Taylor joint spectrum

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Let $\Lambda = \Lambda[e] = \Lambda_n[e]$ be the exterior algebra on $n$ generators $e_1, \ldots, e_n$, with identity $e_0 = 1$. $\Lambda$ is the algebra of forms in $e_1, \ldots, e_n$ with complex coefficients, subject to the collapsing property $e_i e_j + e_j e_i = 0$ $(1 \leq i, j \leq n)$. Let $E_i : \Lambda \to \Lambda$ denote the creation operator, given by $E_i \xi := e_i \xi$ ($\xi \in \Lambda, 1 \leq i \leq n$). If one declares $\{e_{i_1}, \ldots, e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\}$ as an orthonormal basis, the exterior algebra $\Lambda$ becomes a Hilbert space, admitting an orthogonal decomposition $\Lambda = \bigoplus_{k=1}^n \Lambda^k$, where $\dim \Lambda^k = \binom{n}{k}$. Thus, each $\xi \in \Lambda$ admits a unique orthogonal decomposition $\xi = \xi' + \xi''$, where $\xi'$ and $\xi''$ have no $e_i$ contribution. It then readily follows that $E_i^* \xi = \xi'$. Indeed, each $E_i$ is a partial isometry, satisfying $E_i^* E_j + E_j E_i^* = \delta_{ij} (1 \leq i, j \leq n)$.

Let $\mathcal{X}$ be a normed space, let $A = (A_1, \ldots, A_n)$ be a commuting $n$-tuple of bounded
operators on $\mathcal{X}$, and set $\Lambda(\mathcal{X}) := \mathcal{X} \otimes \mathbb{C} \Lambda$. We define $D_A : \Lambda(\mathcal{X}) \to \Lambda(\mathcal{X})$ by $D_A := \sum_{i=1}^{n} A_i \otimes E_i$. Clearly $D_A^2 = 0$, so $\text{Ran } D_A \subseteq \text{Ker } D_A$.

Taylor spectrum. The commuting $n$-tuple $A$ is said to be nonsingular on $\mathcal{X}$ if $\text{Ran } D_A = \text{Ker } D_A$. The Taylor (joint) spectrum of $A$ on $\mathcal{X}$ is the set

$$\sigma_T(A, \mathcal{X}) := \{\lambda \in \mathbb{C}^n : A - \lambda \text{ is singular}\}.$$ 

The decomposition $\Lambda = \bigoplus_{k=1}^{n} \Lambda^k$ gives rise to a cochain complex $K(A, \mathcal{X})$, the so-called Koszul complex associated to $A$ on $\mathcal{X}$, as follows:

$$K(A, \mathcal{X}) : 0 \to \Lambda^0(\mathcal{X}) \xrightarrow{D_A^0} \Lambda^1(\mathcal{X}) \xrightarrow{D_A^1} \cdots \xrightarrow{D_A^{n-1}} \Lambda^n(\mathcal{X}) \to 0,$$

where $D_A^k$ denotes the restriction of $D_A$ to the subspace $\Lambda^k(\mathcal{X})$. Thus, $\sigma_T(A, \mathcal{X}) = \{\lambda \in \mathbb{C}^n : K(A - \lambda, \mathcal{X}) \text{ is not exact}\}$.

J.L. Taylor showed in [Tay1] that if $\mathcal{X}$ is a Banach space, then $\sigma_T(A, \mathcal{X})$ is compact, nonempty, and contained in $\sigma'(A)$, the (joint) algebraic spectrum of $A$ with respect to the commutant of $A$, $(A)' := \{B \in \mathcal{L}(\mathcal{X}) : BA = AB\}$. Moreover, $\sigma_T$ carries an analytic functional calculus with values in the double commutant of $A$, so that in particular $\sigma_T$ possesses the projection property.

**Example 1.** For $n = 1$, $D_A$ admits the following $2 \times 2$ matrix relative to the direct sum decomposition $(\mathcal{X} \otimes e_0) \oplus (\mathcal{X} \otimes e_1)$:

$$D_A = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}.$$ 

Then $\text{Ker } D_A / \text{Ran } D_A = \text{Ker } A \oplus (\mathcal{X} / \text{Ran } A)$. It follows at once that $\sigma_T$ agrees with $\sigma$, the spectrum of $A$. 

2
Example 2. For $n = 2$,

$$D_A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & -A_2 & A_1 & 0 \end{pmatrix}.$$ 

so $\text{Ker } D_A/\text{Ran } D_A = (\text{Ker } A_1 \cap \text{Ker } A_2) \oplus \{(x_1, x_2) : A_2x_1 = A_1x_2\}/\{(A_1x_0, A_2x_0) : x_0 \in \mathcal{X}\} \oplus (\mathcal{X}/(\text{Ran } A_1 + \text{Ran } A_2)).$

Observation 3. Since $\sigma_T$ is defined in terms of the actions of the operators $A_i$ on vectors of $\mathcal{X}$, it is intrinsically “spatial,” as opposed to $\sigma'$, $\sigma''$, and other algebraic joint spectra. $\sigma_T$ contains other well known spatial spectra, like $\sigma_p$ (the point spectrum), $\sigma_{\approx}$ (the approximate point spectrum), and $\sigma_\delta$ (the defect spectrum). Moreover, if $\mathcal{B}$ is a commutative Banach algebra, $a \equiv (a_1, \ldots, a_n)$, with each $a_i \in \mathcal{B}$, and $L_a$ denotes the $n$-tuple of left multiplications by the $a_i$'s, it is not hard to show that $\sigma_T(L_a, \mathcal{B}) = \sigma_E(a)$. As a matter of fact, the same result holds when $\mathcal{B}$ is not commutative, provided the $a_i$’s come from the center of $\mathcal{B}$.

Spectral permanence. When $\mathcal{B}$ is a $C^*$-algebra, say $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$, $\sigma_T(L_a, \mathcal{B}) = \sigma_T(a, \mathcal{H})$ [Cur1]. This fact, known as spectral permanence for the Taylor spectrum, shows that for $C^*$-algebra elements (and also for Hilbert space operators), the nonsingularity of $L_a$ is equivalent to the invertibility of the associated Dirac operator $D_a + D_{a^*}$.

Finite dimensional case. When $\dim \mathcal{X} < \infty$,

$$\sigma_p = \sigma_\ell = \sigma_{\approx} = \sigma_\delta = \sigma_T = \sigma' = \sigma'' = \hat{\sigma},$$
where $\sigma_\ell$, $\sigma_r$ and $\hat{\sigma}$ denote the left, right and polynomially convex spectra, respectively. As a matter of fact, in this case the commuting $n$-tuple $A$ can be simultaneously triangularized as $A_k \equiv (a_{i,j}^{(k)})_{i,j=1}^{\dim \mathcal{X}}$; and

$$\sigma_T(A, \mathcal{X}) = \{ (a_{ii}^{(1)}, \ldots, a_{ii}^{(n)}) : 1 \leq i \leq \dim \mathcal{X} \}.$$ 

**Case of compact operators.** If $A$ is a commuting $n$-tuple of compact operators acting on a Banach space $\mathcal{X}$, then $\sigma_T(A, \mathcal{X})$ is countable, with $(0, \ldots, 0)$ as the only accumulation point. Moreover, $\sigma_\pi(A, \mathcal{X}) = \sigma_\delta(A, \mathcal{X}) = \sigma_T(A, \mathcal{X})$.

**Invariant subspaces.** If $\mathcal{X}$ is a Banach space, $\mathcal{Y}$ is a closed subspace of $\mathcal{X}$, and $A$ is a commuting $n$-tuple leaving $\mathcal{Y}$ invariant, then the union of any two of the sets $\sigma_T(A, \mathcal{X})$, $\sigma_T(A, \mathcal{Y})$ and $\sigma_T(A, \mathcal{X}/\mathcal{Y})$ contains the third [Tay1]. This can be seen by looking at the long cohomology sequence associated to the Koszul complex and the canonical short exact sequence $0 \to \mathcal{Y} \to \mathcal{X} \to \mathcal{X}/\mathcal{Y} \to 0$.

**Additional properties.** In addition to the above mentioned properties of $\sigma_T$, we list below the following facts, which can be found in the survey article [Cur2] and the references therein:

(i) $\sigma_T$ gives rise to a compact nonempty subset $M_{\sigma_T}(B, \mathcal{X})$ of the maximal ideal space of any commutative Banach algebra $B$ containing $A$, in such a way that $\sigma_T(A, \mathcal{X}) = \hat{A}(M_{\sigma_T}(B, \mathcal{X}))$ [Tay1];

(ii) for $n = 2$, $\partial \sigma_T(A, \mathcal{H}) \subseteq \partial \sigma_H(A, \mathcal{H})$, where $\sigma_H := \sigma_\ell \cup \sigma_r$ denotes the Harte spectrum;

(iii) the upper semi-continuity of separate parts holds for the Taylor spectrum;

(iv) every isolated point in $\sigma_H(A)$ is an isolated point of $\sigma_T(A, \mathcal{H})$ (and, a fortiori, an
isolated point of $\sigma_t(A, \mathcal{H}) \cap \sigma_r(A, \mathcal{H})$;

(v) if $0 \in \sigma_T(A, \mathcal{H})$, up to approximate unitary equivalence one can always assume that 
\[ \text{Ran} \ D_A \neq \text{Ker} \ D_A \ [\text{CuFi}] ; \]

(vi) the functional calculus introduced by J.L. Taylor in [Tay2] admits a concrete realization in terms of the Bochner-Martinelli kernel in case $A$ acts on a Hilbert space or on a C*-algebra [Vas];

(vii) M. Putinar established in [Put1] the uniqueness of the functional calculus provided it extends the polynomial calculus.

Fredholm n-tuples. In a way entirely similar to the development of Fredholm theory, one can define the notion of Fredholm n-tuple: a commuting $n$-tuple $A$ is said to be Fredholm on $\mathcal{X}$ if the associated Koszul complex $K(A, \mathcal{X})$ has finite dimensional cohomology spaces. The Taylor essential spectrum of $A$ on $\mathcal{X}$ is then

\[ \sigma_{Te}(A, \mathcal{X}) := \{ \lambda \in \mathbb{C}^n : A - \lambda \text{ is not Fredholm} \} . \]

The Fredholm index of $A$ is defined as the Euler characteristic of $K(A, \mathcal{X})$, e.g., if $n = 2$, 
\[ \text{index}(A) = \dim \text{Ker} \ D^0_A - \dim(\text{Ker} \ D^1_A / \text{Ran} \ D^0_A) + \dim(\mathcal{X} / \text{Ran} \ D^1_A) . \]

In Hilbert space, $\sigma_{Te}(A, \mathcal{H}) = \sigma_{T}(I_A, \mathbb{Q}(\mathcal{H}))$, where $a := \pi(A)$ is the coset of $A$ in the Calkin algebra for $\mathcal{H}$.

Example 4. If $\mathcal{H} = H^2(S^3)$ and $A_i := M_{z_i}$ ($i = 1, 2$), then $\sigma_t(A) = \sigma_{te}(A) = \sigma_{re}(A) = \sigma_{Te}(A) = S^3$, $\sigma_r(A) = \sigma_T(A) = B_4$, and $\text{index}(A - \lambda) = 1$ ($\lambda \in B_4$).

Remark 1. The Taylor spectral and Fredholm theories of multiplication operators acting on Bergman spaces over Reinhardt domains or bounded pseudoconvex domains, or acting
on the Hardy spaces over the Shilov boundary of bounded symmetric domains on several
complex variables have been described in [BeCo], [BCK], [CuMu], [CuSa], [CuYa], [Sal],
[SSU], [Upm] and [Ven]; for Toeplitz operators with $H^\infty$ symbols acting on bounded
pseudoconvex domains, concrete descriptions appear in [EsPu].

Spectral inclusion. If $S$ is a subnormal $n$-tuple acting on $\mathcal{H}$ with minimal normal
extension $N$ acting on $\mathcal{K}$, $\sigma_T(N, \mathcal{K}) \subseteq \sigma_T(S, \mathcal{H}) \subseteq \sigma(N, \mathcal{K})$ [Put2].

Left and right multiplications. For $A$ and $B$ two commuting $n$-tuples of operators on
a Hilbert space $\mathcal{H}$, and $L_A$ and $R_B$ the associated $n$-tuples of left and right multiplication
operators,

$$\sigma_T((L_A, R_B), \mathcal{L}(\mathcal{H})) = \sigma_T(A, \mathcal{H}) \times \sigma_T(B, \mathcal{H}),$$

and

$$\sigma_{T_e}((L_A, R_B), \mathcal{L}(\mathcal{H})) = [\sigma_{T_e}(A, \mathcal{H}) \times \sigma_{T_e}(B, \mathcal{H})] \cup [\sigma_{T_e}(A, \mathcal{H}) \times \sigma_{T_e}(B, \mathcal{H})]$$ [CuFi].

Over the last several years, Taylor spectral theory has received considerable attention;
for further details and information, we refer the reader to the monographs [AmVa], [EsPu]
and [Vas], the survey article [Cur2], and [AlVa]. There is also a parallel local spectral
theory, described in [EsPu], [LaNe] and [Vas].

References

[AlVa] E. Albrecht and F.-H. Vasilescu, Semi-Fredholm complexes, Operator Theory:


