JOINTLY HYPONORMAL PAIRS OF COMMUTING SUBNORMAL OPERATORS NEED NOT BE JOINTLY SUBNORMAL

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Abstract. We construct three different families of commuting pairs of subnormal operators, jointly hyponormal but not admitting commuting normal extensions. Each such family can be used to answer in the negative a 1988 conjecture of R. Curto, P. Muhly and J. Xia. We also obtain a sufficient condition under which joint hyponormality does imply joint subnormality.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S, T \in B(\mathcal{H})$ let $[S, T] := ST - TS$. We say that an $n$-tuple $T = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) hyponormal if the operator matrix

$$[T^*, T] := \begin{pmatrix}
[T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\
[T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\
\vdots & \vdots & \ddots & \vdots \\
[T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n]
\end{pmatrix}$$

is positive on the direct sum of $n$ copies of $\mathcal{H}$ (cf. [Ath], [CMX]). The $n$-tuple $T$ is said to be normal if $T$ is commuting and each $T_i$ is normal, and $T$ is subnormal if $T$ is the restriction of a normal $n$-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal. The Bram-Halmos criterion states that an operator $T \in B(\mathcal{H})$ is subnormal if and only if the $k$-tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$.

For $\alpha \equiv \{a_n\}_{n=0}^\infty$ a bounded sequence of positive real numbers (called weights), let $W_{\alpha} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_{\alpha} e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The moments of $\alpha$ are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{ll}
1 & \text{if } k = 0 \\
\alpha_0^{\frac{k}{2}} \cdots \alpha_{k-1}^{\frac{k}{2}} & \text{if } k > 0.
\end{array} \right.$$}

It is easy to see that $W_{\alpha}$ is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_k, \beta_k \in \ell^\infty(\mathbb{Z}_+^2)$, $k \equiv (k_1, k_2) \in \mathbb{Z}_+^2$ := $\mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.)

We define the 2-variable weighted shift $T \equiv (T_1, T_2)$ by

$$T_1 e_k := \alpha_k e_{k+1},$$

$$T_2 e_k := \beta_k e_{k+1}.$$
where \( \varepsilon_1 := (1, 0) \) and \( \varepsilon_2 := (0, 1) \). Clearly,

\[ T_1 T_2 = T_2 T_1 \iff \beta_{k+\varepsilon_1} = \alpha_{k+\varepsilon_2} \beta_k \quad \text{ (all } k). \tag{1.1} \]

In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts \( W_\alpha \) and \( W_\beta \) gives rise to a 2-variable weighted shift \( T \equiv (T_1, T_2) \), if we let \( \alpha_{(k_1, k_2)} := \alpha_{k_1} \) and \( \beta_{(k_1, k_2)} := \beta_{k_2} \) (all \( k_1, k_2 \in \mathbb{Z}_+ \)). In this case, \( T \) is subnormal (resp. hyponormal) if and only if so are \( T_1 \) and \( T_2 \); in fact, under the canonical identification of \( \ell^2(\mathbb{Z}_+) \) and \( \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \), \( T_1 \cong W_\alpha \otimes I \) and \( T_2 \cong I \otimes W_\beta \), and \( T \) is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [Con, III.8.16]): \( W_\alpha \) is subnormal if and only if there exists a probability measure \( \xi \) supported in \( [0, \|W_\alpha\|^2] \) (called the Berger measure of \( W_\alpha \)) such that \( \gamma_k(\alpha) := \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 = \int t^k \, d\xi(t) \quad (k \geq 1) \). If \( W_\alpha \) is subnormal, and if for \( h \geq 1 \) we let \( \mathcal{M}_h := \bigvee\{e_n : n \geq h\} \) denote the invariant subspace obtained by removing the first \( h \) vectors in the canonical orthonormal basis of \( \ell^2(\mathbb{Z}_+) \), then the Berger measure of \( W_\alpha|_{\mathcal{M}_h} \) is \( \frac{1}{\gamma_h} t^h \, d\xi(t) \).

An important class of subnormal weighted shifts is obtained by considering measures \( \mu \) with exactly two atoms \( t_0 \) and \( t_1 \). These shifts arise naturally in the Subnormal Completion Problem [CuFi3] and in the theory of truncated moment problems (cf. [CuFi1], [CuFi4]). For \( t_0, t_1 \in \mathbb{R}_+ \) with \( t_0 < t_1 \), and \( \rho_0, \rho_1 > 0 \) with \( \rho_0 + \rho_1 = 1 \), the moments of the 2-atomic probability measure \( \xi := \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1} \) (here \( \delta_p \) denotes the point-mass probability measure with support the singleton \( \{p\} \)) satisfy the 2-step recursive relation \( \gamma_{n+2} = \varphi_0 \gamma_n + \varphi_1 \gamma_{n+1} \) \((n \geq 0)\); at the weight level, this can be written as \( \alpha_{n+1}^2 = \alpha_n^2 + \alpha_1 \) \((n \geq 0)\). More generally, any finitely atomic Berger measure corresponds to a recursively generated weighted shift (i.e., one whose moments satisfy an \( r \)-step recursive relation); in fact, \( r = \text{card supp } \xi \). In the special case of \( r = 2 \), the theory of recursively generated weighted shifts makes contact with the work of J. Stampfli in [Sta], in which he proved that given three positive numbers \( \alpha_0 < \alpha_1 < \alpha_2 \), it is always possible to find a subnormal weighted shift, denoted \( W_{(\alpha_0, \alpha_1, \alpha_2)} \), whose first three weights are \( \alpha_0, \alpha_1, \alpha_2 \). In this case, the coefficients of recursion (cf. [CuFi2, Example 3.12], [CuFi3, Section 3], [Cu3, Section 1, p. 81]) are given by

\[ \varphi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^4 - \alpha_0^4} \quad \text{and} \quad \varphi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^4 - \alpha_0^4}, \tag{1.2} \]

the atoms \( t_0 \) and \( t_1 \) are the roots of the equation

\[ t^2 - (\varphi_0 + \varphi_1 t) = 0, \tag{1.3} \]

and the densities \( \rho_0 \) and \( \rho_1 \) uniquely solve the \( 2 \times 2 \) system of equations

\[ \begin{cases} \rho_0 + \rho_1 = 1 \\ \rho_0 t_0 + \rho_1 t_1 = \alpha_0^2. \end{cases} \tag{1.4} \]

We also recall the notion of moment of order \( k \) for a pair \( (\alpha, \beta) \) satisfying (1.1). Given \( k \in \mathbb{Z}_+^2 \), the moment of \( (\alpha, \beta) \) of order \( k \) is

\[ \gamma_k \equiv \gamma_k(\alpha, \beta) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)} \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \]
We remark that, due to the commutativity condition (1.1), $\gamma_k$ can be computed using any non-decreasing path from $(0,0)$ to $(k_1,k_2)$.

**Theorem 1.1. (Berger’s Theorem, 2-variable case) ([JeLu])** A 2-variable weighted shift $T \equiv (T_1,T_2)$ admits a commuting normal extension if and only if there is a probability measure $\mu$ defined on the 2-dimensional rectangle $R = [0,a_1] \times [0,a_2]$ $(a_i := ||T_i||^2)$ such that $\gamma_k = \iint_R t_1^{k_1} t_2^{k_2} d\mu(t_1,t_2)$ (all $k \in \mathbb{Z}_+^2$).

Clearly, each component $T_i$ of a subnormal 2-variable weighted shift $T \equiv (T_1,T_2)$ must be subnormal. For instance, $d\nu_j(t_1) := \frac{1}{(0,\omega_j)} \int_{[0,\omega_j]} t_2^2 d\Phi_{t_1}(t_2)$, where $d\mu(t_1,t_2) \equiv d\Phi_{t_1}(t_2) d\eta(t_1)$ is the canonical disintegration of $\mu$ by horizontal slices. On the other hand, if we only know that each of $T_1, T_2$ is subnormal, and that they commute, the following problem is natural.

**Problem 1.2. (Lifting Problem for Commuting Subnormals)** Find necessary and sufficient conditions on $T_1$ and $T_2$ to guarantee the subnormality of $T \equiv (T_1,T_2)$.

It is well known that the above mentioned necessary conditions do not suffice (cf. [Cu1]). In terms of the marginal measures, the problem can be phrased as a reconstruction-of-measure problem, that is, under what conditions on the single variable measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$ associated with $T_1$ and $T_2$, respectively, does there exist a 2-variable measure $\mu$ correctly interpolating all the powers $t_1^{k_1} t_2^{k_2}$ $(k_1,k_2 \geq 0)$?

To detect hyponormality for 2-variable weighted shifts, there is a simple criterion involving a base point $k$ in $\mathbb{Z}_+^2$ and its five neighboring points in $k + \mathbb{Z}_+^2$ at path distance at most 2 (cf. Figure 1).

**Theorem 1.3. ([Cu1]) (Six-point Test)** Let $T \equiv (T_1,T_2)$ be a 2-variable weighted shift, with weight sequences $\alpha$ and $\beta$. Then

$$[T^*,T] \geq 0 \iff \left( (\alpha_{k+\varepsilon_1} + \varepsilon_1) \right)_{k \in \mathbb{Z}_+^2} \geq 0 \quad (all \ k \in \mathbb{Z}_+^2)$$

$$\iff \left( \frac{\alpha_{k+\varepsilon_1}^2 - \beta_{k+\varepsilon_1}^2}{\alpha_k^2 - \beta_k^2} \right) \geq 0 \quad (all \ k \in \mathbb{Z}_+^2).$$

![Figure 1. Weight diagram used in the Six-point Test](image-url)
Unlike the single variable case, in which there is a clear separation between hyponormality and subnormality (cf. [CuFi3], [Cu3], [CuLe]), much less is known about the multivariable case. In this paper we will construct three conceptually different families of counterexamples to the following conjecture.

**Conjecture 1.4.** ([CMX]) Let $T \equiv (T_1, T_2)$ be a pair of commuting subnormal operators on $\mathcal{H}$. Then $T$ is subnormal if and only if $T$ is hyponormal.

We mention that M. Dritschel and S. McCullough, working independently, have been able to obtain a separate example ([DrMcC]). We shall see in Section 4 that their example is a special case of a general construction that produces nonsubnormal hyponormal pairs with $T_1 \cong T_2$.

We now formulate an improved version of a result due to R. Curto.

**Proposition 1.5.** (Subnormal backward extension of a 1-variable weighted shift) (cf. [Cu2]) Let $T$ be a weighted shift whose restriction $T_M := T|_M$ to $M := \mathcal{V}\{e_1, e_2, \cdots\}$ is subnormal, with associated measure $\mu_M$. Then $T$ is subnormal (with associated measure $\mu$) if and only if

(i) $\frac{1}{t} \in L^1(\mu_M)$

(ii) $\alpha_0 \leq (\|1\|_{L^1(\mu_M)})^{-1}$.

In this case, $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_M(t) + (1 - \alpha_0^2 \|1\|_{L^1(\mu_M)}) d\delta_0(t)$, where $\delta_0$ denotes Dirac measure at 0. In particular, $T$ is never subnormal when $\mu_M(\{0\}) > 0$.

**Proof.** $\Rightarrow$) We first observe that the moments of $T$ and $T_M$ are related by the equation

$$\gamma_k(T_M) \equiv \alpha_1^2 \cdots \alpha_k^2 = \frac{\gamma_{k+1}(T)}{\alpha_0^2}$$

so that

$$\frac{1}{\alpha_0^2} \int t^{k+1} d\mu(t) = \int t^k d\mu_M(t) \quad (\text{all } k \geq 0),$$

that is, $td\mu(t) = \alpha_0^2 d\mu_M(t)$. It follows at once that

$$d\mu(t) = \lambda d\delta_0(t) + \alpha_0^2 \frac{1}{t} d\mu_M(s),$$

where $\lambda \geq 0$. Since $\int d\mu = 1$, we must have $\frac{1}{t} \in L^1(\mu_M)$ and $\alpha_0^2 \|1\|_{L^1(\mu_M)} \leq 1$. Finally, it is straightforward to verify that $\lambda = 1 - \alpha_0^2 \|1\|_{L^1(\mu_M)}$.

$\Leftarrow$) Let

$$d\mu(t) := \frac{\alpha_0^2}{t} d\mu_M(t) + (1 - \alpha_0^2 \|1\|_{L^1(\mu_M)}) d\delta_0(t).$$

By hypotheses, $\mu$ is a positive Borel measure on $[0, \|T\|^2]$. Moreover,

$$\int d\mu = \alpha_0^2 \int \frac{1}{t} d\mu_M + (1 - \alpha_0^2 \|1\|_{L^1(\mu_M)}) \int d\delta_0 = 1,$$

and for $k \geq 1$,

$$\int t^k d\mu(t) = \alpha_0^2 \int t^{k-1} \frac{1}{t} d\mu_M(t) + (1 - \alpha_0^2 \|1\|_{L^1(\mu_M)}) \int t^{k-1} d\delta_0(t) = \alpha_0^2 \int t^{k-1} d\mu_M(t) = \alpha_0^2 \gamma_{k-1}(T_M) = \gamma_k(T).$$

Therefore, $T$ is subnormal, with Berger measure $\mu$. \hfill $\square$
Notation 1.6. The maximum possible value for $\alpha_0$ in Proposition 1.5, namely $(\|\frac{1}{t}\|_{L^1(\mu_{\text{ext}})})^{-1}$, will be denoted by $\alpha_{\text{ext}} \equiv \alpha_{\text{ext}}(\mu_M)$. Observe that shift $(\alpha_{\text{ext}}, \alpha_1, \alpha_2, \cdots)$ is subnormal, with Berger measure $d\mu(t) = \frac{a_0}{2} d\mu_M(t)$. For example, if $B_+$ denotes the Bergman shift on $\ell^2(\mathbb{Z}_+)$, then $B_+|_M$ is subnormal, with Berger measure $d\mu(t) := 2dt$ on $[0,1]$. Then $d\mu_{\text{ext}}(t) = dt$, so in this case the extremal measure $\mu_{\text{ext}}$ is the Berger measure of $B_+$.

More generally, given a (1-variable) subnormal weighted shift $W$ with weight sequence $\eta_1 \leq \eta_2 \leq \cdots$ and Berger measure $\nu$, we let

$$\eta_{\text{ext}} := \left\{ \begin{array}{ll} 0 & \text{if } \frac{1}{t} \notin L^1(\nu) \\ (\|\frac{1}{t}\|_{L^1(\nu)})^{-1} & \text{if } \frac{1}{t} \in L^1(\nu). \end{array} \right.$$

Observe that when the weight sequence $\eta$ is strictly increasing and $\frac{1}{t} \in L^1(\nu)$, we must necessarily have

$$\eta_{\text{ext}} < \eta_1, \quad (1.5)$$

by [Sta, Theorem 6]. On occasion, we will write shift $(\alpha_0, \alpha_1, \cdots)$ to denote the weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. We also denote by $U_+ := \text{shift}(1,1,\cdots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := \text{shift}(a,1,1,\cdots)$. Observe that the Berger measures of $U_+$ and $S_a$ are $\delta_1$ and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where $\delta_p$ denotes the point-mass probability measure with support the singleton $\{p\}$. Finally, we let $B_+$ denote the Bergman shift, whose Berger measure is Lebesgue measure on the interval $[0,1]$; the weights of $B_+$ are given by the formula $\alpha_n := \sqrt{\frac{2n+1}{n+1}} \quad (n \geq 0)$.

We conclude this section with a result that will be needed in Section 3.

Lemma 1.7. (cf. [CuFi3, Theorem 3.10]) For $0 < \alpha_0 < \alpha_1 < \alpha_2$, let $W_{(\alpha_0,\alpha_1,\alpha_2)}$ be the weighted shift described by (1.2), (1.3) and (1.4). Consider now $W_{\eta} := \text{shift}(\alpha_1, \alpha_2, \cdots)$, that is, $W_{\eta}$ is the restriction of $W_{(\alpha_0,\alpha_1,\alpha_2)}$ to $M$. Then $\eta_{\text{ext}} = \alpha_0$.

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2. The First Family of Counterexamples

Recall that a unilateral weighted shift $W_\alpha$ is subnormal if and only if there exists a probability measure $\xi \equiv \xi_\alpha$ supported in $[0,\|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int t^k d\xi(t) \quad (k \geq 1)$. For instance, when $\alpha_1 = \alpha_2 = \cdots = 1$ (i.e., $W_\alpha \equiv \text{shift}(\alpha_0,1,1,\cdots)$), we have $\xi_\alpha = (1-\alpha_0^2)\delta_0 + \alpha_0^2\delta_1$. The proof of the following lemma is straightforward.

Lemma 2.1. Given two 1-variable weight sequences $\alpha$ and $\beta$, the 2-variable weighted shift $(W_\alpha \otimes I, I \otimes W_\beta)$ is always subnormal, with Berger measure $\mu := \xi_\alpha \times \xi_\beta$.

Definition 2.2. Let $\mu$ and $\nu$ be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ on $X := \mathbb{R}_+$, if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on $\mathbb{R}_+$.

Definition 2.3. Let $\mu$ be a probability measure on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, and assume that $\frac{1}{t} \in L^1(\mu)$. The extremal measure $\mu_{\text{ext}}$ (which is also a probability measure) on $X \times Y$ is given by $d\mu_{\text{ext}}(s,t) := (1-\delta_0(t))\int_{[1]}^{1} \frac{1}{t\|1\|_{L^1(\mu)}} d\mu(s,t)$.
Definition 2.4. Given a measure $\mu$ on $X \times Y$, the marginal measure $\mu^X$ is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto $X$. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. Observe that if $\mu$ is a probability measure, then so is $\mu^X$.

Lemma 2.5. Let $\mu$ be the Berger measure of a 2-variable weighted shift $T$ and let $\nu$ be the Berger measure of shift $(\alpha_0, \alpha_{10}, \cdots)$. Then $\nu = \mu^X$. As a consequence, $\int f(s) \, d\mu(s,t) = \int f(s) \, d\mu^X(s)$ for all $f \in C(X)$.

Proof. Observe that $\int s^i \, d\nu(s) = \gamma_0 = \int s^i \, d\mu(s,t)$ for all $i \geq 0$. It follows that $\int f(s) \, d\nu(s) = \int f(s) \, d\mu(s,t)$ for all $f \in C(X)$. Then, for any Borel set $E \subseteq X$, we have

$$\nu(E) = \int \chi_E \, d\nu = \int \chi_{E \times Y} \, d\mu = \mu(E \times Y) = \mu^X(E),$$

as desired. The second assertion follows immediately from what we have established.

Corollary 2.6. Let $\mu$ be the Berger measure of a 2-variable weighted shift $T$. For $j \geq 1$, let $d\mu_j(s,t) := \frac{1}{\gamma_0} \nu^j d\mu_j(s,t)$. Then the Berger measure of shift $(\alpha_{0j}, \alpha_{1j}, \cdots)$ is $\nu_j \equiv \mu_j^X$.

Example 2.7. Let $\mu := \xi \times \eta$ be a probability product measure on $X \times Y$. Then $\mu^X = \xi$.

Lemma 2.8. Let $\mu$ and $\omega$ be two measures on $X \times Y$, and assume that $\mu \leq \omega$. Then $\mu^X \leq \omega^X$.

Proof. Straightforward from Definition 2.4.

Proposition 2.9. (Subnormal backward extension of a 2-variable weighted shift) Consider the following 2-variable weighted shift (see Figure 2), and let $M$ be the subspace associated to indices $k$ with $k_2 \geq 1$. Assume that $T_M := T |_M$ is subnormal with associated measure $\mu_M$ and that $W_0 := \text{shift}(\alpha_0, \alpha_{10}, \cdots)$ is subnormal with associated measure $\nu$. Then $T$ is subnormal if and only if

(i) $\frac{1}{t} \in L^1(\mu_M)$;
(ii) $\beta_0^2 \leq (\frac{1}{t})^{-1} L^1(\mu_M)$;
(iii) $\beta_0^2 \frac{1}{t} L^1(\mu_M)(\mu_M)_{ext} \leq \nu$.

Moreover, if $\beta_0^2 \frac{1}{t} L^1(\mu_M) = 1$, then $(\mu_M)_{ext} = \nu$. In the case when $T$ is subnormal, the Berger measure $\mu$ of $T$ is given by

$$d\mu(s,t) = \beta_0^2 \frac{1}{t} L^1(\mu_M) d(\mu_M)_{ext}(s,t) + (d\nu(s) - \beta_0^2 \frac{1}{t} L^1(\mu_M)) d(\mu_M)_{ext}(s,t) d\delta_0(t).$$

Proof. ($\Rightarrow$) First, observe that the moments of $T$ and $T_M$ are related as follows:

$$\gamma_k(\alpha_j)(T) = \beta_0^2 \gamma_k(T_M) \ (\text{all } k \in \mathbb{Z}^2_+),$$

so under the assumption that $T$ is subnormal we must have

$$\int s^i t^j (d\mu)(s,t) = \int s^i t^j d\mu(s,t) = \gamma_{i,j+1}(T) = \beta_0^2 \gamma_{i,j} \int s^i t^j d\mu_M(s,t).$$

Thus $t d\mu(s,t) = \beta_0^2 d\mu_M(s,t)$ and $\mu_M(E \times \{0\}) = 0$ for all $E \subseteq X$. It follows at once that

$$\int \frac{1}{t} d\mu_M(s,t) = \int \left[ \int (s \geq 0) \frac{1}{t} d\mu_M(s,t) = \frac{1}{\beta_0^2} \int (s \geq 0) \frac{1}{t} t d\mu(s,t) \right] \leq \frac{1}{\beta_0^2},$$
Figure 2. Weight diagram of the 2-variable weighted shift in Proposition 2.9

which establishes parts (i) and (ii). As for part (iii), let $E \subseteq X$ and $F \subseteq Y$ be two arbitrary Borel sets. Then

$$
\begin{align*}
\beta \left\| \frac{1}{t} \right\|_{L^1(\mu_\mathcal{M})} (\mu_\mathcal{M})_{ext}(E \times F) &= \beta_0 \left\| \frac{1}{t} \right\|_{L^1(\mu_\mathcal{M})} \int E \times F (1 - \delta_0(t)) \frac{1}{t} \left\| \frac{1}{t} \right\|_{L^1(\mu)} d\mu_\mathcal{M}(s, t) \\
&= \int E \times (F \setminus \{0\}) \frac{1}{t} \beta_0^2 d\mu_\mathcal{M}(s, t) = \mu_\mathcal{M}(E \times (F \setminus \{0\})) \\
&\leq \mu(E \times F),
\end{align*}
$$

and by Lemmas 2.8 and 2.5, $\beta_0 \left\| \frac{1}{t} \right\|_{L^1(\mu_\mathcal{M})} (\mu_\mathcal{M})_{ext}^X \leq \mu^X = \nu$. Finally, observe that when $\beta_0 \left\| \frac{1}{t} \right\|_{L^1(\mu_\mathcal{M})} = 1$, the inequality in (2.2) becomes an equality, and therefore $(\mu_\mathcal{M})_{ext}^X = \nu$. 

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\((\Longleftrightarrow)\) Assume that (i), (ii) and (iii) hold, and let\[
\mu := \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_M)} (\mu_M)_{\text{ext}} + \left| \nu - \beta_{00}^2 \right\|_{L^1(\mu_M)} (\mu_M)^X \times \delta_0.
\]
Of course, if \(\beta_{00} \parallel \frac{1}{t} \parallel_{L^1(\mu_M)} = 1\), then \(\mu = (\mu_M)_{\text{ext}}\), since the total mass of the second summand is zero. We now compute the moments of \(\mu\) and verify that they agree with the moments of \(T\). If \(j > 0\), then
\[
\begin{align*}
\int \int s^i t^j \ d\mu(s, t) &= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_M)} \int \int s^i t^j \ d(\mu_M)_{\text{ext}}(s, t) \\
&= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_M)} \int \int s^i t^j (1 - \delta_0(t)) \frac{1}{t} \left\| 1 \right\|_{L^1(\mu)} d\mu_M(s, t) \\
&= \beta_{00}^2 \int \int s^i t^{j-1} d\mu_M(s, t) = \beta_{00}^2 \gamma(i, j - 1) (T_M) = \gamma(i, j)(T),
\end{align*}
\]
as desired. When \(j = 0\), we have
\[
\begin{align*}
\int \int s^i \ d\mu(s, t) &= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_M)} \int \int s^i \ d(\mu_M)_{\text{ext}}(s, t) \\
&\quad + \int s^i \ d\nu(\nu - \beta_{00}^2) \left\| \frac{1}{t} \right\|_{L^1(\mu_M)} (\mu_M)^X(s) \\
&= \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_M)} \int s^i \ d(\mu_M)_{\text{ext}}(s) \\
&\quad + \int s^i \ d\nu(s) - \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_M)} \int s^i \ d(\mu_M)^X_{\text{ext}}(s) \\
&\quad \text{(using Lemma 2.5 for the first term)} \\
&= \int s^i \ d\nu(s) = \gamma(i, 0)(T),
\end{align*}
\]
as desired. It follows that \(T\) is subnormal, with Berger measure \(\mu\). \(\square\)

We are now ready to exhibit our first family of counterexamples to Conjecture 1.4. Consider the 2-variable weighted shift given by Figure 3, where \(\max\{a, x, y, \frac{ay}{x}\} < 1\).

**Proposition 2.10.** The 2-variable weighted shift \(T\) given by Figure 3 is hyponormal if and only if \(y \leq x \sqrt{\frac{1 - x^2}{x^2 - 2a^2x^2 + a^4}}\).

**Proof.** By the Six-point Test (Theorem 1.3), to show the joint hyponormality of \(T\) it is enough to check that
\[
H := \begin{pmatrix}
1 - x^2 & \frac{a^2y}{x} - yx \\
\frac{a^2y}{x} - yx & 1 - y^2
\end{pmatrix} \geq 0.
\]
Since \(x < 1\), the positivity of \(H\) is equivalent to \(\det H \geq 0\), i.e.,
\[
(1 - x^2)(1 - y^2) \geq \left(\frac{a^2y}{x} - yx\right)^2,
\]
which in turn is equivalent to \(y \leq x \sqrt{\frac{1 - x^2}{x^2 - 2a^2x^2 + a^4}}\) (observe that \(x^2 - 2a^2x^2 + a^4 = x^2(1 - x^2) + (x^2 - a^2)^2 > 0\)). \(\square\)
Figure 3. Weight diagram of the 2-variable weighted shift in Propositions 2.10 and 2.11

Proposition 2.11. The 2-variable weighted shift $T$ given by Figure 3 is subnormal if and only if $y \leq \sqrt{\frac{1-x^2}{1-a^2}}$.

Proof. From Figure 3, it is obvious that $T_M \cong (S_a \otimes I, I \otimes U_+)$ (recall that $S_a := shift(a,1,1,\cdots)$ and $U_+$ is the (unweighted) unilateral shift). By Lemma 2.1, $T_M$ is subnormal, with Berger measure
µₘ := [(1 − a²)δ₀ + a²δ₁] × δ₁. By Proposition 2.9,

\[ \mathbf{T} \text{ is subnormal } \iff \beta_{\delta_0}^2 \left\| \frac{1}{t} \right\|_{L^1(µₘ)}^X (µₘ)_{ext} \leq ν \]

\[ \iff y^2[(1 − a^2)δ₀ + a^2δ₁] \leq (1 − x^2)δ₀ + x^2δ₁ \]

\[ \iff y^2(1 − a^2) \leq 1 − x^2 \text{ and } ay \leq x \]

\[ \iff y \leq \min\left\{ \frac{x}{a}, \sqrt{\frac{1 − x^2}{1 − a^2}} \right\} \]

\[ \iff y \leq \sqrt{\frac{1 − x^2}{1 − a^2}} \text{ (since } \max\{a, x, y, ay\} < 1). \]

□

We summarize the results in Propositions 2.10 and 2.11 as follows.

**Theorem 2.12.** The 2-variable weighted shift \( \mathbf{T} \) given by Figure 3 is hyponormal and not subnormal if and only if

\[ \sqrt{\frac{1 − x^2}{1 − a^2}} < y \leq x\sqrt{\frac{1 − x^2}{x^2 + a^2 − 2a^2x^2}} \text{ (see Figure 4).} \]

![Figure 4](image-url)

**Figure 4.** Regions of hyponormality and subnormality for the 2-variable weighted shift in Theorem 2.12

**Remark 2.13.** As exemplified in Figure 4, observe that for \( x > a, \sqrt{\frac{1 − x^2}{1 − a^2}} < x\sqrt{\frac{1 − x^2}{x^2 + a^2 − 2a^2x^2}} < \frac{x}{a} \); for, if \( a < x \) we have

\[ a^4 < a^2x^2 \Rightarrow x^2 + a^4 − 2a^2x^2 < (1 − a^2)x^2 \]

\[ \Rightarrow \frac{1 − x^2}{1 − a^2} < \frac{x^2(1 − x^2)}{x^2 + a^4 − 2a^2x^2}. \]
and
\[
a^2(1-a^2) < x^2(1-a^2) \Rightarrow a^2 + a^2x^2 < x^2 + a^4
\]
\[
\Rightarrow a^2(1-x^2) < x^2 + a^4 - 2a^2x^2
\]
\[
\Rightarrow \frac{x^2(1-x^2)}{x^2 + a^4 - 2a^2x^2} < \frac{x^2}{a^2},
\]
as desired.

### 3. The Second Family of Counterexamples

**Construction of the family.** Let \(0 < a, b < 1\) and let \(\{\xi_k\}_{k=0}^{\infty}\) and \(\{\eta_k\}_{k=0}^{\infty}\) be two strictly increasing weight sequences. Consider the 2-variable weighted shift \(T \equiv (T_1, T_2)\) on \(\ell^2(\mathbb{Z}^2_+)\) given by the double-indexed weight sequences

\[
\alpha(k) := \begin{cases} 
\xi_k & \text{if } k_1 \geq 1 \text{ or } k_2 \geq 1 \\
\eta_k & \text{if } k_1 = 0 \text{ and } k_2 = 0 
\end{cases}
\]

and

\[
\beta(k) := \begin{cases} 
\eta_k & \text{if } k_1 \geq 1 \text{ or } k_2 \geq 1 \\
b & \text{if } k_1 = 0 \text{ and } k_2 = 0,
\end{cases}
\]

where \(W_\xi\) and \(W_\eta\) are two single-variable subnormal weighted shifts with Berger measures \(\nu\) and \(\omega\), resp., and

\[
a\eta_0 = b\xi_0
\]

(to guarantee the commutativity of \(T_1\) and \(T_2\), cf. (1.1)). \(T\) can be represented by the following weight diagram (Figure 5). It is then clear that \(T_1\) and \(T_2\) are subnormal provided \(a \leq \xi_{\text{ext}}(\nu,\omega)\) and \(b \leq \eta_{\text{ext}}(\omega,\nu)\), where, as usual, \(\mathcal{M} := \vee\{e_1, e_2, \cdots\}\); in particular, \(a < \xi_1\) and \(b < \eta_1\).

**Proposition 3.1.** The 2-variable weighted shift \(T\) defined by (3.1) and (3.2) is subnormal only if \(a \leq s\), where \(s := \sqrt{\frac{\xi_0\xi_2\eta_0^2}{\xi_0\eta_0^4 + \xi_0^2\eta_0^4 - \xi_0^2\eta_0^2}}\).

**Proof.** Suppose that \(T\) above is subnormal, and let \(\mu\) be the associated Berger measure. Then the following partial moment matrix \(M\), corresponding to the moments of \(\mu\) associated with the monomials \(1, s, t\) and \(ts\), must be positive semi-definite:

\[
M := \begin{pmatrix}
1 & a^2 & b^2 & a^2\xi^2_0 \\
a^2 & a^2\xi^2_0 & a^2\eta^2_0 & a^2\eta^2_0\xi^2_1 \\
b^2 & a^2\eta^2_0 & a^2\eta^2_0\xi^2_1 & a^2\eta^2_0\eta^2_1 \\
a^2\eta^2_0 & a^2\eta^2_0\xi^2_1 & a^2\eta^2_0\eta^2_1 & a^2\eta^2_0\xi^2_1\eta^2_1
\end{pmatrix}.
\]

Now, using *Mathematica* we obtain

\[
\det(M) \geq 0 \quad \Leftrightarrow \quad a^0\eta^0_0(\xi^2_0 - \xi^2_2)(\eta^2_0 - \eta^2_2)(a^2\xi^2_0\eta^2_0 - a^2\xi^2_1\eta^2_0 - a^2\xi^2_0\eta^2_1 + \xi^2_0\xi^2_1\eta^2_1) \geq 0
\]
\[
\Leftrightarrow \quad a^2\xi^2_0\eta^2_0 - a^2\xi^2_1\eta^2_0 - a^2\xi^2_0\eta^2_1 + \xi^2_0\xi^2_1\eta^2_1 \geq 0
\]
\[
\Leftrightarrow \quad a \leq \sqrt{\frac{\xi^2_0\xi^2_1\eta^2_1}{\xi^2_1\eta^2_0 + \xi^2_0\eta^2_1 - \xi^2_0\eta^2_0}} = s.
\]

\[\square\]
**Figure 5**

**Proposition 3.2.** The 2-variable weighted shift $T$ defined by (3.1) and (3.2) is hyponormal if and only if $a \leq h$, where $h := \xi_0 \sqrt{\frac{\xi_1^2 - \xi_0^2}{\xi_0^2 + \xi_1^2 - 2\xi_0^2\eta_0^2}}$.

**Proof.** From the definition of $T$ and the Six-point Test (Theorem 1.3), it is clear that all we need is for the following matrix to be positive semi-definite:

$$L := \begin{pmatrix} \xi_1^2 - a^2 & \xi_0\eta_0 - ab \\ \xi_0\eta_0 - ab & \eta_1^2 - b^2 \end{pmatrix}.$$
Observe that
\[
\det L \geq 0 \iff \xi_1^2 \eta_0^2 - \xi_1^2 \eta_0^2 - \xi_0^2 \eta_0^2 - a^2 \eta_0^2 + 2ab \xi_0 \eta_0 \geq 0
\]
\[
\iff \xi_1^2 \eta_0^2 - \xi_1^2 \eta_0^2 - \xi_0^2 \eta_0^2 - a^2 \eta_0^2 + 2a^2 \eta_0^2 \geq 0 \quad \text{(using } b \xi_0 = a \eta_0; \text{ cf. (3.3)})
\]
\[
\iff a^2 \leq \frac{\xi_0^2 (\xi_1^2 \eta_0^2 - \xi_2^2 \eta_0^2)}{\xi_0^2 \eta_0^2 + \xi_1^2 \eta_0^2 - 2 \xi_2^2 \eta_0^2} = h^2
\]
(observe that \(\xi_0^2 \eta_0^2 + \xi_1^2 \eta_0^2 - 2 \xi_2^2 \eta_0^2 = \xi_0^2 (\eta_0^2 - \eta_0^2) + (\xi_1^2 - \xi_2^2) \eta_0^2 > 0\), because the weight sequences are strictly increasing by hypothesis). Thus, \(a \leq h\) is clearly a necessary condition for the hyponormality of \(T\). Now, a straightforward calculation shows that \(h < \xi_1\); for,
\[
\xi_1^2 - h^2 = \frac{\eta_0^2 (\xi_1^2 - \xi_2^2)^2}{\xi_0^2 \eta_0^2 + \xi_1^2 \eta_0^2 - 2 \xi_2^2 \eta_0^2} > 0.
\]
\[(3.4)\]
It follows that \(a \leq h\) implies \(a < \xi_1\), and therefore \(L \geq 0\) by the Nested Determinant Test [Atk]. Thus, the condition \(a \leq h\) is also sufficient for the hyponormality of \(T\), and the proof is complete.

It follows from Propositions 3.1 and 3.2 that to ascertain the existence of a nonsubnormal, hyponormal 2-variable weighted shift \(T\) (with \(T_1\) and \(T_2\) subnormal), it suffices to show that for appropriate choices of \(\xi_0, \xi_1, \eta_0\) and \(\eta_1\), it is possible to obtain \(s < h\), while keeping \(a \leq \xi_{\text{ext}}(\nu, M)\) and \(b \equiv \frac{\eta_0}{\xi_0} \leq \eta_{\text{ext}}(\omega, M)\). Now,
\[
h^2 - s^2 = \frac{\xi_0^2 \eta_0^2 (\xi_1^2 - \xi_2^2)(\eta_0^2 - \eta_0^2)}{(\xi_0^2 \eta_0^2 + \xi_1^2 \eta_0^2 - 2 \xi_2^2 \eta_0^2)(\xi_1^2 \eta_0^2 + \xi_2^2 \eta_0^2 - \xi_0^2 \eta_0^2)} > 0.
\]
Therefore, it suffices to prove the existence of strictly increasing weight sequences \(\{\xi_i\}\) and \(\{\eta_j\}\) such that

\(\text{(i)} \ a \leq h\) (hyponormality of \(T\))

\(\text{(ii)} \ a > s\) (nonsubnormality of \(T\))

\(\text{(iii)} \ a \leq \xi_{\text{ext}}(\nu, M)\) (subnormality of \(T_1\))

\(\text{(iv)} \ a \leq s_2 := \frac{\xi_0}{\eta_0} \eta_{\text{ext}}(\omega, M)\) (subnormality of \(T_2\)).

We now seek to determine the relative positions of \(h, s, s_2, \xi_0, \xi_{\text{ext}}(\nu, M)\) and \(\xi_1\) in the positive real axis.

\textbf{Claim 1:} \(\xi_0 \leq \xi_{\text{ext}}(\nu, M)\). This is straightforward from the fact that \(shift(\xi_0, \xi_1, \cdots)\) is subnormal.

\textbf{Claim 2:} \(\xi_0 < s\). For,
\[
s^2 - \xi_0^2 = \frac{\xi_0^2 \xi_1^2 \eta_1^2}{\xi_0^2 \eta_0^2 + \xi_1^2 \eta_1^2 - \xi_2^2 \eta_0^2} - \xi_0^2 = \frac{\xi_0^2 (\xi_1^2 - \xi_2^2)(\eta_1^2 - \eta_0^2)}{\xi_0^2 \eta_0^2 + \xi_1^2 \eta_1^2 - \xi_2^2 \eta_0^2} > 0.
\]

\textbf{Claim 3:} \(h < \xi_1\). This was established in the proof of Proposition 3.2, cf. (3.4).

\textbf{Claim 4:} \(h \leq s_2\) whenever \(\eta_0 \leq u := \frac{\xi_1^2 (\eta_1^2 - \eta_0^2) + 2 \xi_0^2 \eta_0^2 - \sqrt{\eta_1^2 - \eta_0^2})(\xi_1^2 (\eta_1^2 - \eta_0^2) + 4 \xi_0^2 \eta_0^2 (\xi_1^2 - \xi_2^2))}{2 \xi_0^2}\). Since
\[
s^2 - h^2 = \frac{\xi_0^2 \{\xi_0^2 \eta_0^4 - [\xi_1^2 (\eta_1^2 - \eta_0^2) + 2 \xi_0^2 \eta_0^2] \eta_0^2 + \xi_0^2 \eta_0^2 \eta_1^2\}}{\eta_0^2 (\xi_0^2 \eta_1^2 + \xi_1^2 \eta_0^2 - 2 \xi_2^2 \eta_0^2)}
\]

it follows that \(h \leq s_2\) if and only if the quadratic form
\[
q(t) \equiv At^2 + Bt + C
\]
\[
= \xi_0^2 t^2 - [\xi_1^2 (\eta_1^2 - \eta_0^2) + 2 \xi_0^2 \eta_0^2] t + \xi_0^2 \eta_0^2 \eta_1^2
\]
is nonnegative. Since \( A \) and \( C \) are positive, and \( B \) is negative, we need to study the discriminant, \( \Delta := B^2 - 4AC \). Now,
\[
\Delta = (\xi_1^2(\eta_1^2 - \eta_e^2) + 2\xi_0^2\eta_e^2 - 4\xi_0\eta_e^2\eta_1^2)
\]
\[
= (\eta_1^2 - \eta_e^2)[\xi_1^2\eta_1^2 - \eta_e^2(2\xi_0^2 - \xi_1^2)^2],
\]
so \( \Delta \geq 0 \iff \xi_1^2\eta_1^2 - \eta_e^2(2\xi_0^2 - \xi_1^2)^2 \geq 0 \). Since \( \xi_1^2\eta_1^2 - \eta_e^2(2\xi_0^2 - \xi_1^2)^2 = \xi_1^2(\eta_1^2 - \eta_e^2) + 4\xi_0^2\eta_e^2(\xi_1^2 - \xi_1^2) \), we see that \( \Delta \) is always positive. We conclude that \( q \geq 0 \) on the interval \([0, t_1]\), where \( t_1 := \frac{-B-\sqrt{\Delta}}{2A} \) is the leftmost zero of \( q \). Finally, a straightforward calculation shows that \( t_1 = u \).

We now summarize what we have so far. For \( \eta_0 \leq u \) we have
\[
\begin{cases}
\xi_0 < s < h \leq s_2 \\
h < \xi_1 \\
\xi_{\text{ext}}(\nu_M) < \xi_1 \text{ (by (1.5))}.
\end{cases}
\]
Thus, if we can ensure that \( h \leq \xi_{\text{ext}}(\nu_M) \), the construction of the example will be complete by taking \( a \) such that \( s < a \leq h \). Now, since \( h \leq s_2 \), an easy way to accomplish this is to build shift \((\xi_0, \xi_1, \cdots)\) in such a way that \( \xi_{\text{ext}}(\nu_M) = s_2 \). To do this, we appeal to Lemma 1.7, that is, we first build a 2-step recursively generated weighted shift whose first three weights are \( s_2, \xi_1 \) and \( \xi_2 \), and we then consider the shift \( W_{\phi}(\xi_1, \xi_2, \xi_3) \), where \( \xi_3 \) is given by \( \xi_3 := \frac{\phi_0}{\xi_2} + \phi_1 \) obtained from the equation \( \gamma_4 = \phi_0\gamma_2 + \phi_1\gamma_3 \). Observe that the extremal value of \( W(\xi_1, \xi_2, \xi_3) \) is \( s_2 \), and that \( \xi_0 < s_2 \), so the subnormality of \( W_{\phi}(\xi_1, \xi_2, \xi_3) \) is guaranteed. This completes the construction of the example.

**Theorem 3.3.** Let \( T \equiv (T_1, T_2) \) be the 2-variable weighted shift defined by (3.1) and (3.2), let
\[
\begin{align*}
 h &:= \xi_0\sqrt{\frac{\xi_1^2\eta_1^2 - \xi_1^2\eta_e^2}{\xi_1^2\eta_1^2 + \xi_1^4\eta_0^2 - 2\xi_0^2\eta_0}}, \\
 s &:= \sqrt{\frac{\xi_0^2\xi_1^2\eta_1^2}{\xi_1^4\eta_0^2 + \xi_1^4\eta_1^2 - 2\xi_0^2\eta_0}}, \\
 s_2 &:= \frac{\xi_0^2\eta_e^2}{\eta_e^2}, \text{ where } \eta_e \equiv \eta_{\text{ext}}(\omega_M), \\
 u &:= \frac{\xi_0^2\eta_1^2}{\xi_1^2(\eta_1^2 - \eta_e^2) + \xi_0^2\eta_0^2}, \text{ and} \\
 v &:= \frac{\xi_1^2(\eta_1^2 - \eta_e^2)^2 + 2\xi_0^2\eta_e^2 - \sqrt{(\eta_1^2 - \eta_e^2)^2\xi_1^4(\eta_1^2 - \eta_e^2) + 4\xi_0^2\eta_e^2(\xi_1^2 - \xi_1^2)}}{2\xi_0^2}.
\end{align*}
\]
Assume further that, as above, \( s_2 = \xi_{\text{ext}}(\nu_M) \) and \( \eta_0 \leq \min\{u, v\} \). Finally, choose \( a \) such that \( s < a \leq h \). Then
(i) \( T_1T_2 = T_2T_1 \);
(ii) \( T_1 \) is subnormal;
(iii) \( T_2 \) is subnormal;
(iv) \( T \) is hyponormal; and
(v) \( T \) is not subnormal.
Example 3.4. For a concrete numerical example, let $d\omega_\mathcal{M}(t) := 2dt$ on $[{1\over 2}; 1]$, so that $\|{1\over t}\|_{L^1(\omega_\mathcal{M})} = 2\ln 2$. It follows that $\eta_e \equiv \eta_\text{ext}(\omega_\mathcal{M}) = {1\over \sqrt{2\ln 2}}$ and $\eta_1 = \sqrt{3\over 2}$. Now take $\xi_0 := \frac{1}{2}$ and $\xi_1 := 1$. Then $u = {1\over 4(2\ln 2-1)} \approx 0.647$ and $v = {1\over 6}(6\ln 2-2-\sqrt{2}(3\ln 2-2)(3\ln 2-1)) \approx 0.523$, so we can take $\eta_0 := \frac{1}{2}$.

With this choice of $\eta_0$ we obtain $s = \frac{\sqrt{5}}{2} \approx 0.707$, $h = \frac{1}{2}\sqrt{\frac{1}{3}} \approx 0.742$ and $s_2 = \eta_e = \frac{1}{\sqrt{2\ln 2}} \approx 0.849$. We can then take $a \in (s, h)$, for instance $a := 0.72$. To build the weighted shift $W_\xi$ we start with $s_2$, $\xi_1$ and $\xi_2 := \frac{\sqrt{2}}{\sqrt{2}-1}$ to obtain $\varphi_0 = \frac{1}{1-2\ln 2}$ and $\varphi_1 = \frac{1}{1-2\ln 2}$. This gives $\xi_3 = \frac{1}{2}\sqrt{\frac{16\ln 2-5}{2\ln 2-1}} \approx 1.985$. The 2-atomic measure $\nu_\mathcal{M}$ for $W(\xi_1, \xi_2, \xi_3)$ has atoms $t_0 \approx 0.659$ and $t_1 \approx 3.93$ and densities $\rho_0 \approx 0.981$ and $\rho_1 \approx 0.019$. With these values we can compute $\|{1\over t}\|_{L^1(\nu_\mathcal{M})} \approx 1.494$; observe that $\xi_0 = \frac{1}{2} \leq \xi_3(\nu_\mathcal{M}) = \frac{1}{\sqrt{\|{1\over t}\|_{L^1(\nu_\mathcal{M})}}} \approx 0.818$. By Proposition 1.5, the measure associated to $shift(\xi_0, \xi_1, \xi_2, \cdots)$ is $d\nu(t) = \frac{1}{4\pi}(\rho_0t_0(t) + \rho_1t_1(t)) + (1 - \frac{1}{4}\|{1\over t}\|_{L^1(\nu_\mathcal{M})})d\delta_0(t)$.

4. The Third Family of Counterexamples

Construction of the family. Let us consider the following 2-variable weighted shift (see Figure 6), where

\[
\begin{align*}
(i) & \quad 0 < \xi_1 < \xi_2 < \cdots < \xi_n \not\to 1; \\
(ii) & \quad W_\xi := shift(\xi_1, \xi_2, \cdots) \text{ is subnormal with Berger measure } \nu; \\
(iii) & \quad \frac{1}{s} \in L^1(\nu) \text{ (this implies that } \frac{1}{s} \in L^1(\nu), \text{ by Jensen’s inequality);} \\
(iv) & \quad \xi_e \equiv \xi_\text{ext} := \left(\int \frac{1}{s^2}d\nu(s)\right)^{-1/2}; \\
(v) & \quad a \leq \frac{1}{\xi_e}\left(\int \frac{1}{s^2}d\nu(s)\right)^{-1/2}; \\
(vi) & \quad b \leq \xi_e^2 \text{ (this implies the condition } b < \xi_e; \text{ and} \\
(vii) & \quad a^2 \leq b^2 + \xi_e^2. \\
\end{align*}
\]

(Recall that $\xi_e$ is the maximum possible value for $\xi_0$ in Proposition 1.5.)

Observe that $T_1 \cong T_2$ and that $T_1T_2 = T_2T_1$. We claim that $T_1$ (and therefore $T_2$) is subnormal. For, the choice of $\xi_e$ immediately implies that $shift(\xi_e, \xi_1, \xi_2, \cdots)$ is subnormal, with Berger measure $d\nu_e(s) := \frac{\xi_e^2}{s}d\nu(s)$ (cf. Proposition 1.5). Another application of Proposition 1.5 shows that $shift(a, \xi_e, \xi_1, \cdots)$ is subnormal if and only if $\frac{1}{s} \in L^1(\nu_e)$ (i.e., $\frac{1}{s^2} \in L^1(\nu)$, which is true by (4.1)(iii)) and $a^2\xi_e^2 \int \frac{1}{s^2}d\nu(s) \leq 1$, which holds by (4.1)(v)). This implies that the restriction of $T_1$ to $\{e_{(i,0)} : i \geq 0\}$ is subnormal. Moreover, the subnormality of $T_1$ when restricted to $\{e_{(i,j)} : i \geq 0\} (j > 0)$ requires that $b \leq \xi_e$, which holds by (4.1)(vi).

For a concrete numerical example, consider the probability measure $d\nu(s) := 3s^2ds$ on the interval $[0, 1]$. The measure $\nu$ corresponds to a subnormal weighted shift with weights $\xi_1 = \sqrt{3\over 4}$, $\xi_2 = \sqrt{4\over 5}$, $\xi_3 = \sqrt{5\over 6}$, \ldots; indeed, in this case $W_\xi$ is the restriction of the Bergman shift $B_+$ to the invariant subspace $\mathcal{M}_2$ obtained by removing the first two basis vectors in the canonical orthonormal basis of
Clearly \( \frac{1}{\sqrt{z}} \in L^1(\nu) \), and \( \int \frac{1}{\sqrt{z}} d\nu(s) = 3 \); moreover, \( \int \frac{1}{s} d\nu(s) = \frac{3}{2} \), so in this case \( \xi_e = \sqrt{\frac{2}{3}} \).

Choosing \( a = \sqrt{\frac{1}{2}} \) and \( b = \sqrt{\frac{2}{3}} \) we see that all conditions in (4.1) are satisfied (cf. Corollary 4.4).

**Proposition 4.1.** The 2-variable weighted shift \( T \) given by Figure 6 is hyponormal.

**Proof.** Since the restriction of \( T \) to \( \sqrt{\{e_{ij} : i, j \geq 1\}} \) is clearly subnormal (being unitarily equivalent to \( W_{\xi} \otimes I, I \otimes W_{\xi} \)), and since the weight diagram of \( T \) is symmetric with respect to the diagonal \( j = i \), it suffices to apply the Six-point Test (Theorem 1.3) to \( k = (i, 0) \), with \( i \geq 0 \).

**Case 1:** \( k = (0, 0) \). Here we have

\[
\begin{pmatrix}
    \xi_e^2 - a^2 & b^2 - a^2 \\
    b^2 - a^2 & \xi_e^2 - a^2
\end{pmatrix} \geq 0 \iff (\xi_e^2 - a^2)^2 \geq (b^2 - a^2)^2 \\
\iff \xi_e^2 - a^2 \geq |b^2 - a^2|.
\]
When $b \leq a$, the last condition is equivalent to $2a^2 \leq b^2 + \xi_e^2$, which holds by (4.1)(vii). When $b > a$, the condition is equivalent to $\xi_e \geq b$, which is guaranteed by (4.1)(vi).

**Case 2:** $k = (1,0)$. Here

\[
\begin{pmatrix}
\xi_1^2 - \xi_e^2 & \xi_1^2 - b \xi_e \\
\xi_1^2 - b \xi_e & \xi_1^2 - b^2
\end{pmatrix} \geq 0 \iff (\xi_1^2 - \xi_e^2)(\xi_1^2 - b^2) \geq (\xi_1^2 - b \xi_e)^2
\]

\[
\iff \xi_1^2 - b^2 \geq (\xi_1^2 - \xi_e^2) \frac{b^2}{\xi_e^2} \iff b \leq \xi_e,
\]

which again is guaranteed by (4.1)(vi).

**Case 3:** $k = (n + 1, 0)$ ($n \geq 0$). Here

\[
\begin{pmatrix}
\xi_{n+1}^2 - \xi_n^2 & \xi_{n+1}^2 - b \xi_n \\
\xi_{n+1}^2 - b \xi_n & \xi_{n+1}^2 - b^2
\end{pmatrix} \geq 0
\]

\[
\iff (\xi_{n+1}^2 - \xi_n^2)(\xi_{n+1}^2 - b^2) \geq (\xi_{n+1}^2 - b \xi_n)^2
\]

\[
\iff \frac{\xi_{n+1}^2 - \xi_n^2}{\xi_e^2} \frac{\xi_{n+1}^2 - b^2}{\xi_e^2} \geq 0
\]

\[
\iff b \leq \frac{\xi_{n+1}^2}{\xi_n} (\text{all } n \geq 1).
\]

Since the sequence $\{\xi_n\}$ increases to 1, the last inequality in (4.2) is equivalent to $b \leq \xi_1 \xi_e$, which holds by (4.1)(vi).

The proof is now complete.

**Proposition 4.2.** The 2-variable weighted shift $T$ given by Figure 6 is not subnormal if $p < 0$, where $p := \xi_e^2 \xi_1^4 + 4a^2 b^2 \xi_e^2 - b^2 \xi_1^4 - a^2 b^2 \xi_e^2 - a^2 b^4 - 2a^2 \xi_1^4$.

**Proof.** Assume that $T$ is subnormal, and consider the moment matrix associated to the monomials $1$, $x$, $y$ and $xy$ (cf. [CuFi4], [CuFi5]), that is,

\[
M := \begin{pmatrix}
1 & a^2 & a^2 b^2 & a^2 \xi_1^2 \\
a^2 & a^2 \xi_e^2 & a^2 b^2 \xi_1^2 & a^2 b^2 \xi_e^2 \\
a^2 b^2 & a^2 b^2 \xi_1^2 & a^2 b^2 \xi_e^2 & a^2 b^2 \xi_1^2 \\
a^2 b^2 \xi_1^2 & a^2 b^2 \xi_e^2 & a^2 b^2 \xi_1^2 & a^2 b^2 \xi_1^2 \\
\end{pmatrix}.
\]

In the presence of a representing measure, it is well known that $M$ must be positive semi-definite, so in particular $\det M \geq 0$. Now, a straightforward calculation shows that

\[
\det M = a^6 b^2 (\xi_e^2 - b^2) (\xi_e^2 \xi_1^4 - \xi_e^2 a^2 b^2 - 2 a^2 \xi_1^4 - b^2 \xi_1^4 + 4 a^2 b^2 \xi_1^4 - b^4 a^2) = a^6 b^2 (\xi_e^2 - b^2)^2.
\]

It follows that $p \geq 0$. Therefore, $T$ is not subnormal whenever $p < 0$, as desired.

**Theorem 4.3.** Let $a > 0$ be such that $\sqrt{\frac{\xi_e^2}{a^2}} < a \leq \sqrt{\frac{\xi_e^2 + \xi_1^4}{2}}$ and $a \leq \frac{1}{\nu}(\int \frac{1}{\nu} d\nu(s))^{-1/2}$, and define $b := \sqrt{\frac{2a^2 - \xi_e^2}{\xi_e^2}}$. Then the 2-variable weighted shift $T \equiv (T_1, T_2)$ satisfies (4.1)(i)-(vii), is hyponormal, and is not subnormal.

**Proof.** Observe that the condition $\sqrt{\frac{\xi_e^2}{a^2}} < a$ guarantees that $2a^2 > \xi_e^2$ (so $b$ is well defined), and that the condition $a \leq \sqrt{\frac{\xi_e^2 + \xi_1^4}{2}}$ is equivalent to $2a^2 - \xi_e^2 \leq \xi_1^4$ (so $b$ satisfies (4.1)(vi)). Moreover,
$a^2 = \frac{b^2 + \xi_2^2}{2}$ trivially, so (4.1)(vii) also holds. It follows that $T$ is hyponormal, by Proposition 4.1.

To break subnormality, by Proposition 4.2 it suffices to show that $p$ is negative. Since $b^2 = 2a^2 - \xi_2^2$, we have

$$p = \xi_e^2 \xi_1^4 - \xi_e^2 a^2 (2a^2 - \xi_2^2) - 2a^2 \xi_1^4 - (2a^2 - \xi_2^2) \xi_1^4 + 4a^2 (2a^2 - \xi_2^2) \xi_1^4 - (2a^2 - \xi_2^2)^2 a^2$$

$$= -2 (\xi_1^4 - a^2)^2 (2a^2 - \xi_2^2) < 0,$$

as desired. The proof is now complete. \hfill \square

**Corollary 4.4.** [DrMcC] Let $dv(s) := 3s^2 ds$ on $[0, 1]$ and choose $a = \sqrt{\frac{1}{2}}$ and $b = \sqrt{\frac{3}{2}}$. Then the 2-variable weighted shift $T$ given by Figure 6 is commuting, has subnormal components, is hyponormal, but is not subnormal.

**Proof.** By Theorem 4.3 and the comments preceding Proposition 4.1, it suffices to verify that $\sqrt{\frac{\xi_2^2 + \xi_5^2}{2}} < a \leq \sqrt{\frac{\xi_2^4 + \xi_5^4}{4}}$. Since $\xi_e = \sqrt{\frac{2}{3}}$ and $a = \sqrt{\frac{1}{2}}$, the result follows by a straightforward calculation. \hfill \square

5. **An Instance When Hyponormality Suffices**

In this section we will prove that under a suitable condition hyponormality does imply subnormality for commuting pairs of subnormal operators. We begin with an elementary result of independent interest.

**Lemma 5.1.** Let $\nu$ be a probability measure on $[0, 1]$, and let $\gamma_n \equiv \gamma_n(\nu) := \int s^n d\nu(s)$ ($n \geq 0$) be the moments of $\nu$. The sequence $\{\gamma_n\}_{n=0}^\infty$ is bounded below if and only if $\nu$ has an atom at $\{1\}$.

**Proof.** ($\Rightarrow$) Let $\rho := \nu(\{1\}) > 0$ and write $\nu \equiv (1 - \rho) \eta + \rho \delta_1$, where $\eta$ is a probability measure on $[0, 1]$ with $\eta(\{1\}) = 0$. It follows that $\gamma_n(\nu) \geq \rho \int s^n d\delta_1(s) = \rho (\text{all } n \geq 0)$, so $\{\gamma_n\}$ is bounded below by $\rho$.

($\Leftarrow$) Suppose $\nu(\{1\}) = 0$, let $f_n(s) := s^n$ ($0 \leq s \leq 1$, $n \geq 0$), and consider the sequence of nonnegative functions $\{f_n\}_{n \geq 0}$. Clearly $f_n \not\to \chi_{\{1\}}$ pointwise, and $|f_n| \leq 1$ (all $n \geq 0$). By the Lebesgue Dominated Convergence Theorem, $\lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \int s^n d\nu(s) = \lim_{n \to \infty} \int s^n d\nu(s) = \nu(\{1\}) = 0$. Therefore, $\{\gamma_n\}$ is not bounded below.

We now consider the 2-variable weighted shift $T$ given by Figure 7, where $W_\xi := \text{shift}(\xi_0, \xi_1, \cdots)$ is a subnormal contraction with associated measure $\nu$, and $y \leq 1$.

It is clear that $T_1 T_2 = T_2 T_1$, and that $T_1$ is subnormal (being the orthogonal direct sum of $W_\xi$ and copies of $U_\pm$). To ensure the subnormality of $T_2$, we must impose the condition $\sqrt{n} \leq 1$ (all $n \geq 0$), i.e., $y^2 \leq \gamma_n$ (all $n \geq 0$), where $\gamma_n \equiv \gamma_n(\nu)$. Notice that this condition also guarantees the boundedness of $T$.

**Theorem 5.2.** Let $T$ be the 2-variable weighted shift given by Figure 7, and assume that $T$ is hyponormal. Then $T$ is subnormal.

**Proof.** We apply the Six-point Test (Theorem 1.3) to an arbitrary lattice point of the form $(n, 0)$. Since $T$ is hyponormal by hypothesis, we must have $(\xi_{n+1}^2 - \xi_n^2)(1 - \frac{y^2}{\gamma_n}) \geq \left(\frac{y}{\sqrt{\gamma_{n+1}}} - \frac{\sqrt{\gamma_n}}{\sqrt{\gamma_{n+1}}}\right)^2$, or equivalently $(\xi_{n+1}^2 - \xi_n^2)(1 - \frac{y^2}{\gamma_n}) \geq \frac{y^2}{\gamma_n} \left(1 - \xi_n\right)^2$, that is, $y^2 \leq \left(\frac{\xi_{n+1}^2 - \xi_n^2}{\gamma_{n+1} \xi_n - \xi_n^2}\right)^2 \gamma_n$. Since $\xi_n^2 + \frac{1}{\xi_n^2} \geq 2 \geq 0$ and $\frac{\xi_{n+1}^2 - \xi_n^2}{\xi_{n+1}^2 \xi_n - \xi_n^2} = \frac{\xi_{n+1}^2 - \xi_n^2}{\xi_{n+1}^2 \xi_n - \xi_n^2}$, it follows that $\frac{\xi_{n+1}^2 - \xi_n^2}{\xi_{n+1}^2 \xi_n - \xi_n^2} \leq 1$, so $0 < y^2 \leq \gamma_n$ (all $n \geq 0$).
Thus, \( \{\gamma_n\} \) is bounded below, and by Lemma 5.1 we can write \( \nu = (1 - \rho)\eta + \rho\delta_1 \), with \( \rho := \nu(\{1\}) \) and \( \eta(\{1\}) = 0 \). It follows that \( y^2 \leq \rho \). Thus, \( y^2\delta_1 \leq \nu \). By Proposition 2.9, \( T \) is subnormal. \( \square \)

**Remark 5.3.** Theorem 5.2 (and its proof) reveals that for the 2-variable weighted shift given by Figure 7, the subnormality of \( T_2 \) is equivalent to the subnormality of \( T \), which in turn is equivalent to the hyponormality of \( T \).

**References**


[DrMcC] M. Dritschel and S. McCullough, private communication.


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