Taylor joint spectrum

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Let $\Lambda \equiv \Lambda[e] \equiv \Lambda_n[e]$ be the exterior algebra on n generators $e_1, ..., e_n$, with identity $e_0 \equiv 1$. Λ is the algebra of forms in $e_1, ..., e_n$ with complex coefficients, subject to the collapsing property $e_i e_j + e_j e_i = 0$ $(1 \le i, j \le n)$. Let $E_i : \Lambda \to \Lambda$ denote the creation operator, given by $E_i \xi := e_i \xi$ $(\xi \in \Lambda, 1 \le i \le n)$. If one declares $\{e_{i_1}, ..., e_{i_k} : 1 \le i_1 < ... < i_k \le n\}$ as an orthonormal basis, the exterior algebra Λ becomes a Hilbert space, admitting an orthogonal decomposition $\Lambda = \bigoplus_{k=1}^{n} \Lambda^k$, where $\dim \Lambda^k = \begin{pmatrix} n \\ k \end{pmatrix}$. Thus, each $\xi \in \Lambda$ admits a unique orthogonal decomposition $\xi = e_i \xi' + \xi''$, where ξ' and ξ'' have no e_i contribution. It then readily follows that $E_i^* \xi = \xi'$. Indeed, each E_i is a partial isometry, satisfying $E_i^* E_j + E_j E_i^* = \delta_{ij}$ $(1 \le i, j \le n)$.

Let $\mathcal{X}$ be a normed space, let $A \equiv (A_1, ..., A_n)$ be a commuting n-tuple of bounded operators on $\mathcal{X}$, and set $\Lambda(\mathcal{X}) := \mathcal{X} \otimes_{\mathbf{C}} \Lambda$. We define $D_A : \Lambda(\mathcal{X}) \to \Lambda(\mathcal{X})$ by $D_A := \sum_{i=1}^n A_i \otimes E_i$. Clearly $D_A^2 = 0$, so $Ran\ D_A \subseteq\ Ker\ D_A$.

Taylor spectrum. The commuting n-tuple A is said to be *nonsingular* on $\mathcal{X}$ if $Ran\ D_A = Ker\ D_A$. The *Taylor (joint) spectrum* of A on $\mathcal{X}$ is the set

$$\sigma_T(A, \mathcal{X}) := \{\lambda \in \mathbf{C}^n : A - \lambda \text{ is singular}\}.$$

The decomposition $\Lambda = \bigoplus_{k=1}^n \Lambda^k$ gives rise to a cochain complex $K(A, \mathcal{X})$, the so-called Koszul complex associated to A on $\mathcal{X}$, as follows:

$$K(A, \mathcal{X}) : 0 \to \Lambda^0(\mathcal{X}) \stackrel{D_A^0}{\to} \Lambda^1(\mathcal{X}) \stackrel{D_A^1}{\to} \ldots \stackrel{D_A^{n-1}}{\to} \Lambda^n(\mathcal{X}) \to 0,$$

where D_A^k denotes the restriction of D_A to the subspace $\Lambda^k(\mathcal{X})$. Thus, $\sigma_T(A, \mathcal{X}) = \{\lambda \in \mathbf{C}^n : K(A - \lambda, \mathcal{X}) \text{ is not exact}\}$.

J.L. Taylor showed in [Tay1] that if $\mathcal{X}$ is a Banach space, then $\sigma_T(A, \mathcal{X})$ is compact, nonempty, and contained in $\sigma'(A)$, the (joint) algebraic spectrum of A with respect to the commutant of A, $(A)' := \{B \in \mathcal{L}(\mathcal{X}) : BA = AB\}$. Moreover, σ_T carries an analytic functional calculus with values in the double commutant of A, so that in particular σ_T possesses the projection property.

Example 1. *For $n = 1$, D_A admits the following 2×2 matrix relative to the direct sum decomposition $(\mathcal{X} \otimes e_0) \bigoplus (\mathcal{X} \otimes e_1)$:*

$$D_A = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}.$$

Then $Ker\ D_A / Ran\ D_A = Ker\ A \bigoplus (\mathcal{X}/Ran\ A)$. It follows at once that σ_T agrees with σ, the spectrum of A.

Example 2. *For $n = 2$,*

$$D_A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & -A_2 & A_1 & 0 \end{pmatrix},$$

so $Ker\ D_A/Ran\ D_A = (Ker\ A_1 \bigcap Ker\ A_2) \bigoplus \{(x_1, x_2) : A_2x_1 = A_1x_2\}/\{(A_1x_0, A_2x_0) : x_0 \in \mathcal{X}\} \bigoplus (\mathcal{X}/(Ran\ A_1 + Ran\ A_2))$.

Observation 3. *Since σ_T is defined in terms of the actions of the operators A_i on vectors of $\mathcal{X}$, it is intrinsically "spatial," as opposed to σ', σ'', and other algebraic joint spectra. σ_T contains other well known spatial spectra, like σ_p (the point spectrum), σ_π (the approximate point spectrum), and σ_δ (the defect spectrum). Moreover, if $\mathcal{B}$ is a commutative Banach algebra, $a \equiv (a_1, ..., a_n)$, with each $a_i \in \mathcal{B}$, and L_a denotes the n-tuple of left multiplications by the a_i's, it is not hard to show that $\sigma_T(L_a, \mathcal{B}) = \sigma_{\mathcal{B}}(a)$. As a matter of fact, the same result holds when $\mathcal{B}$ is not commutative, provided the a_i's come from the center of $\mathcal{B}$.*

Spectral permanence. When $\mathcal{B}$ is a C*-algebra, say $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$, $\sigma_T(L_a, \mathcal{B}) = \sigma_T(a, \mathcal{H})$ [Cur1]. This fact, known as spectral permanence for the Taylor spectrum, shows that for C*-algebra elements (and also for Hilbert space operators), the nonsingularity of L_a is equivalent to the invertibility of the associated Dirac operator $D_a + D_{a^*}^t$.

Finite dimensional case. When $\dim \mathcal{X} < \infty$,

$$\sigma_p = \sigma_\ell = \sigma_\pi = \sigma_\delta = \sigma_r = \sigma_T = \sigma' = \sigma'' = \hat{\sigma},$$

where σ_ℓ, σ_r and $\hat{\sigma}$ denote the left, right and polynomially convex spectra, respectively. As a matter of fact, in this case the commuting n-tuple A can be simultaneously triangularized as $A_k \equiv (a_{i,j}^{(k)})_{i,j=1}^{\dim \mathcal{X}}$, and

$$\sigma_T(A, \mathcal{X}) = \{(a_{ii}^{(1)}, ..., a_{ii}^{(n)}) : 1 \leq i \leq \dim \mathcal{X}\}.$$

Case of compact operators. If A is a commuting n-tuple of compact operators acting on a Banach space $\mathcal{X}$, then $\sigma_T(A, \mathcal{X})$ is countable, with $(0, ..., 0)$ as the only accumulation point. Moreover, $\sigma_\pi(A, \mathcal{X}) = \sigma_\delta(A, \mathcal{X}) = \sigma_T(A, \mathcal{X})$.

Invariant subspaces. If $\mathcal{X}$ is a Banach space, $\mathcal{Y}$ is a closed subspace of $\mathcal{X}$, and A is a commuting n-tuple leaving $\mathcal{Y}$ invariant, then the union of any two of the sets $\sigma_T(A, \mathcal{X})$, $\sigma_T(A, \mathcal{Y})$ and $\sigma_T(A, \mathcal{X}/\mathcal{Y})$ contains the third [Tay1]. This can be seen by looking at the long cohomology sequence associated to the Koszul complex and the canonical short exact sequence $0 \to \mathcal{Y} \to \mathcal{X} \to \mathcal{X}/\mathcal{Y} \to 0$.

Additional properties. In addition to the above mentioned properties of σ_T, we list below the following facts, which can be found in the survey article [Cur2] and the references therein:

(i) σ_T gives rise to a compact nonempty subset $M_{\sigma_T}(\mathcal{B}, \mathcal{X})$ of the maximal ideal space of any commutative Banach algebra $\mathcal{B}$ containing A, in such a way that $\sigma_T(A, \mathcal{X}) = \hat{A}(M_{\sigma_T}(\mathcal{B}, \mathcal{X}))$ [Tay1];

(ii) for $n = 2$, $\partial\sigma_T(A, \mathcal{H}) \subseteq \partial\sigma_H(A, \mathcal{H})$, where $\sigma_H := \sigma_\ell \bigcup \sigma_r$ denotes the Harte spectrum;

(iii) the upper semicontinuity of separate parts holds for the Taylor spectrum;

(iv) every isolated point in $\sigma_{\mathcal{B}}(A)$ is an isolated point of $\sigma_T(A, \mathcal{H})$ (and, a fortiori, an isolated point of $\sigma_\ell(A,\mathcal{H})\bigcap\sigma_r(A,\mathcal{H})$);

(v) if $0\in\sigma_T(A,\mathcal{H})$, up to approximate unitary equivalence one can always assume that $\overline{Ran\ D_A}\neq Ker\ D_A$ [CuFi];

(vi) the functional calculus introduced by J.L. Taylor in [Tay2] admits a concrete realization in terms of the Bochner-Martinelli kernel in case A acts on a Hilbert space or on a C*-algebra [Vas];

(vii) M. Putinar established in [Put1] the uniqueness of the functional calculus provided it extends the polynomial calculus.

Fredholm n-tuples. In a way entirely similar to the development of Fredholm theory, one can define the notion of Fredholm n-tuple: a commuting n-tuple A is said to be *Fredholm* on $\mathcal{X}$ if the associated Koszul complex $K(A,\mathcal{X})$ has finite dimensional cohomology spaces. The Taylor *essential* spectrum of A on $\mathcal{X}$ is then

$$\sigma_{Te}(A,\mathcal{X}):=\{\lambda\in\mathbf{C}^n : A-\lambda \text{ is not Fredholm}\}.$$

The *Fredholm index* of A is defined as the Euler characteristic of $K(A,\mathcal{X})$, e.g., if $n=2$, $index(A)=\dim\ Ker\ D_A^0-\dim(Ker\ D_A^1\ /\ Ran D_A^0)+\dim(\mathcal{X}\ /\ Ran\ D_A^1)$. In Hilbert space, $\sigma_{Te}(A,\mathcal{H})=\sigma_T(L_a,\mathcal{Q}(\mathcal{H}))$, where $a:=\pi(A)$ is the coset of A in the Calkin algebra for $\mathcal{H}$.

Example 4. *If $\mathcal{H}=H^2(S^3)$ and $A_i:=M_{z_i}$ $(i=1,2)$, then $\sigma_\ell(A)=\sigma_{\ell e}(A)=\sigma_{re}(A)=\sigma_{Te}(A)=S^3$, $\sigma_r(A)=\sigma_T(A)=\mathbf{B}_4$, and $index(A-\lambda)=1$ $(\lambda\in\mathbf{B}_4)$.*

Remark 1. *The Taylor spectral and Fredholm theories of multiplication operators acting on Bergman spaces over Reinhardt domains or bounded pseudoconvex domains, or acting* *on the Hardy spaces over the Shilov boundary of bounded symmetric domains on several complex variables have been described in [BeCo], [BCK], [CuMu], [CuSa], [CuYa], [Sal], [SSU], [Upm] and [Ven]; for Toeplitz operators with H^∞ symbols acting on bounded pseudoconvex domains, concrete descriptions appear in [EsPu].*

Spectral inclusion. If S is a subnormal n-tuple acting on $\mathcal{H}$ with minimal normal extension N acting on $\mathcal{K}$, $\sigma_T(N,\mathcal{K}) \subseteq \sigma_T(S,\mathcal{H}) \subseteq \hat{\sigma}(N,\mathcal{K})$ [Put2].

Left and right multiplications. For A and B two commuting n-tuples of operators on a Hilbert space $\mathcal{H}$, and L_A and R_B the associated n-tuples of left and right multiplication operators,

$$\sigma_T((L_A, R_B), \mathcal{L}(\mathcal{H})) = \sigma_T(A,\mathcal{H}) \times \sigma_T(B,\mathcal{H}),$$

and

$$\sigma_{Te}((L_A, R_B), \mathcal{L}(\mathcal{H}))$$

$$= [\sigma_{Te}(A,\mathcal{H}) \times \sigma_T(B,\mathcal{H})] \bigcup [\sigma_T(A,\mathcal{H}) \times \sigma_{Te}(B,\mathcal{H})] \text{ [CuFi]}.$$

Over the last several years, Taylor spectral theory has received considerable attention; for further details and information, we refer the reader to the monographs [AmVa], [EsPu] and [Vas], the survey article [Cur2], and [AlVa]. There is also a parallel *local* spectral theory, described in [EsPu], [LaNe] and [Vas].

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