Truncated Moment Problems: The Interplay Between Functional Analysis, Algebraic Geometry and Optimization

(joint work with L.A. Fialkow, H.M. Möller and S. Yoo)

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A) Low-order polynomial approx. on subintervals of decreasing size

Commonly used Newton-Cotes formulas

T	$n=1$	$\int_a^b f(x)\,dx$	$=$	$\frac{h}{2}[f(a)+f(b)]-\frac{h^3}{12}f''(\xi)$
S	$n=2$	$\int_a^b f(x)\,dx$	$=$	$\frac{h}{3}[f(a)+4f(\frac{a+b}{2})+f(b)]-\frac{h^5}{90}f^{(4)}(\xi)$
$\frac{3}{8}$	$n=3$	$\int_a^b f(x)\,dx$	$=$	$\begin{cases} \frac{3h}{8}[f(a)+3f(a+h)+3f(b-h)+f(b)] \\ -\frac{3h^5}{80}f^{(4)}(\xi) \end{cases}$
	$n=4$	$\int_a^b f(x)\,dx$	$=$	$\begin{cases} \frac{2h}{45}[7f(a)+32f(a+h)+12f(\frac{a+b}{2}) \\ +32f(b-h)+7f(b)]-\frac{8h^7}{945}f^{(6)}(\xi) \end{cases}$

B) Polynomial approximation of increasing degree, using fewer, strategically-placed nodes

Definition

A quadrature (or cubature) rule of size p and precision m is a numerical integration formula which uses p nodes, is exact for all polynomials of degree at most m, and fails to recover the integral of some polynomial of degree $m+1$.

Gaussian Quadrature (size n, precision $2n-1$)

$\int_{-1}^{1} f(t)\, dt = \sum_{j=0}^{n-1} \rho_j f(t_j^{(n)})$ for every polynomial $f \in \mathbf{R}_{2n-1}[t]$

(Gaussian means minimum number of nodes possible)

Interpolating Equations:

$$\sum_{j=0}^{n-1} \rho_j t_j^k = \int_{-1}^{1} t^k \, dt = \begin{cases} 0 & k = 1, 3, ..., 2n-1 \\ \\ \frac{2}{k+1} & k = 0, 2, ..., 2n-2 \end{cases}$$

Example: $n = 2$

$$\begin{cases} \rho_0 + \rho_1 & = & 2 \\ \rho_0 t_0 + \rho_1 t_1 & = & 0 \\ \rho_0 t_0^2 + \rho_1 t_1^2 & = & \frac{2}{3} \\ \rho_0 t_0^3 + \rho_1 t_1^3 & = & 0 \end{cases}$$

$\rho_0 = \rho_1 = 1;\ t_0 = -\frac{\sqrt{3}}{3},\ t_1 = \frac{\sqrt{3}}{3}.$

$$\int_{-1}^{1} \sum_{k=0}^{3} a_k t^k = \sum_{j=0}^{1} \rho_j \sum_{k=0}^{3} a_k t_j^k$$

NA textbooks prove this by using orthogonal Legendre polynomials

($t_0 < \ldots < t_{n-1}$ are the zeros of the nth Legendre polynomial)

(RC-L. Fialkow, 1990) Can do this as follows:

$\gamma_0 := 2,\ \gamma_1 := 0,\ \gamma_2 := \frac{2}{3},\ \gamma_3 := 0,\ \gamma_4 := \frac{2}{5}$, etc.

Assume n even, and form the Hankel matrix

$$H(n) := \begin{pmatrix} 2 & 0 & \frac{2}{3} & \cdots & 0 & \frac{2}{n+1} \\ 0 & \frac{2}{3} & 0 & \cdots & \frac{2}{n+1} & 0 \\ \frac{2}{3} & 0 & & \cdots & 0 & \frac{2}{n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{2}{n+1} & 0 & \cdots & \frac{2}{2n-1} & 0 \\ \frac{2}{n+1} & 0 & \frac{2}{n+3} & \cdots & 0 & ? \end{pmatrix},$$

label the columns $\mathit{1}, T, T^2, \ldots,$

require that $T^n = \varphi_0 \mathit{1} + \ldots + \varphi_{n-1} T^{n-1}$,

build the polynomial

$$g(t) := t^n - (\varphi_0 + \ldots + \varphi_{n-1} t^{n-1}),$$

(non-iterative construction of Legendre polynomials) find its zeros ($t_0 < \ldots < t_{n-1}$),

and

compute the densities using the Vandermonde system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ t_0^{n-1} & t_1^{n-1} & \cdots & t_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{n-1} \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \cdots \\ \gamma_{n-1} \end{pmatrix}.$$

To solve the Gaussian quadrature problem, RC and Fialkow's basic idea was to augment the original Hankel matrix by one row and one column at a time, preserving the rank (which a fortiori preserves positivity):

$$H(n) \prec H(n+1) \prec \ldots H(\infty)$$

Then define

$$\langle p, q \rangle_{H(\infty)} := (H(\infty)\widehat{p}, \widehat{q})_{\ell_2},$$

and show that

$$\langle p, q \rangle_{H(\infty)} = \int p\bar{q} \; d\mu$$

for some finitely atomic rep. meas., with supp $\mu = \mathcal{Z}(g)$.

The Truncated Real Moment Problem

Given a family of real numbers β: $\beta_0, \beta_1, \ldots, \beta_{2n}$ with $\beta_0 > 0$, the **TMP** entails finding a positive Borel measure μ supported in the real line $\mathbb{R}$ such that

$$\beta_i = \int t^i \, d\mu \qquad (0 \le i \le 2n);$$

μ is called a **representing measure** for β.

Theorem

*FULL MP (***Hamburger, 1920***)*

$$\exists \mu \Leftrightarrow A(n) := (\beta_{i+j})_{i,j=0}^{n} \equiv \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \ddots & \cdots \\ \beta_2 & \beta_3 & \ddots & \ddots & \cdots \\ \beta_3 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \ge 0 \ \forall \quad n \ge 0.$$

Theorem

FULL MP (**Stieltjes, 1894**)

$$\exists \mu \text{ with } \operatorname{supp} \mu \subseteq [0, +\infty)$$

$$\Leftrightarrow (\beta_{i+j})_{i,j=0}^{n} \geq 0 \text{ and } (\beta_{i+j+1})_{i,j=0}^{n} \geq 0 \ \forall \ \ n \geq 0.$$

$$\begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \ddots & \cdots \\ \beta_2 & \beta_3 & \ddots & \ddots & \cdots \\ \beta_3 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots \\ \beta_2 & \beta_3 & \beta_4 & \ddots & \cdots \\ \beta_3 & \beta_4 & \ddots & \ddots & \cdots \\ \beta_4 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ & & \text{(localizing matrix)} & & \end{pmatrix} \geq 0$$

The Truncated Complex Moment Problem

- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,2n}, \ldots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 2n);$$

  μ is called a **rep. meas.** for γ.
- In earlier joint work with L. Fialkow,
- We have introduced an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0,\infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a,b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**. For simplicity, in this talk we focus on **one complex variable or two real variables**, although several results have multivariable versions.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We have fully resolved, among others, the cases

$$\bar{Z} = \alpha 1 + \beta Z$$

and

$$Z^k = p_{k-1}(Z, \bar{Z}) \;\; (1 \leq k \leq [\frac{n}{2}] + 1; \deg p_{k-1} \leq k-1).$$

- We obtain applications to quadrature problems in numerical analysis.
- We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on $\mathbb{R}^d$.

Applications

- Subnormal Operator Theory (unilateral weighted shifts)

  For $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$, the weighted shift W_α is subnormal if and only if the moment problem $\alpha_0^2 \alpha_1^2 \cdots \alpha_{k-1}^2 = \int s^k d\mu(s)$ is soluble.
- Physics (determination of contours)
- Computer Science (image recognition and reconstruction)
- Geography (location of proposed distribution centers)
- Probability (reconstruction of p.d.f.'s)

- Environmental Science (oil spills, via quadrature domains)
- Engineering (tomography)
- Optimization (finding the global minimum of a real polynomial in several real variables - J. Lasserre)
- Function Theory (a dilation-type structure theorem in Fejér-Riesz factorization theory - S. McCullough)
- Geophysics (inverse problems, cross sections)

Typical Problem: Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.

Basic Positivity Condition

$\mathcal{P}_n$: polynomials p in z and $\bar{z}$, $\deg p \leq n$

Given $p \in \mathcal{P}_n$, $p(z,\bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$,

$$0 \leq \int \mid p(z,\bar{z}) \mid^2 d\mu(z,\bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(z,\bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell,j+k}.$$

- To understand this **"matricial" positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots$$

Define $M[i,j]$ as in

$$M[3,2] := \begin{pmatrix} \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} \end{pmatrix}$$

Then

$$(\textbf{``matricial'' positivity}) \quad \sum_{ijk\ell} a_{ij}\bar{a}_{k\ell}\gamma_{i+\ell,j+k} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0,0] & M[0,1] & ... & M[0,n] \\ M[1,0] & M[1,1] & ... & M[1,n] \\ ... & ... & ... & \ldots \\ M[n,0] & M[n,1] & \ldots & M[n,n] \end{pmatrix} \geq 0.$$

For example,

- $$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

- $$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Similarly, one can build $M(\infty)$.

In the real case, $\mathcal{M}(n)_{ij} := \gamma_{i+j},\ i,j \in \mathbb{Z}_+^2$.

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{06}, \ldots, \gamma_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. But this can also be done for $n = 2$.

Moment Problems and Nonnegative Polynomials (FULL MP Case)

- $\mathcal{M} := \{\gamma \equiv \gamma^{(\infty)} : \gamma \text{ admits a rep. meas. } \mu\}$
- $\mathcal{B}_+ := \{\gamma \equiv \gamma^{(\infty)} : M(\infty)(\gamma) \geq 0\}$

  Clearly, $\mathcal{M} \subseteq \mathcal{B}_+$

- (Berg, Christensen and Ressel) $\gamma \in \mathcal{B}_+$, γ bounded $\Rightarrow \gamma \in \mathcal{M}$
- (Berg and Maserick) $\gamma \in \mathcal{B}_+$, γ exponentially bounded $\Rightarrow \gamma \in \mathcal{M}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ finite rank $\Rightarrow \gamma \in \mathcal{M}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ flat $\Rightarrow \gamma \in \mathcal{M}$

- $\mathcal{P}_+$: nonnegative poly's

  Σ^2 : sums of squares of poly's

  Clearly, $\Sigma^2 \subseteq \mathcal{P}_+$

Duality

For C a cone in $\mathbb{R}^{\mathbb{Z}_+^2}$, we let

$$C^* := \{\xi \in \mathbb{R}^{\mathbb{Z}_+^2} : \text{supp}(\xi) \text{ is finite and } \langle p, \xi \rangle \geq 0 \text{ for all } p \in C\}.$$

- (Riesz-Haviland) $\mathcal{P}_+^* = \mathcal{M}$

  For, consider the Riesz functional $\Lambda_\gamma(p) := p(\gamma) \equiv \langle p, \gamma \rangle$, which induces a map $\mathcal{M} \to \mathcal{P}_+^*$ $(\gamma \mapsto \Lambda_\gamma)$; Haviland's Theorem says that this maps is onto, that is, there exists μ r.m. for γ if and only if $\Lambda_\gamma \geq 0$ on $\mathcal{P}_+$.

  There exists a version of this result for TMP, as we will see shortly.

Can one localize the support of a representing measure?

J. Stochel solved the **full** moment problem on planar curves of degree at most 2. Paraphrasing Stochel's work (i.e., translating from the language of *moment sequences* into the language of moment matrices), we consider the following property of a polynomial P:

$$\beta^{(\infty)} \text{ has a rep. meas. supported in } P(x,y) = 0 \qquad (A)$$

$$\text{if and only if } \mathcal{M}(\infty)(\beta) \geq 0 \text{ and } P(X,Y) = 0 \text{ in } \mathcal{C}_{\mathcal{M}(\infty)}.$$

Theorem

(Stochel, 1992) If $\deg P \leq 2$*, then* P *satisfies* (A)*.*

Stochel also proved that there exist polynomials of degree 3 that do not satisfy (A).

The link between TMP and FMP is provided by another result of Stochel (2001):

Theorem

$\beta^{(\infty)}$ has a rep. meas. supported in a closed set $K \subseteq \mathbb{R}^2$ if and only if, for each n, $\beta^{(2n)}$ has a rep. meas. supported in K.

A Version of Riesz-Haviland for TMP

Given a moment sequence β, the Riesz functional is

$$L_\beta(p) := p(\beta) \;\; (p \in \mathbb{C}[z, \bar{z}]).$$

Recall the Riesz-Haviland Theorem:

$\exists\ \mu$ rep. meas. for $\beta \Leftrightarrow L \equiv L_\beta \geq 0$ on $\mathcal{P}_+$.

- For TMP, the natural analogue won't work.
- We say that the Riesz functional L is K*-positive* if

$p \in \mathcal{P}$ and $p|K \geq 0 \Rightarrow L(p) \geq 0.$

Consider the case

$d = 1,\ K = \mathbb{R}$, and

$$M(2) := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0.$$

In this case,

$$\mathrm{L}(\mathrm{a}_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) := a_0 + a_1 + a_2 + a_3 + 2a_4$$

One proves that L is K-positive, but β has no representing measure.

In TMP, K-positivity is a necessary (but not sufficient) condition for a K-representing measure μ.

THEOREM (TMP VERSION OF RIESZ-HAVILAND)

(RC-LF, 2007) $\beta \equiv \beta^{(2n)}$ *admits a K-representing measure if and only if L_β admits a K-positive linear extension* $L : \mathcal{P}_{2n+2} \longmapsto \mathbb{R}$.

This Theorem implies the classical Riesz-Haviland, via Stochel's Theorem.

$$\begin{array}{ccc} \mathcal{P} & & \\ \vdots & & \\ \uparrow & & \\ \mathcal{P}_{2n+2} & & \\ & \color{red}{K\text{-pos.}} & \\ \uparrow & \searrow & \\ \mathcal{P}_{2n} & \overset{\color{red}{K\text{-pos.}}}{\longrightarrow} & \mathbb{R} \end{array}$$

In general it is quite difficult to directly verify that an extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ is K-positive. One approach to establishing K-positivity or the existence of representing measures is through extensions of moment matrices.

Positivity of Block Matrices

Theorem

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases}.$$

Moreover, rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ =rank $A \Leftrightarrow C = W^*AW$.

Corollary

Assume $\operatorname{rank}\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A$. *Then*

$$A \geq 0 \Leftrightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

We say that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is a flat extension of A. *Observe that*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix}.$$

Corollary

Assume that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Then

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - W^*AW \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix}^* \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}^* \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}$$

(sum-of-squares representation).

Functional Calculus

For $p \in \mathcal{P}_n$, $p(z,\bar{z}) \equiv \sum_{0 \le i+j \le n} a_{ij} \bar{z}^i z^j$, let $\widehat{p}$ denote the vector of coefficients and define

$$p(Z,\bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j \equiv M(n)\widehat{p}.$$

If there exists a rep. meas. μ, then

$$p(Z,\bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(Recursiveness) If $p, q, pq \in \mathcal{P}_n$, **and** $p(Z,\bar{Z}) = 0$,

then $(pq)(Z,\bar{Z}) = 0$.

Singular TMP; Real Case

- Given a finite family of moments, build moment matrix.
- Identify all column relations $p(Z, \bar{Z}) = 0$, i.e., $M(n)\widehat{p} = 0$.
- Build algebraic variety

$$\mathcal{V} := \bigcap_{p \in \mathcal{P}_n,\ \widehat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p.$$

- Always true: in the presence of a measure,

$$\text{supp } \mu \subseteq \mathcal{V}.$$

Therefore,

$$r := \text{rank}\,\mathcal{M}(n) \leq \text{card supp } \mu \leq v := \text{card}\,\mathcal{V}(\gamma).$$

Thus, if the variety is finite there's a natural candidate for supp μ, i.e., supp $\mu = \mathcal{V}(\gamma)$

(**It is possible for the inclusion** supp $\mu \subseteq \mathcal{V}$ **to be proper**.)

First Existence Criterion

Theorem

(RC-L. Fialkow, 1998) *Let γ be a truncated moment sequence. TFAE:*

(i) γ has a rep. meas.;

(ii) γ has a rep. meas. with moments of all orders;

(iii) γ has a compactly supported rep. meas.;

(iv) γ has a finitely atomic rep. meas. (with at most $(n+2)(2n+3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n+k+1)$ (here $k \leq 2n^2+6n+6$).

Case of Flat Data

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

γ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

Theorem

(RC-L. Fialkow, 1996) If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

Theorem

(RC-L. Fialkow, 1996) The truncated moment sequence γ has a rank $M(n)$*-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

To find μ concretely, let $r :=$rank $M(n)$ and look for the relation

$$Z^r = c_0 1 + c_1 Z + \ldots + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + \ldots + c_{r-1} z^{r-1})$$

and solve the Vandermonde equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

M. Laurent has been able to use techniques from algebraic geometry to obtain an alternative proof of the Flat Extension Theorem.

Once the matrix $M(n)$ has been extended to $M(\infty)$, one observes that $\ker M(\infty)$ is a polynomial ideal.

For an ideal $\mathcal{I}$, let

$$V(\mathcal{I}) := \{z \in \mathbb{C}^n : \ p(z) = 0 \text{ for all } f \in \mathcal{I}\};$$

$V(\mathcal{I})$ is the complex variety associated to $\mathcal{I}$.

Both $V(\mathcal{I})$ and

$$\sqrt{\mathcal{I}} := \{p : p^k \in \mathcal{I} \text{ for some integer } k \geq 1\}$$

are again ideals, and they both contain $\mathcal{I}$.

$\mathcal{I}$ is called radical if $\mathcal{I} = \sqrt{\mathcal{I}}$.

Hilbert Nullstellensatz. $\sqrt{\mathcal{I}} = I(V(\mathcal{I}))$.

Corollary. If $\mathcal{I}$ is radical, then every polynomial that vanishes in $V(\mathcal{I})$ belongs to $\mathcal{I}$.

(Laurent, 2005) $\ker M(\infty)$ is a radical ideal.

Corollary. If $M(\infty) \geq 0$ and $\operatorname{rank} M(\infty) < \infty$, then the cardinality of $V(\ker M(\infty))$ is $\operatorname{rank} M(\infty)$.

The Quartic Moment Problem

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A\,1$ (Dirac measure)
- $\bar{Z} = A\,1 + B\,Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A\,1 + B\,Z + C\,\bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$\begin{aligned} D &= 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \text{ and } C = \bar{B} \\ &\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2 \\ &\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}. \end{aligned}$$

With $x := \mathrm{Re}[z]$ and $y := \mathrm{Im}[z]$, and using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

(a)	$\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$	parabola; $y = x^2$
(b)	$\bar{W}^2 = -4i1 + W^2$	hyperbola; $yx = 1$
(c)	$\bar{W}^2 = W^2$	pair of intersect. lines; $yx = 0$
(d)	$\bar{W}W = 1$	unit circle; $x^2 + y^2 = 1$.

Theorem QUARTIC

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ *be given, and assume* $M(2) \geq 0$ *and* $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ *is a basis for* $\mathcal{C}_{M(2)}$. *Then* $\gamma^{(4)}$ *admits a rep. meas.* μ. *Moreover, it is possible to find* μ *with* card supp $\mu =$ rank $M(2)$, *except in some cases when* $\mathcal{V}(\gamma^{(4)})$ *is a pair of intersecting lines, in which cases there exist* μ *with* card supp $\mu \leq 6$.

Corollary

Assume that $M(2) \geq 0$, $M(2)$ *singular, and that* rank $M(2) \leq$ card $\mathcal{V}(\gamma^{(4)})$. *Then* $M(2)$ *admits a representing measure.*

The Case of Invertible $M(2)$

(L. Fialkow and J. Nie, 2010) Consider a quartic moment problem with invertible $M(2)$. Then there exists a representing measure.

The proof is abstract, using convex analysis.

(RC and S. Yoo, 2013) Concrete construction of a representing measure, when $M(2)$ is invertible. Moreover, there exists a 6-atomic representing measure, that is, $M(2)$ admits a flat extension $M(3)$.

An Application to Optimization

Consider the problem

$$p^* := inf\ p(x) \text{ subject to } h_1 \geq 0, \cdots, h_m \geq 0;$$

that is, we try to minimize the values of the polynomial p over the semialgebraic set F determined by the polynomials $h_1, \cdots, h_m$.
Let $d_0 := [(degp)/2]$ and $d_i := [(degh_i)/2]$. For
$t \geq max\{d_0, d_1, \cdots, d_m\}$, consider the associated optimization problem

An Application to Optimization, cont.

$$p_t^* := inf\ p^T \beta$$

subject to

$$\beta_0 = 1,\ M(t)[\beta] \geq 0 \text{ and } M_{h_j}(t - d_j)[\beta] \geq 0\ (j = 1, \cdots, m).$$

This is a semidefinite program. One proves that

$$p_t^* \leq p_{t+1}^* \leq p^*.$$

That is, the sequence $(p_t^*)_t$ approximates the absolute minimum p^* from below.

An Application to Optimization, cont.

J. Lasserre was able to use the Flat Extension Theorem to prove that the sequence converges to p^* when the semialgebraic set F is compact. Hence, the above mentioned semidefinite program can be used to approximate the minimum value of p over F.

Moreover, in the $0/1$ and grid cases, Lasserre was able to prove finite convergence. The significant outcome of this is that for certain optimization problems, the Flat Extension theorem allows one to establish finite stopping times for suitable algorithms.

Extremal Real MP; $r = v$

The *algebraic variety* of β is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \widehat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$. If β admits a representing measure μ, then

$$p \in \mathcal{P}_n \text{ satisfies } \widehat{p} \in \ker \mathcal{M}(n) \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus supp $\mu \subseteq \mathcal{V}$, so $r :=$ rank $\mathcal{M}(n)$ and $v :=$ card $\mathcal{V}$ satisfy

$$r \leq \text{card supp } \mu \leq v.$$

Basic necessary conditions for the existence of a representing measure

$$\text{(Positivity) } \mathcal{M}(n) \geq 0 \tag{9.1}$$

$$\text{(Consistency) } p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0 \tag{9.2}$$

$$\text{(Variety Condition) } r \leq v, \text{ i.e., } \operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \mathcal{V}. \tag{9.3}$$

Consistency implies

$$\text{(Recursiveness) } p, q, pq \in \mathcal{P}_n,\ \widehat{p} \in \ker \mathcal{M}(n) \Longrightarrow \widehat{pq} \in \ker \mathcal{M}(n). \tag{9.4}$$

(ideal-like property)

Theorem EXT

(RC-LF and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ **extremal**, *i.e.,* $r = v$, *the following are equivalent:*

(i) β *has a representing measure;*

(ii) β *has a unique representing measure, which is* rank $\mathcal{M}(n)$*-atomic (minimal);*

(iii) $\mathcal{M}(n) \geq 0$ *and* β *is consistent.*

Other Results

RC-Fialkow have used Truncated Moment Theory to estimate the number and location of the zeros of a prescribed polynomial; for example, to show that the polynomial

$$p(z) \equiv z^{2n} + az^{2n-1} - az - 1 \quad (0 < a < 1)$$

has $2n$ distinct zeros, all in the unit circle.

Several authors have used techniques from real algebra to develop structure theorems for positive polynomials on certain noncompact sets $K_{\mathcal{Q}}$: Kuhlmann-Marshall, Powers-Reznick, Powers-Scheiderer, Prestel, Putinar, Scheiderer, Schmüdgen.

They then derived moment theorems for measures supported on $K_{\mathcal{Q}}$.

Pausing to Regroup

- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
- Strategy: Build positive extension, repeat, and eventually extremal

  $\text{rank } M(n) \leq \text{rank } M(n+1) \leq \text{card } \mathcal{V}(M(n+1)) \leq \text{card } \mathcal{V}(M(n))$

- General case.

Cubic Column Relations

Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$.

Recall

Theorem A

(RC-L. Fialkow) If $M(n)$ admits a column relation of the form $Z^k = p_{k-1}(Z, \bar{Z})$ $(1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and $\deg\ p_{k-1} \leq k-1)$, then $M(n)$ admits a flat extension $M(n+1)$, and therefore a representing measure.

Now, if $k = 3$, Theorem A can be used only if $n \geq 4$. Thus, one strategy is to somehow extend $M(3)$ to $M(4)$ and preserve the column relation $Z^3 = p_2(Z, \bar{Z})$. This requires checking that the C block in the extension satisfies the Toeplitz condition, something highly nontrivial.

Here's a different approach:

We'd like to study the case of harmonic poly's: $q(z,\bar{z}) := f(z) - \overline{g(z)}$, with deg $q = 3$.

Recall that rank $M(n) \leq$ card $\mathcal{Z}(q)$

so of special interest is the case when card $\mathcal{Z}(q) \geq 7$, since otherwise the TMP admits a flat extension, or has no representing measure. In the case when $g(z) \equiv z$, we have

Lemma

(Wilmshurst '98, Sarason-Crofoot, '99, Khavinson-Swiatek, '03)

$$\text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7.$$

Bézout's Theorem predicts card $\mathcal{Z}(f(z) - \bar{z}) \leq 9$

- To get 7 points is not easy, as most complex cubic harmonic poly's tend to have 5 or fewer zeros. One way to maximize the number of zeros is to impose symmetry conditions on the zero set K. Also, the substitution $w = z + b/3$ (which produces an equivalent TMP) transforms a cubic $z^3 + bz^2 + cz + d$ into $w^3 + \tilde{c}w + \tilde{d}$; WLOG, we always assume that there's no quadratic term in the analytic piece.

- Now, for a poly of the form $z^3 + \alpha z + \beta \bar{z}$, it is clear that $0 \in K$ and that $z \in K \Rightarrow -z \in K$. Another natural condition is to require that K be symmetric with respect to the line $y = x$, which in complex notation is $z = i\bar{z}$. When this is required, we obtain $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$. Thus, the column relation becomes $Z^3 = itZ + u\bar{Z}$, with $t, u \in \mathbb{R}$.

Under these conditions, one needs to find only two points, one on the line $y = x$, the other outside that line.

We thus consider the harmonic polynomial $q_7(z,\bar{z}) := z^3 - itz - u\bar{z}$.

Proposition

(RC-S. Yoo, '09) card $\mathcal{Z}(q_7) = 7$. *In fact, for* $0 < u < |t| < 2u$,

$$\mathcal{Z}(q_7) = \{0, p+iq, q+ip, -p-iq, -q-ip, r+ir, -r-ir\},$$

where $p, q, r > 0$, $p^2 + q^2 = u$ *and* $r^2 = \frac{t-u}{2}$.

To prove this result, we first identify the two real poly's

Re $q_7 = x^3 - 3xy^2 + ty - ux$ and Im $q_7 = -y^3 + 3x^2y - tx + uy$ and calculate $Resultant(Req_7, Imq_7, y)$, which is the determinant of the Sylvester matrix, i.e.,

$$\det \begin{pmatrix} -3x & t & x^3 - ux & 0 & 0 \\ 0 & -3x & t & x^3 - ux & 0 \\ 0 & 0 & -3x & t & x^3 - ux \\ -1 & 0 & 3x^2 + u & -tx & 0 \\ 0 & -1 & 0 & 3x^2 + u & -tx \end{pmatrix}$$

$$= \quad x\left(u - t + 2x^2\right)\left(u + t + 2x^2\right)\left(16x^4 - 16x^2u + t^2\right).$$

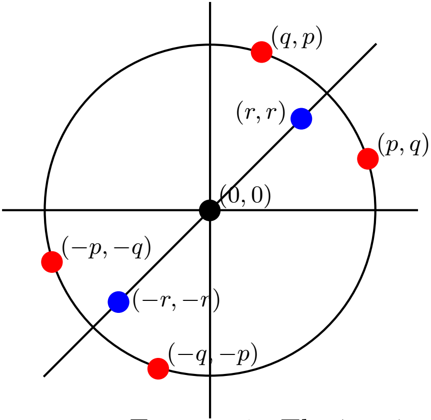

FIGURE 1. The 7-point set $\mathcal{Z}(q_7)$, where $r = \sqrt{\frac{t-u}{2}}$, $p = \frac{1}{2}(2u + \sqrt{4u^2 - t^2})$ and $p^2 + q^2 = u$

The fact that q_7 has the maximum number of zeros predicted by the Lemma is significant to us, in that each sextic TMP with invertible $M(2)$ and a column relation of the form $q_7(Z, \bar{Z}) = 0$ either does not admit a representing measure or is necessarily extremal.

As a consequence, the existence of a representing measure will be established once we prove that such a TMP is consistent. This means that for each poly p of degree at most 6 that vanishes on $\mathcal{Z}(q_7)$ we must verify that $\Lambda(p) = 0$.

Since rank $M(3) = 7$, there must be another column relation besides $q_7(Z, \bar{Z}) = 0$. Clearly the columns

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \bar{Z}Z^2$$

must be linearly independent (otherwise $M(3)$ would be a flat extension of $M(2)$), so the new column relation must involve $\bar{Z}Z^2$ and $\bar{Z}^2Z$. An analysis using the properties of the functional calculus shows that, **in the presence of a representing measure**, the new column relation must be

$$\bar{Z}^2Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0.$$

Notation

In what follows, $\mathbb{C}_6[z,\bar{z}]$ will denote the space of complex polynomials in z and $\bar{z}$ of degree at most 6, and let

$$\begin{aligned} q_{LC}(z,\bar{z}) &:= \bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z} \\ &= i(z - i\bar{z})(\bar{z}z - u). \end{aligned}$$

Observe that the zero set of q_{LC} is the union of a line and a circle, and that $\mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC})$.

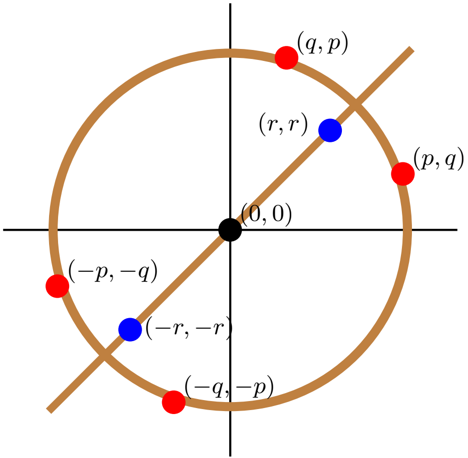

FIGURE 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$

Main Theorem (RC & S. Yoo, J. Funct. Anal., 2014)

Let $M(3) \geq 0$, with $M(2) > 0$ and $q_7(Z, \bar{Z}) = 0$. There exists a representing measure for $M(3)$ if and only if

$$\begin{cases} \Lambda(q_{LC}) & = & 0 \\ \Lambda(zq_{LC}) & = & 0. \end{cases} \tag{10.1}$$

Equivalently,

$$\begin{cases} Re\ \gamma_{12} - Im\ \gamma_{12} = u(Re\ \gamma_{01} - Im\ \gamma_{01}) & = & 0 \\ \gamma_{22} = (t+u)\gamma_{11} - 2u\ Im\ \gamma_{02} & = & 0. \end{cases}$$

Equivalently,

$$q_{LC}(Z, \bar{Z}) = 0 \tag{10.2}$$

($\Longleftarrow$) On $\mathcal{Z}(q_7)$ we have $z^3 = itz + u\bar{z}$. Using this relation and (10.1), we can prove that $\Lambda(\bar{z}^i z^j q_{LC}) = 0$ for all $0 \leq i + j \leq 3$. For example,

$$\begin{aligned} \bar{z}q_{LC} - izq_{LC} &= (\bar{z} - iz)(\bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z}) \\ &= -uz^2 + \bar{z}z^3 - u\bar{z}^2 + \bar{z}^3 z \\ &= -uz^2 + \bar{z}(itz + u\bar{z}) - u\bar{z}^2 + (-it\bar{z} + uz)z \\ &= 0, \end{aligned}$$

and therefore $\Lambda(\bar{z}q_{LC}) = i\Lambda(zq_{LC}) = 0$. It follows that for $f, g, h \in \mathbb{C}_3[z, \bar{z}]$ we have $\Lambda(fq_7 + g\bar{q}_7 + hq_{LC}) = 0$. Consistency will be established once we show that all degree-six polynomials vanishing in $\mathcal{Z}(q_7)$ are of the form $fq_7 + g\bar{q}_7 + hq_{LC}$.

Fundamental Theorem of Linear Algebra

Let T be a linear transformation from $\mathcal{X}$ to $\mathcal{Y}$, and consider the exact sequence

$$0 \to \ker\ T \hookrightarrow \mathcal{X} \to \text{Ran}\ T \to 0.$$

Then

$$\dim\ \ker\ T - \dim\ \mathcal{X} + \dim\ RanT = 0.$$

Equivalently,

$$\dim\ \mathcal{X} = \dim\ \ker\ T + \dim\ \text{Ran}\ T.$$

Proposition (Representation of Polynomials)

Let $\mathcal{P}_6 := \{p \in \mathbb{C}_6[z,\bar{z}] : p|_{\mathcal{Z}(q_7)} \equiv 0\}$ *and let*
$\mathcal{I} := \{p \in \mathbb{C}_6[z,\bar{z}] : p = fq_7 + g\bar{q}_7 + hq_{LC}$ *for some* $f, g, h \in \mathbb{C}_3[z,\bar{z}]\}$*.*
Then $\mathcal{P}_6 = \mathcal{I}$*.*

Proof. Clearly, $\mathcal{I} \subseteq \mathcal{P}_6$. We shall show that $\dim \mathcal{I} = \dim \mathcal{P}_6$. Let $T : \mathbb{C}^{30} \longrightarrow \mathbb{C}_6[z,\bar{z}]$ be given by

$$(a_{00}, \cdots, a_{30}, b_{00}, \cdots, b_{30}, c_{00}, \cdots, c_{30}) \longmapsto$$

$$\begin{aligned} &(a_{00} + a_{01}z + a_{10}\bar{z} + \cdots + a_{30}\bar{z}^3)q_7 \\ &+(b_{00} + b_{01}z + b_{10}\bar{z} + \cdots + b_{30}\bar{z}^3)\bar{q}_7 \\ &+(c_{00} + c_{01}z + c_{10}\bar{z} + \cdots + c_{30}\bar{z}^3)q_{LC}. \end{aligned}$$

Recall that $30 = \dim \ \mathbb{C}^{30} = \dim \ \ker \ T + \dim \operatorname{Ran} \ T$, and observe that $\mathcal{I} = \operatorname{Ran} \ T$, so that $\dim \ \mathcal{I} = \operatorname{rank} \ T$.

To determine $\operatorname{rank} \ T$, we first determine $\dim \ \ker \ T$. Using Gaussian elimination, we prove that $\dim \ \ker \ T = 9$ whenever $ut \neq 0$. It follows that $\operatorname{rank} \ T = 30 - 9 = 21$, that is, $\dim \ \mathcal{I} = 21$.

Now consider the evaluation map $S : \mathbb{C}_6[z, \bar{z}] \longrightarrow \mathbb{C}^7$ given by

$$\begin{aligned} S(p(z, \bar{z})) \quad := \quad & (p(w_0, \bar{w}_0), p(w_1, \bar{w}_1), p(w_2, \bar{w}_2), \\ & p(w_3, \bar{w}_3), p(w_4, \bar{w}_4), p(w_5, \bar{w}_5), p(w_6, \bar{w}_6)). \end{aligned}$$

Again, $\dim \ \ker \ S + \dim \text{Ran} \ \ S = \dim \ \mathbb{C}_6[z, \bar{z}] = 28$. Using Lagrange Interpolation, it is easy to verify that S is onto, i.e., $\text{rank} \ \ S = 7$. Moreover, $\ker \ \ S = \mathcal{P}_6$. Since $\dim \ \mathbb{C}_6[z, \bar{z}] = 28$, it follows that $\dim \ \ker \ \ S = 21$, and a fortiori that $\dim \ \ \mathcal{P}_6 = 21$.

Therefore, $\dim \ \mathcal{I} = 21 = \dim \ \ \mathcal{P}_6$, and since $\mathcal{I} \subseteq \mathcal{P}_6$, we have established that $\mathcal{I} = \mathcal{P}_6$, as desired.

Yet Another Approach to TMP: The Division Algorithm

Division Algorithm in $\mathbb{R}[x_1, \cdots, x_n]$

Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$ and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $f_1, \cdots, f_s$.

Furthermore, if $a_i f_i \neq 0$, then we have

$$multideg(f) \geq multideg(a_i f_i).$$

- The Division Algorithm works as follows: we identify sufficiently many polynomials $f_1, \cdots, f_s$ vanishing on $\mathcal{V}(\beta)$, and simultaneously in the kernel of the Riesz functional L_β. By the Division Algorithm, any polynomial f vanishing on $\mathcal{V}(\beta)$ can be written as $f = a_1 f_1 + \cdots + a_s f_s + r$, which readily implies that r must also vanish on $\mathcal{V}(\beta)$. Due to the divisibility condition on the monomials of r, and the characteristics of $\mathcal{V}(\beta)$, which generate an invertible Vandermonde matrix, we then prove that $r \equiv 0$.
- With some additional work, it is then possible to prove that $f \in \ker\ L_\beta$, which establishes the Consistency of β.

A New Tool: Rank Reduction

Given a point $(a, b) \in \mathbb{R}^2$ we let $\mathbf{v} \equiv \mathbf{v}_{(a,b)}$ denote the row vector

$$(1, a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3)$$

We also let $\delta_{(a,b)}$ denote the point mass at (a, b). It is easy to see that the moment matrix associated with $\delta_{(a,b)}$ is $\mathbf{vv}^T$, that is, the matrix whose entries are $M(3)_{ij} = a^i b^j$. For this moment matrix, $r = 1$ and $\mathcal{V} = \{(a, b)\}$.

Theorem

Assume $M(3) \geq 0$, $M(2) > 0$, rank $M(3) = 7$ and card $\mathcal{V} \geq 8$. Assume also that $M(3)$ satisfies the Consistency Property. Then $M(3)$ admits a flat extension $M(4)$; that is, there exists a representing measure μ with card supp $\mu = 7$.

Sketch of Proof. WLOG, assume

$$\mathcal{V} = \{(x_1, y_1), \ldots, (x_8, y_8)\}.$$

Also assume that in $M(3)$ the first seven columns are linearly independent. Now form the Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & x_1^3 & x_1^2 y_1 & x_1 y_1^2 & y_1^3 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & x_2^3 & x_2^2 y_2 & x_2 y_2^2 & y_2^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_8 & y_8 & x_8^2 & x_8 y_8 & y_8^2 & x_8^3 & x_8^2 y_8 & x_8 y_8^2 & y_8^3 \end{pmatrix}.$$

This is an 8×10 matrix, with rank 7. It follows that exactly seven rows are linearly independent, so one of them must be a linear combination of the other seven, say

$$R_j = \sum_{i \neq j} \lambda_i R_i.$$

The row R_j must be associated with a point $(x_j, y_j) \in \mathcal{V}$. To single out this point, we will denote it by (a, b). Now let

$$\mathcal{V}' := \mathcal{V} \setminus \{(a, b)\}.$$

Claim. No conic goes through $\mathcal{V}'$. For, let

$$C(x, y) \equiv C_{00} + C_{10}x + C_{01}y + C_{20}x^2 + C_{11}xy + C_{02}y^2$$

be such a conic, that is, if $\widehat{C}$ denotes the vector of coefficients of C regarded as a 10-vector, then

$$R_i \widehat{C} = C(x_i, y_i) = 0,$$

and therefore

$$\sum_{i \neq j} \lambda_i R_i \widehat{C} = 0;$$

that is, $C(a,b) \equiv R_j \widehat{C} = 0$, which implies that C also vanishes on (a,b), and a fortiori C vanishes on the entire algebraic variety $\mathcal{V}$. Then, by the Consistency Property, $C(X,Y) = 0$, that is, the moment matrix $M(3)$ admits a quadratic column relation, a contradiction to the fact that $M(2) > 0$.

We now define

$$\widetilde{M(3)} := M(3) - \rho \mathbf{v}\mathbf{v}^T,$$

where $\mathbf{v}$ is the row vector associated with the point (a,b). By the above argument, $\widetilde{M(2)}$ has rank 6.

We wish to prove that rank $\widetilde{M(3)} = 6$ for some positive value of ρ. If we do this, then $\widetilde{M(3)}$ will be a flat extension of $\widetilde{M(2)}$, and we will have a 6-atomic measure for $\widetilde{M(3)}$, and therefore a 7-atomic measure for $M(3)$, since $M(3) = \widetilde{M(3)} + \rho \mathbf{v}\mathbf{v}^T$.

Let λ denote the nonzero eigenvalue of $\mathbf{v}\mathbf{v}^T$, and let $\mathcal{B}$ be the basis of the column space of $M(3)$. Then

$$\det \widetilde{M(3)}_{\mathcal{B}} = \det M(3)_{\mathcal{B}} - \rho\lambda \det(M(3)_{\mathcal{B}}|_{\{2,3,4,5,6,7\}}).$$

, where $\lambda := 1 + a^2 + b^2 + \cdots + a^6 + a^4 b^2 + a^2 b^4 + b^6$. Thus, with

$$\rho := \frac{\det M(3)_{\mathcal{B}}}{\lambda \det(M(3)_{\mathcal{B}}|_{\{2,3,4,5,6,7\}})},$$

we successfully reduce the rank.

Problem

Assume $M(3) \geq 0$*,* $M(2) > 0$*, rank* $M(3) = 8$*, card* $\mathcal{V} = 9$*. Under what conditions does the moment sequence admit a representing measure?*

Using the above mentioned new tools, we have been able to build an explicit algorithm that provides a solution to this Problem.

Our techniques also allow us to deal with

Problem

Assume $M(3) \geq 0$*,* $M(2) > 0$*, rank* $M(3) = 8$*, card* $\mathcal{V} = \infty$*. Under what conditions does the moment sequence admit a representing measure?*

Table: Sextic TMP in terms of r and v

r_3	v_3	$v_3 - r_3$	Max Ext		Solution Presented in
7	7	0	$\mathcal{M}(4)$	extremal	RC-Yoo Div Alg paper
7	8	1	$\mathcal{M}(5)$		Rank Red Theorem
7	9	2	$\mathcal{M}(6)$		Rank Red Theorem
7	∞	N/A	N/A		Rank Red Theorem
8	8	0	$\mathcal{M}(4)$	extremal	RC-Yoo Div Alg paper
8	9	1	$\mathcal{M}(5)$		New algorithm
8	∞	N/A	N/A		New algorithm
9	∞	N/A	N/A		RC-Yoo JFA paper; L. Fialkow
10	∞	N/A	N/A		unknown

In the specific case of $r = 8$ and $v = 9$, one must have the algebraic variety $\mathcal{V}$ of $\mathcal{M}(3)$ as the intersection of two cubics C_1 and C_2 in general position. We then use the Cayley-Bacharach Theorem:

Assume that two cubics C_1 and C_2 in the projective plane meet in nine (different) points (that is $C_1 \bigcap C_2 = \mathcal{V}$). Then every cubic C that passes through any eight of the points in $\mathcal{V}$ also passes through the ninth point.

Summary

- Given a finite family of moments, build moment matrix
- Identify all column relations, and build algebraic variety $\mathcal{V}$
- Always true: $r \leq$ card supp $\mu \leq v$
- Finite rank case; flat case
- Quartic Case
- Extremal case (must check Consistency)
- Harmonic cubic poly's in Sextic Case
- Some instances of the general singular case
- But the invertible case is still a big mystery...

Summary

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