Abrahamse's Theorem for Matrix-valued Symbols and Subnormal Toeplitz Completions

Raúl E. Curto

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(with In Sung Hwang and Woo Young Lee; two related papers have appeared in Adv. Math., J. Funct. Anal.)

raul-curto@uiowa.edu http://www.math.uiowa.edu/ ~rcurto

Abstract

Reformulation of Halmos's Problem 5: Which subnormal Toeplitz operators with matrix-valued symbols are either normal or analytic ?

M. Abrahamse (1976): Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If T_φ is subnormal, then T_φ is either normal or analytic.

In this talk we will discuss a matrix-valued version of Abrahamse's Theorem and then apply this result to solve the following subnormal Toeplitz completion problem:

Find the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{bmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{bmatrix} \quad (\alpha, \beta \in \mathbb{D})$$

so that A becomes subnormal, where b_λ is a Blaschke factor of the form

$$b_\lambda(z) := \frac{z - \lambda}{1 - \overline{\lambda} z} \ \ (\lambda \in \mathbb{D}).$$

A Subnormal Toeplitz Completion Problem

Problem. Given two Blaschke products b_α and b_β $(\alpha, \beta \in \mathbb{D})$, find necessary and sufficient conditions on φ, ψ rational to make

$$G := \begin{bmatrix} T_{\overline{b}_\alpha} & T_\varphi \\ T_\psi & T_{\overline{b}_\beta} \end{bmatrix} \tag{1}$$

subnormal.

Main Idea: Think of G as a block Toeplitz operator

Motivation: A Simple Case

Let U_+ be the unilateral shift on H^2. Find the unspecified *Toeplitz* entries ? of the partial block Toeplitz matrix

$$A := \begin{bmatrix} U_+^* & ? \\ ? & U_+^* \end{bmatrix}$$

so that A becomes subnormal.

A Related Completion Problem

Proposition

$\begin{bmatrix} T_z & ? \\ ? & T_{\overline{z}} \end{bmatrix}$ *is never hyponormal, if ? is Toeplitz.*

Recall that the related dilation problem

$$A := \begin{bmatrix} U_+^* & ? \\ ? & ? \end{bmatrix}$$

admits the canonical solution

$$A := \begin{bmatrix} U_+^* & 0 \\ I - U_+ U_+^* & U_+ \end{bmatrix}.$$

(But of course the $(1,2)$-entry is not Toeplitz.)

One can write down a couple of nontrivial subnormal Toeplitz completions, as follows:

$$A := \begin{bmatrix} U_+^* & U_+ \\ U_+ & U_+^* \end{bmatrix}.$$

and

$$A := \begin{bmatrix} U_+^* & \alpha U_+^* + \sqrt{1+|\alpha|^2} U_+ \\ \alpha U_+^* + \sqrt{1+|\alpha|^2} U_+ & U_+^* \end{bmatrix}.$$

How general are these solutions?

Notation and Preliminaries

$L^\infty \equiv L^\infty(\mathbb{T})$; $H^\infty \equiv H^\infty(\mathbb{T})$; $L^2 \equiv L^2(\mathbb{T})$; $H^2 \equiv H^2(\mathbb{T})$,
$P : L^2 \to H^2$ orthogonal projection
$T \in \mathcal{L}(\mathcal{H})$: algebra of bounded operators on a Hilbert space $\mathcal{H}$

- normal if $T^*T = TT^*$
- quasinormal if T commutes with T^*T
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$
- 2-hyponormal if (T, T^2) is (jointly) hyponormal ($k \geq 1$)

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] \\ [T^*, T^2] & [T^{*2}, T^2] \end{pmatrix} \geq 0$$

quasinormal $\Rightarrow$ subnormal $\Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal

For $\varphi \in L^\infty$, the Toeplitz operator with symbol φ is $T_\varphi : H^2 \to H^2$, given by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2)$$

T_φ is said to be *analytic* if $\varphi \in H^\infty$

Halmos's Problem 5 (1970):

Is every subnormal Toeplitz operator either normal or analytic?

C. Cowen and J. Long (1984): No

- C. Cowen (1988)

  $\varphi \in L^\infty$, $\varphi = \bar{f} + g \ \ (f, g \in H^2)$

$$T_\varphi \textit{ is hyponormal } \Leftrightarrow \ f = c + T_{\bar{h}} g,$$

  for some $c \in \mathbb{C}$, $h \in H^\infty$, $\|h\|_\infty \leq 1$.

- Nakazi-Takahashi (1993)

  For $\varphi \in L^\infty$, let

$$\mathcal{E}(\varphi) := \{k \in H^\infty : \|k\|_\infty \leq 1 \textit{ and } \varphi - k\bar{\varphi} \in H^\infty\}.$$

  Then

$$T_\varphi \textit{ is hyponormal } \Leftrightarrow \ \mathcal{E}(\varphi) \neq \emptyset.$$

Natural Questions:

1) When is T_φ subnormal?

At present, there's no known characterization of subnormality in terms of the symbol φ.

2) Characterize 2-hyponormality for Toeplitz operators

Sample Result:

Theorem

(RC and WY Lee, 2001) Every 2-hyponormal **trigonometric** *Toeplitz operator is subnormal.*

- Cowen-Long (1984)
  If $T_\varphi \cong W_\alpha$ is hyponormal, with α strictly increasing, then there exists $\beta \in (0,1)$ such that
  $$\alpha_k = \sqrt{1-\beta^{2k+2}}\,\|T_\varphi\| \quad (\textit{all } k).$$
- Cowen-Long (1984)
  Let $0 < \alpha < 1$, let $\psi : \mathbb{D} \to E$ be conformal, where E is the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$, and let
  $$\varphi := \frac{\psi + \alpha\bar{\psi}}{1-\alpha^2}.$$

Question

Let φ be the Cowen & Long symbol. Does it follow that $T_\varphi \cong T_\eta$ for some $\eta \in H^\infty$?

Question

A Reformulation of Halmos's Problem 5

Let T_φ be a non-normal subnormal Toeplitz operator. Does it follow that $T_\varphi \cong T_\eta$ for some $\psi \in H^\infty$?

These two questions remain open.

Functions of bounded type and Abrahamse's Theorem

$\varphi \in H^\infty$ is of *bounded type* (or in the Nevanlinna class) if

$$\varphi := \frac{\psi_1}{\psi_2}.$$

(Abrahamse, 1976) Assume φ or $\bar{\varphi}$ is of bounded type. If T_φ is hyponormal and $\ker[T_\varphi^*, T_\varphi]$ is invariant for T_φ, then T_φ is normal or analytic.

Thus, the answer to Halmos's Problem 5 is affirmative if φ is of bounded type.

$\varphi \in L^\infty$ is of *bounded type* (or in the Nevanlinna class) if

$$\varphi := \frac{\psi_1}{\psi_2},$$

with $\psi_1, \psi_2 \in H^\infty$.

Theorem

(RC & WY Lee, 2001) If $T_\varphi \cong W_\alpha$ and T_φ is 2-hyponormal, then T_φ is subnormal.

Theorem

(RC & WY Lee, 2001) If T_φ is 2-hyponormal and if φ or $\bar{\varphi}$ is of bounded type, then T_φ is normal or analytic, so in particular T_φ is subnormal.

(this generalizes Abrahamse's Theorem)

Block Toeplitz Operators

$M_n := M_{n\times n}$ $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$ $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ For $\Phi \in L^\infty_{M_n}$, $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes the *block* Toeplitz operator with symbol Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$.

A *block Hankel* operator with symbol $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where $J_n(f)(z) := \overline{z} I_n f(\overline{z})$ for $f \in L^2_{\mathbb{C}^n}$.

We easily see that

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \dots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \dots & T_{\varphi_{nn}} \end{bmatrix} \quad \text{and} \quad H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \dots & H_{\varphi_{1n}} \\ & \vdots & \\ H_{\varphi_{n1}} & \dots & H_{\varphi_{nn}} \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \dots & \varphi_{nn} \end{bmatrix} \in L^\infty_{M_n}.$$

For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if $\Theta^*\Theta = I_m$ almost everywhere on $\mathbb{T}$. Given $\Phi, \Psi \in L^\infty_{M_n}$,

$$T^*_\Phi = T_{\Phi^*}$$

$$H^*_\Phi = H_{\widetilde{\Phi}} \quad \text{(recall that } \widetilde{\Phi}(z) := \Phi^*(\overline{z})\text{)}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}$$

Block Toeplitz operators have been studied by D.Z. Arov, E. Basor, A. Böttcher, R.G. Douglas, H. Dym, I. Feldman, I. Gohberg, S. Grudsky, C. Gu, W. Helton, J. Hendricks, I.S. Hwang, D.-O. Kang, M.A. Kaashoek, I. Koltracht, W.Y. Lee, A. Rogozhin, D. Rutherford, I. Spitkovsky, H. Woerdeman, D. Zheng, Y. Zucker, and others.

R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, Amer. Math. Soc., 1980.

$\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$ is of *bounded type* if each entry φ_{ij} is of b.t.
Φ is *rational* if each entry φ_{ij} is a rational function.

The *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by

$$S := T_{zI_n}.$$

The Beurling-Lax-Halmos Theorem. *A nonzero subspace $\mathcal{M}$ of $H^2_{\mathbb{C}^n}$ is invariant for S if and only if $\mathcal{M} = \Theta H^2_{\mathbb{C}^m}$, where Θ is an inner matrix function. Furthermore, Θ is unique up to a unitary constant right factor.*

As a consequence, if $\ker H_\Phi \neq \{0\}$, then

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ.

Normality of Block Toeplitz Operators

(C. Gu, J. Hendricks and D. Rutherford, 2006) Let $\Phi \equiv \Phi_+ + \Phi_-^*$ be normal. If $\det \Phi_+$ is not identically zero then

$$T_\Phi \text{ is normal} \iff \Phi_+ - \Phi_+(0) = \big(\Phi_- - \Phi_-(0)\big)\, U \qquad (2)$$

for some constant unitary matrix U.

Hyponormality of Block Toeplitz Operators

(C. Gu, J. Hendricks and D. Rutherford, 2006) For $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \left\{ K \in H^\infty_{M_n} : \ ||K||_\infty \leq 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n} \right\}.$$

Then T_Φ is hyponormal if and only if Φ is normal (i.e. $\Phi^*\Phi = \Phi\Phi^*$) and $\mathcal{E}(\Phi)$ is nonempty.

Theorem

(Gu, Hendricks and Rutherford, 2006) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

1. *Φ is of bounded type;*
2. *$\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;*
3. *$\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.*

Definition: Θ and A are right coprime if they do not have a common nontrivial right factor.

Example

Let $\Phi := \left(\begin{smallmatrix} z & z \\ z & z \end{smallmatrix}\right)$ then we can write

$$\Phi = \Theta A^* = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

but $\Theta := \left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$ and $A := \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ are not right coprime because $\frac{1}{\sqrt{2}} \left[\begin{smallmatrix} z & -z \\ 1 & 1 \end{smallmatrix}\right]$ is a common right inner divisor, i.e.,

$$\Theta = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix}$$

$$A = \sqrt{2} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix}.$$

Given a matrix-valued symbol Φ, write

$$\Phi \equiv \Phi_-^* + \Phi_+ = \Theta A^*.$$

If Φ and Φ^* are of bounded type, then

$$\Phi_+ = \Theta_1 A^*$$

and

$$\Phi_- = \Theta_2 B^*,$$

where Θ_1 and A are right coprime, and Θ_2 and B are also right coprime.

Necessary Condition for Hyponormality:

Theorem

$\Theta_2 | \Theta_1$.

Thus, WLOG, we can always assume:

$\Phi_+ = \Theta_2 \Theta_0 A^*$

$\Phi_- = \Theta_2 B^*$

If Φ is rational, then each Θ_i is a finite Blaschke product. In general, Θ and A need not be (right) coprime! (Recall that $\Phi = \Theta A^*$.)

Recall: Θ and A are right coprime if they do not have a common nontrivial right factor.

Abrahamse's Theorem for Block Toeplitz Operators

Theorem (CHKL, 2013)

(Abrahamse's Theorem for Matrix-Valued Rational Symbols) Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Thus we may write $\Phi_- = \Theta B^* \quad \text{(right coprime factorization)}$. Assume that Θ has a nonconstant diagonal-constant inner divisor. If

(i) *T_Φ is hyponormal;*

(ii) *$\ker [T_\Phi^*, T_\Phi]$ is invariant for T_Φ,*

then T_Φ is normal. Hence, if T_Φ is subnormal then T_Φ is normal.

Corollary (CHKL, 2013)

Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. We may write

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization).}$$

Assume that Θ has a nonconstant diagonal-constant inner divisor. Then the following are equivalent:

1. *T_Φ is 2-hyponormal;*
2. *T_Φ is subnormal;*
3. *T_Φ is normal.*

Corollary

Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ *is a matrix-valued rational function. We may write*

$$\Phi_- = B^* \Theta,$$

where $\Theta := \theta I_n$ *with a finite Blaschke product* θ. *Assume that* $B(\alpha)$ *is invertible for each* $\alpha \in \mathcal{Z}(\theta)$. *If* T_Φ *is subnormal then* T_Φ *is normal or analytic.*

Quasinormal Block Toeplitz Operators

Yakubovich's Theorem (2006). If $T \in \mathcal{B}(\mathcal{H})$ is a pure subnormal operator with finite rank self-commutator and without point masses then it is unitarily equivalent to a Toeplitz operator T_Φ with a matrix-valued analytic rational symbol Φ.

On the other hand, Ito and Wong proved in 1972 that every quasinormal Toeplitz operator is either normal or analytic, i.e., the answer to the Halmos's Problem 5 is affirmative for quasinormal Toeplitz operators.

However, this is not true for the cases of matrix-valued symbols: indeed, if

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z} + 2z \\ \overline{z} + 2z & \overline{z} \end{bmatrix}. \tag{3}$$

then T_Φ is quasinormal, but it is neither normal nor analytic. But since

$$T_\Phi = \begin{bmatrix} U_+^* & U_+^* + 2U_+ \\ U_+^* + 2U_+ & U_+^* \end{bmatrix},$$

it follows that if

$$W := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

then W is unitary and

$$W^* T_\Phi W = 2 \begin{bmatrix} U_+^* + U_+ & 0 \\ 0 & -U_+ \end{bmatrix},$$

which says that T_Φ is unitarily equivalent to a direct sum of a normal operator, $2(U_+^* + U_+)$, and an analytic Toeplitz operator, $-2U_+$.

Theorem (CHKL, 2013)

Every pure quasinormal operator with finite rank self-commutator is unitarily equivalent to a Toeplitz operator with a matrix-valued analytic rational symbol.

Corollary (CHKL, 2013)

Every pure quasinormal Toeplitz operator with a matrix-valued rational symbol is unitarily equivalent to an analytic Toeplitz operator.

A Subnormal Toeplitz Completion Problem

Problem. For $\lambda \in \mathbb{D}$, let b_λ be a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$. Complete the unspecified *rational* Toeplitz operators (i.e., the unknown entries are rational Toeplitz operators) of the partial block Toeplitz matrix

$$G := \begin{bmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{bmatrix} \quad (\alpha, \beta \in \mathbb{D}) \tag{4}$$

to make G subnormal.

We begin with:

Lemma

Let

$$\Phi := \begin{bmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{bmatrix} \quad (\varphi, \psi \in L^\infty)$$

be such that T_Φ is hyponormal. Then $\alpha = \beta$.

Proof. If T_Φ is hyponormal then Φ is normal, so that a straightforward calculation gives $|\varphi| = |\psi|$. Also, by the characterization of hyponormality due to Gu-Hendricks-Rutherford, there exists a matrix function $K \equiv \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right] \in \mathcal{E}(\Phi)$, i.e., $||K||_\infty \le 1$ such that $\Phi - K\Phi^* \in H^\infty_{M_2}$, i.e.,

$$\begin{bmatrix} \overline{b}_\alpha & \overline{\varphi_-} \\ \overline{\psi_-} & \overline{b}_\beta \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & \overline{\psi_+} \\ \overline{\varphi_+} & 0 \end{bmatrix} \in H^2_{M_2}, \tag{5}$$

which implies that

$$H_{\overline{b}_\alpha} = H_{k_2\overline{\varphi_+}} = H_{\overline{\varphi_+}} T_{k_2} \quad \text{and} \quad H_{\overline{b}_\beta} = H_{k_3\overline{\psi_+}} = H_{\overline{\psi_+}} T_{k_3}.$$

If $\overline{\varphi_+}$ is not of bounded type then $\ker H_{\overline{\varphi_+}} = \{0\}$, so that $k_2 = 0$, a contradiction; and if $\overline{\psi_+}$ is not of bounded type then $\ker H_{\overline{\psi_+}} = \{0\}$, so that $k_3 = 0$, a contradiction. Thus $\overline{\varphi_+}$ and $\overline{\psi_+}$ are of bounded type, so that Φ^* is of bounded type. Since T_Φ is hyponormal, it follows that Φ is also of bounded type. Thus we can write

$$\varphi_- := \theta_0 \overline{a} \quad \text{and} \quad \psi_- := \theta_1 \overline{b} \quad (a \in \mathcal{H}_{z\theta_0}, b \in \mathcal{H}_{z\theta_1}),$$

where θ_0 and θ_1 are inner, a and θ_0 are coprime and b and θ_1 are coprime.

On the other hand, by (5), we have

$$\begin{cases} \overline{b}_\alpha - k_2\overline{\varphi_+} \in H^2, & \overline{\theta}_1 b - k_4\overline{\varphi_+} \in H^2 \\ \overline{b}_\beta - k_3\overline{\psi_+} \in H^2, & \overline{\theta}_0 a - k_1\overline{\psi_+} \in H^2, \end{cases} \tag{6}$$

which implies that the following Toeplitz operators are all hyponormal (by Cowen's Theorem):

$$T_{\overline{b}_\alpha+\varphi_+}, \quad T_{\overline{\theta}_1 b+\varphi_+}, \quad T_{\overline{b}_\beta+\psi_+}, \quad T_{\overline{\theta}_0 a+\psi_+}. \tag{7}$$

We can then write

$$\varphi_+ = \theta_1\theta_3\overline{d} \quad \text{and} \quad \psi_+ = \theta_0\theta_2\overline{c} \quad (d \in \mathcal{H}_{z\theta_1\theta_3}, c \in \mathcal{H}_{z\theta_0\theta_2}), \tag{8}$$

where θ_2 and θ_3 are inner, d and $\theta_1\theta_3$ are coprime, and c and $\theta_0\theta_2$ are coprime. In particular, $d(\alpha) \neq 0$ and $c(\beta) \neq 0$. We now claim that

$$\alpha = \beta. \tag{9}$$

Assume to the contrary that $\alpha \neq \beta$. Since Φ is normal, i.e., $\Phi\Phi^* = \Phi^*\Phi$, we have

$$\begin{bmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{bmatrix} \begin{bmatrix} b_\alpha & \overline{\psi} \\ \overline{\varphi} & b_\beta \end{bmatrix} = \begin{bmatrix} b_\alpha & \overline{\psi} \\ \overline{\varphi} & b_\beta \end{bmatrix} \begin{bmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{bmatrix},$$

which gives

$$\overline{b}_\alpha \overline{\psi} + \varphi b_\beta = b_\alpha \varphi + \overline{\psi} \overline{b}_\beta, \quad \text{i.e.,} \quad (b_\alpha - b_\beta)(\psi + \overline{b}_\alpha \overline{b}_\beta \overline{\varphi}) = 0\,,$$

which implies that $\psi = -\overline{b}_\alpha \overline{b}_\beta \overline{\varphi}$ since $\alpha \neq \beta$. We put

$$\varphi'_- := P_{\mathcal{H}(b_\alpha b_\beta)}(\varphi_-) \quad \text{and} \quad \varphi''_- := P_{b_\alpha b_\beta H^2}(\varphi_-).$$

We then have

$$\psi_+ = -\overline{b}_\alpha \overline{b}_\beta \varphi''_- \quad \text{and} \quad \psi_- = -b_\alpha b_\beta(\varphi_+ + \overline{\varphi'_-}). \tag{10}$$

It thus follows from (10) that

$$\theta_1 \overline{b} = \psi_- = -b_\alpha b_\beta(\varphi_+ + \overline{\varphi'_-}), \quad \text{so that } \overline{b} = -b_\alpha b_\beta(\theta_3 \overline{d} + \overline{\theta_1 \varphi'_-}) \in \overline{H^2} \tag{11}$$

which gives

$$\theta_3 \overline{d} + \overline{\theta_1 \varphi'_-} \in \overline{H^2}, \quad \text{and hence, } d \in \theta_3 H^2,$$

which implies that θ_3 is a constant because θ_3 and d are coprime. We therefore have $\varphi_+ = \theta_1 \overline{d}$. It thus follows from (7) that

$$\theta_1 = b_\alpha \theta_1' \quad \text{(some inner function } \theta_1'\text{)}.$$

But since by (11),

$$\overline{b} = -b_\alpha b_\beta \overline{(d + \theta_1 \varphi_-')} \in \overline{H^2},$$

so that

$$d + \theta_1 \varphi_-' \in b_\alpha b_\beta H^2,$$

which implies that $d(\alpha) = 0$, a contradiction because θ_1 and d are coprime. This proves that $\alpha = \beta$. $\square$

Theorem (RC, IS Hwang and WY Lee, 2013)

Let $\varphi, \psi \in L^\infty$ be rational and consider

$$G := \begin{bmatrix} T_{\overline{b}_\alpha} & T_\varphi \\ T_\psi & T_{\overline{b}_\beta} \end{bmatrix}.$$

Then the following statements are equivalent.

1. *G is normal.*
2. *G is subnormal.*
3. *G is 2-hyponormal.*
4. *G is hyponormal and ker $[G^*, G]$ is invariant for G.*

Theorem (cont.)

5. $b_\alpha = b_\beta =: \omega$ *and the following condition holds:*

$$\varphi = e^{i\delta_1}\omega + \zeta$$

and

$$\psi = e^{i\delta_2}\varphi$$

with $\zeta \in \mathbb{C}$; $\delta_1, \delta_2 \in [0, 2\pi)$, *except in an exceptional case.*

Corollary

Let

$$A := \begin{bmatrix} U^* & U^* + 2U \\ U^* + 2U & U^* \end{bmatrix},$$

where $U \equiv T_z$ *is the unilateral shift on* H^2*. Then* A *is a quasinormal (therefore subnormal) completion of* $\left[\begin{smallmatrix} U^* & ? \\ ? & U^* \end{smallmatrix}\right]$*, and* A *is not normal.*

Remark

(Example of exceptional case) If

$$\Phi := \begin{bmatrix} \overline{z}^p & \overline{z}^p + 2z^p \\ \overline{z}^p + 2z^p & \overline{z}^p \end{bmatrix} \quad (p = 1, 2, \cdots)$$

then a straightforward calculation shows that T_Φ is quasinormal, but not normal. We note, however, that $T_\Phi \cong N \oplus T_A$, where N is normal and $A \in H^\infty_{M_k}$ ($\cong$ denotes unitary equivalence).

Remark (cont.)

In fact,

$$T_\Phi \cong \begin{bmatrix} T_{\overline{z}^p + z^p} & 0 \\ 0 & -T_{z^p} \end{bmatrix}.$$

From this viewpoint, we might expect that this is not a coincidence.

Thus we propose:

Conjecture

Every subnormal rational Toeplitz operator is unitarily equivalent to a direct sum of a normal operator and an analytic Toeplitz operator.

Concluding Remarks

Problem (Open Problem)

Assume that T_φ is subnormal, with finite rank self-commutator. Does it follow that T_φ is normal or analytic?

Partial Answer:

Theorem

(CHL, 2010): Suppose T_ϕ is a hyponormal Toeplitz operator with finite rank self-commutator. If $\ker\,[T_\phi^*, T_\phi]$ and $b\ker\,[T_\phi^*, T_\phi]$ (for some $b \in \mathcal{E}(\phi)$) are invariant under T_ϕ, then T_ϕ is normal or analytic.

Remark

We have seen that a 2-hyponormal Toeplitz completion of $\begin{bmatrix} T_{\overline{z}} & ? \\ ? & T_{\overline{z}} \end{bmatrix}$ is automatically normal. Thus, for matrices $\begin{bmatrix} T_{\overline{z}} & T_{\varphi} \\ T_{\psi} & T_{\overline{z}} \end{bmatrix}$ $(\varphi, \psi \in L^{\infty})$ there is no gap between 2-hyponormality and subnormality. However, there does exist a gap between hyponormality and 2-hyponormality. Let

$$\Phi := \begin{bmatrix} \overline{z} & \overline{z}^2 + 2z^2 \\ \overline{z}^2 + 2z^2 & \overline{z} \end{bmatrix}.$$

Remark (cont.)

Clearly, Φ is normal, and if we let

$$K := \begin{bmatrix} \frac{1}{2} & \frac{z}{2} \\ \frac{z}{2} & \frac{1}{2} \end{bmatrix}$$

then $\Phi - K\,\Phi^* \in H^2_{M_2}$ and $||K||_\infty = 1$, so that T_Φ is hyponormal.

However, by our results, T_Φ is not 2-hyponormal. $\square$

Problem

Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued function, and assume that T_Φ is subnormal. Does T_Φ admit an orthogonal decomposition of the form $\begin{pmatrix} T_{\Phi_1} & 0 \\ 0 & T_{\Phi_2} \end{pmatrix}$, with T_{Φ_1} normal and T_{Φ_2} analytic?