ it follows that } \mathcal{M}(n) \geq 0.$$

The algebraic variety of β is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$.

- If β admits a representing measure μ , then

$$p \in \mathcal{P}_n \text{ satisfies } \hat{p} \in \ker \mathcal{M}(n) \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus $\text{supp } \mu \subseteq \mathcal{V}$, so $r := \text{rank } \mathcal{M}(n)$ and $v := \text{card } \mathcal{V}$ satisfy

$$r \leq \text{card } \text{supp } \mu \leq v.$$

If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$.

BASIC NECESSARY CONDITIONS FOR THE EXISTENCE OF A REPRESENTING MEASURE

$$\text{(Positivity)} \quad \mathcal{M}(n) \geq 0 \quad (12.3)$$

$$\text{(Consistency)} \quad p \in \mathcal{P}_{2n}, p|_{\mathcal{V}} \equiv 0 \implies \Lambda(p) = 0 \quad (12.4)$$

$$\text{(Variety Condition)} \quad r \leq v, \text{ i.e., } \text{rank } \mathcal{M}(n) \leq \text{card } \mathcal{V}. \quad (12.5)$$

Consistency implies

$$\text{(Recursiveness)} \quad p, q, pq \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n) \implies \hat{p}q \in \ker \mathcal{M}(n). \quad (12.6)$$

Previous results:

- For $d = 1$ (the T *Hamburger* MP for \mathbb{R}), positivity and recursiveness are sufficient
- For $d = 2$, there exists $\mathcal{M}(3) > 0$ for which β has no representing measure
- In general, Positivity, Consistency and the Variety Condition are not sufficient.

QUESTION C

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, β is *consistent*, and $r \leq v$, does β admit a representing measure?

More generally, the following question remained unsolved until very recently.

QUESTION RG

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, *recursively generated*, and $r \leq v$, does β admit a representing measure?

- RC-LF: If $d = 2$ and $\mathcal{M}(n)\hat{p} = 0$ for some p with $\deg p \leq 2$, then Question RG has an affirmative answer. (Theorem Quartic)
- RC-LF-MM: If $d = 2$, $\widehat{y - x^3} \in \ker \mathcal{M}(n)$ and $r = v \leq 7$, then Question RG has an affirmative answer.
- RC-LF-MM: If $d = 2$, $\widehat{y - x^3} \in \ker \mathcal{M}(n)$ and $r = v = 8$, then Question RG has a negative answer.

The next result gives an affirmative answer to Question C in the *extremal* case, i.e., $r = v$.

THEOREM EXT

(RC-LF and M. Möller, 2005) For $\beta \equiv \beta^{(2^n)}$ **extremal**, i.e., $r = v$, the following are equivalent:

- (i) β has a representing measure;
- (ii) β has a unique representing measure, which is rank $\mathcal{M}(n)$ -atomic (minimal);
- (iii) $\mathcal{M}(n) \geq 0$ and β is consistent.

- In many cases, the conditions of Theorem EXT provide a concrete solution to the extremal case of TMP. Indeed, only elementary linear algebra is required to verify that $\mathcal{M}(n) \geq 0$, to compute rank $\mathcal{M}(n)$, and to identify the column relations which define \mathcal{V} .
- If the points of \mathcal{V} can be computed exactly, then only elementary linear algebra is required to verify that β is consistent.
- Question RG is significant because recursiveness is generally a simpler condition to work with than consistency. For example, we often have $M(n) \geq 0$ and $M(n-1) > 0$ (positive definite), in which case $M(n)$ is obviously recursively generated, but we do not know whether $r \leq v$ implies that $M(n)$ is consistent in this case.

- The extremal case is inherent in TMP:
C. Bayer and J. Teichmann (2006) (extending a classical theorem of V. Tchakaloff and its successive generalizations by I.P. Mysovskikh, M. Putinar and RC-LF) recently proved that if $\beta^{(2n)}$ has a representing measure, then it has a **finitely atomic** representing measure;
- RC-LF showed that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $M(n)$ admits an extension to a positive moment matrix $M(n+k)$ (for some $k \geq 0$, which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $M(n+k+1)$);

- in many instances, $M(n + k + 1)$ is an extremal moment matrix for which there is a computable rank $M(n + k)$ -atomic representing measure μ . Clearly, μ is also a finitely atomic representing measure for $\beta^{(2n)}$, and every finitely atomic representing measure for $\beta^{(2n)}$ arises in this way.
- there exist extremal TCMP of arbitrarily large degree
- Moment theory can sometimes be used to estimate the number and location of the zeros of a prescribed polynomial; for example, we can use TMP techniques to show that the polynomial

$$p(z) \equiv z^{2n} + az^{2n-1} - az - 1 \quad (0 < a < 1)$$

has $2n$ distinct zeros, all in the unit circle.

REAL IDEALS AND NECESSARY CONDITIONS

- If $\beta^{(2n)}$ has a representing measure, then the Riesz functional

$$\Lambda : \mathcal{P}_{2n} \rightarrow \mathbb{R}, \quad \Lambda(x^i) := \beta_i \equiv \int_{\mathbb{R}^d} x^i d\mu \quad (|i| \leq 2n),$$

is square positive, that is,

$$p \in \mathcal{P}_n \Rightarrow \Lambda(p^2) \geq 0.$$

- If we assume that for a representing measure μ all moments

$$\int_{\mathbb{R}^d} x^i d\mu, \quad i \in \mathbb{Z}_+^d$$

are convergent, then we can extend Λ to \mathcal{P} by letting

$$\Lambda(x^i) := \int_{\mathbb{R}^d} x^i d\mu, \quad i \in \mathbb{Z}_+^d,$$

thus obtaining a square positive functional over \mathcal{P} .

Under this assumption the set

$$\mathcal{I} := \{p \in \mathcal{P} : \Lambda(p^2) = 0\}$$

is a *real ideal*, i.e., it is an ideal ($p_1, p_2 \in \mathcal{I} \Rightarrow p_1 + p_2 \in \mathcal{I}$ and $p \in \mathcal{I}, q \in \mathcal{P} \Rightarrow pq \in \mathcal{I}$) and satisfies one of the two equivalent conditions:

- (i) For $s \in \mathbb{Z}_+, p_1, \dots, p_s \in \mathcal{P} : \sum_{i=1}^s p_i^2 \in \mathcal{I} \Rightarrow \{p_1, \dots, p_s\} \subseteq \mathcal{I}$
- (ii) There exists $G \subseteq \mathbb{R}^d$ such that for all $p \in \mathcal{P} : p|_G \equiv 0 \Rightarrow p \in \mathcal{I}$.

If \mathcal{I} is a real ideal, then one may take for $G \subseteq \mathbb{R}^d$ the real variety

$$V_{\mathbb{R}}(\mathcal{I}) := \{w \in \mathbb{R}^d : f(w) = 0 \quad (\text{all } f \in \mathcal{I})\}.$$

But one may also take any subset G of $V_{\mathbb{R}}(\mathcal{I})$ containing sufficiently many points, such that

$$p \in \mathcal{P}, p|_G \equiv 0 \Rightarrow p|_{V_{\mathbb{R}}(\mathcal{I})} \equiv 0.$$

For instance, if the real variety is a (real) line, one may take for G a subset of infinitely many points on that line. On the other hand, if $V_{\mathbb{R}}(\mathcal{I})$ is a finite set of points, then necessarily $G = V_{\mathbb{R}}(\mathcal{I})$.

If \mathcal{I} is an ideal, its subset $\mathcal{I}_k := \mathcal{I} \cap \mathcal{P}_k$ is an \mathbb{R} -vector subspace of \mathcal{P}_k . One can then introduce the *Hilbert function* of \mathcal{I} by

$$H_{\mathcal{I}}(k) := \dim \mathcal{P}_k - \dim \mathcal{I}_k, \quad k \in \mathbb{Z}_+.$$

- Both $k \mapsto \dim \mathcal{I}_k$ and $k \mapsto H_{\mathcal{I}}(k)$ are nondecreasing functions.
- For sufficiently large k , say $k \geq k_0$, $H_{\mathcal{I}}(k)$ becomes a polynomial in k , the so called *Hilbert polynomial of \mathcal{I}* , whose degree equals the dimension of \mathcal{I} .

Assume now that $\beta^{(2n)}$ admits a representing measure μ . Then, irrespective of whether the Riesz functional $\Lambda : \mathcal{P}_n \rightarrow \mathbb{R}$ can be extended to a square positive functional $\Lambda : \mathcal{P} \rightarrow \mathbb{R}$, we can define the ideal

$$\mathcal{I}(\mu) := \{p \in \mathcal{P} : p|_{\text{supp } \mu} \equiv 0\}. \quad (13.1)$$

Since $\text{supp } \mu \subseteq \mathbb{R}^d$, $\mathcal{I}(\mu)$ is a real ideal, which we will call the *real ideal of* $\beta^{(2n)}$.

LEMMA

Assume $\beta^{(2n)}$ has a representing measure, and let $\mathcal{I}(\mu)$ be its real ideal.

Then

$$\{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\} = \mathcal{I}(\mu) \cap \mathcal{P}_n. \quad (13.2)$$

If t_1, \dots, t_N denote the monomials $x^i \in \mathcal{P}_n$ in degree-lexicographic order, then the row vectors of $\mathcal{M}(n)$ and the row vectors of

$W_n := \{(t_1(w), \dots, t_N(w)) : w \in \text{supp } \mu\}$ span the same subspace of \mathbb{R}^N ;
in particular, $\text{rank } \mathcal{M}(n) = H_{\mathcal{I}(\mu)}(n)$.

LEMMA

Let $\Lambda : \mathcal{P}_{2n} \rightarrow \mathbb{R}$ be a linear functional and let $\mathcal{V} \equiv \{w_1, \dots, w_m\} \subseteq \mathbb{R}^d$.

The following statements are equivalent.

- (a) There exist $\alpha_i \in \mathbb{R}$ such that $\Lambda(p) = \sum_{i=1}^m \alpha_i p(w_i)$ (all $p \in \mathcal{P}_{2n}$).
- (b) (*Consistency*) If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = 0$.

- Recall that

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, p(X)=0} \mathcal{Z}_p.$$

Let $\mathcal{P}_n|_{\mathcal{V}}$ denote the restriction to \mathcal{V} of the polynomials in \mathcal{P}_n , and consider the mapping $\varphi_\beta : \mathcal{C}_{M(n)} \rightarrow \mathcal{P}_n|_{\mathcal{V}}$ given by $p(X) \mapsto p|_{\mathcal{V}}$. The map φ_β is well-defined, and β has a representing measure μ , then φ_β is 1-1.

PROPOSITION

Let β, φ_β and $M(n)(\beta)$ be as before. Then

$$\beta \text{ consistent} \implies \varphi_\beta \text{ 1-1} \implies M(n)(\beta) \text{ recursively generated.}$$

PROPOSITION

For $d = 2$ (the plane), if $M(n)(\beta)$ is recursively generated and \mathcal{V}_β is a *proper, infinite irreducible curve*, then β is consistent.

SOLUTION OF THE EXTREMAL MOMENT PROBLEM

Assume that $\beta \equiv \beta^{(2n)}$ is extremal, i.e., $r := \text{rank } M(n)$ and $v := \text{card } \mathcal{V}$ satisfy $r = v$. Let $\mathcal{V} \equiv \{w_1, \dots, w_r\}$ denote the distinct points of \mathcal{V} . If μ is a representing measure for β , then since $\text{supp } \mu \subseteq \mathcal{V}$ and $r \leq \text{card } \text{supp } \mu \leq v$, the extremal hypothesis $r = v$ implies that $\text{supp } \mu = \mathcal{V}$. Thus μ is necessarily is of the form

$$\mu = \sum_{i=1}^r \rho_i \delta_{w_i}. \quad (15.1)$$

Let p_1, \dots, p_r be polynomials in \mathcal{P}_n such that $\mathcal{B} \equiv \{p_1(X), \dots, p_r(X)\}$ is a basis for the column space of $\mathcal{M}(n)$, and set

$$W \equiv W_{\mathcal{B}} := \begin{pmatrix} p_1(w_1) & \cdot & \cdot & \cdot & p_1(w_r) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_r(w_1) & \cdot & \cdot & \cdot & p_r(w_r) \end{pmatrix}.$$

Also, recall that if $\mathcal{P}_n|_{\mathcal{V}}$ denote the restriction to \mathcal{V} of the polynomials in \mathcal{P}_n , the map $\varphi_{\mathcal{B}} : \mathcal{C}_{\mathcal{M}(n)} \rightarrow \mathcal{P}_n|_{\mathcal{V}}$ is given by $p(X) \mapsto p|_{\mathcal{V}}$.

LEMMA

The following are equivalent for β extremal:

- i) $\varphi_{\mathcal{B}}$ is 1-1, i.e., $p \in \mathcal{P}_n$, $p|_{\mathcal{V}} \equiv 0 \implies p(X) = 0$ in $\mathcal{C}_{\mathcal{M}}(n)$;*
- ii) For any basis \mathcal{B} of $\mathcal{C}_{\mathcal{M}}(n)$, $W_{\mathcal{B}}$ is invertible;*
- iii) There exists a basis \mathcal{B} for $\mathcal{C}_{\mathcal{M}}(n)$ such that $W_{\mathcal{B}}$ is invertible.*

THEOREM

(RC-LF and M. Möller, 2005) For $\beta \equiv \beta^{(2^n)}$ extremal, TFAE:

- (i) β has a representing measure;*
- (ii) β has a unique representing measure, which is a rank $\mathcal{M}(n)$ -atomic;*
- (iii) For some (respectively, for every) basis \mathcal{B} for $\mathcal{C}_{\mathcal{M}}(n)$, $W_{\mathcal{B}}$ is invertible and $\mu_{\mathcal{B}}$ is a representing measure for β ;*
- (iv) β is consistent and $\mathcal{M}(n) \geq 0$.*

EXAMPLE

Consider

$$\mathcal{M}(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & 2000 \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & 2000 & 338881 \end{pmatrix}.$$

We have $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$, $r = 8$,

$$Y = X^3, \tag{16.1}$$

and

$$Y^3 = q(X, Y). \tag{16.2}$$

where $q(x, y) := -2285x + 5720y - 34441yx^2 + 578y^2x$. Here $\nu = 9$. Thus, $\mathcal{M}(3)$ is positive, recursively generated, $r < \nu$, and the minimal representing measure for $\beta^{(6)}$ is ν -atomic.