DISINTEGRATION-OF-MEASURE TECHNIQUES FOR COMMUTING MULTIVARIABLE WEIGHTED SHIFTS

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Abstract. We employ techniques from the theory of disintegration of measures to study the Lifting Problem for commuting n-tuples of subnormal weighted shifts. We obtain a new necessary condition for the existence of a lifting, and generate new pathology associated with bringing together the Berger measures associated to each individual weighted shift. For subnormal 2-variable weighted shifts, we then find the precise relation between the Berger measure of the pair and the Berger measures of the shifts associated to horizontal rows and vertical columns of weights.

1. Introduction

In this paper, we consider an old problem in operator theory, the so-called Lifting Problem for Commuting Subnormals: given a commuting pair of subnormal operators on a Hilbert space $\mathcal{H}$, one asks whether the pair is the restriction of a commuting normal pair acting on a larger Hilbert space $\mathcal{K}$. One instance of the problem is to consider multivariable weighted shifts; in this case, the existence of a lifting to a larger space is equivalent to the existence of a positive regular Borel probability measure on $\mathbf{R}_+^2$ (the so-called Berger measure for the pair), interpolating the moments generated by the weight sequences. In turn, each of the subnormal components of the pair comes equipped with a countable family of canonical probability measures supported in the nonnegative real axis $\mathbf{R}_+$, obtained as the solution of power moment problems with data directly linked to the actual weights. From this perspective, the Lifting Problem consists of "compatibly gluing together" the individual measures on $\mathbf{R}_+$ to produce a measure μ on $\mathbf{R}_+^2$ such that μ satisfies the required properties to be the Berger measure of the pair.

Using techniques from the theory of disintegration of measures, we generate interesting pathology associated with bringing together the Berger measures associated to each individual weighted shift. Our study reveals some significant obstructions to the Lifting Problem for Commuting Subnormals, while at the same time producing new necessary conditions for the existence of the lifting.

1991 *Mathematics Subject Classification.* Primary 47B20, 47B37, 47A13, 28A50; Secondary 44A60, 47-04, 47A20.

Key words and phrases. Disintegration of measures, subnormal pairs, jointly hyponormal pairs, 2-variable weighted shifts, n-atomic measures.

Research partially supported by NSF Grants DMS-0099357 and DMS-0400741 .

We now briefly discuss our main results. In Theorem 3.1 we give a complete and concrete description of the Berger measure of each horizontal and vertical slice of a subnormal 2-variable weighted shift; in Theorem 3.3, we use this representation to show that the Berger measures of horizontal and vertical slices are linearly order with respect to absolute continuity. The presence of this linear order is essential for joint subnormality, as we show in Proposition 4.7 that it is possible for a 2-variable weighted shift to have 1-atomic Berger measures in all horizontal and vertical slices without being even hyponormal.

We then look at 2-variable weighted shifts with 2-atomic Berger measures in all horizontal and vertical slices, and exhibit a family where hyponormality and subnormality differ (Proposition 4.9). Next, we prove that allowing flexibility in the support of the Berger measures of the horizontal slices does not generally help (Theorem 4.15). Finally, we produce a striking example of a commuting, hyponormal, 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, such that T_1 and T_2 are subnormal, with mutually absolutely continuous Berger measures for the horizontal and vertical slices, and such that $\mathbf{T}$ does not admit a lifting (Proposition 4.18).

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S, T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [3], [16]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal. The Bram-Halmos criterion states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only the k-tuple $(T, T^2, ..., T^k)$ is hyponormal for all $k \geq 1$.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of α are given by

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot ... \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq ....$ Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-*variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \text{ (all } \mathbf{k}). \tag{1.1}$$

In an entirely similar way one can define multivariable (i.e., n-variable) weighted shifts. Many of the results in this paper (formulated for $n = 2$) have straightforward generalizations to the case $n > 2$. We do not devote any effort to pursue those generalizations, since we believe the substance is in the analysis of the case $n = 2$.

Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+^2$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, $T_1 \cong W_\alpha \bigotimes I$ and $T_2 \cong I \bigotimes W_\beta$, and $\mathbf{T}$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [7, III.8.16]):

$$W_\alpha \text{ is subnormal } \Leftrightarrow \text{ there exists a regular Borel probability measure } \xi$$

$$\text{with } supp\xi \subseteq [0, \|W_\alpha\|^2] \text{ and such that } \gamma_k(\alpha) := \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 = \int t^k \, d\xi(t) \;\; (k \geq 1).$$

As a consequence, if W_α is subnormal, and if for $h \geq 1$ we let $\mathcal{M}_h := \bigvee\{e_n : n \geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$. For $h = 1$, one has the following result.

Proposition 1.1. *(Subnormal backward extension of a* 1*-variable weighted shift) (cf* [9, Proposition 8], [19, Proposition 1.5]*) Let* W_α *be a weighted shift whose restriction* $W_\alpha|_{\mathcal{M}}$ *to* $\mathcal{M} := \vee\{e_1, e_2, \cdots\}$ *is subnormal, with associated measure* $\mu_{\mathcal{M}}$*. Then* W_α *is subnormal (with associated measure* μ*) if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$
(ii) $\alpha_0^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$.
In this case, $d\mu(t) = \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) + (1 - \alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}) d\delta_0(t)$*, where* δ_0 *denotes Dirac measure at* 0*. In particular,* W_α *is never subnormal when* $\mu_{\mathcal{M}}(\{0\}) > 0$.

To state Berger's Theorem in the 2-variable case we need to recall the notion of moment of order $\mathbf{k}$ for a pair (α, β) satisfying (1.1). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of

(α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \left\{ \begin{array}{cc} 1 & \text{if } \mathbf{k} = 0 \\ \alpha^2_{(0,0)} \cdot ... \cdot \alpha^2_{(k_1-1,0)} & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta^2_{(0,0)} \cdot ... \cdot \beta^2_{(0,k_2-1)} & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha^2_{(0,0)} \cdot ... \cdot \alpha^2_{(k_1-1,0)} \cdot \beta^2_{(k_1,0)} \cdot ... \cdot \beta^2_{(k_1,k_2-1)} & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{array} \right\}.$$

We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).

Theorem 1.2. *(Berger's Theorem, 2-variable case) ([26]) A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a regular Borel probability measure μ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ ($a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \iint_R \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) := \iint_R t_1^{k_1} t_2^{k_2}\, d\mu(t_1, t_2)$ (all $\mathbf{k} \in \mathbb{Z}^2_+$).*

Clearly, each component T_i of a subnormal 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ must be subnormal. For instance, $T_1 \cong \bigoplus_{j=0}^{\infty} W_{\alpha^{(j)}}$, where $\alpha_i^{(j)} := \alpha_{(i,j)}$ and each $W_{\alpha^{(j)}}$ has associated Berger measure $d\xi_j(t_1) := \{\frac{1}{\gamma_{0j}} \int_{[0,a_2]} t_2^j \, d\Phi_{t_1}(t_2)\}\, d\xi(t_1)$, where $d\mu(t_1, t_2) \equiv d\Phi_{t_1}(t_2) d\xi(t_1)$ is the canonical disintegration of μ by vertical slices.

On the other hand, if we only know that each of T_1 and T_2 is subnormal, and that T_1 and T_2 commute, the following problem is natural.

Problem 1.3. *(Lifting Problem for Commuting Subnormals) Find necessary and sufficient conditions on T_1 and T_2 to guarantee the subnormality of $\mathbf{T} \equiv (T_1, T_2)$.*

It is well known that the above mentioned necessary conditions do not suffice (cf. [10]). In terms of the *marginal* measures, the problem can be phrased as a reconstruction-of-measure problem, that is, under what conditions on the single variable measures $\{\xi_j\}_{j=0}^{\infty}$ and $\{\eta_i\}_{i=0}^{\infty}$ associated with T_1 and T_2, respectively, does there exist a 2-variable measure μ correctly interpolating all the powers $t_1^{k_1} t_2^{k_2}$ $(k_1, k_2 \geq 0)$?

To detect hyponormality for 2-variable weighted shifts, there is a simple criterion involving a base point $\mathbf{k}$ in $\mathbb{Z}^2_+$ and its five neighboring points in $\mathbf{k} + \mathbb{Z}^2_+$ at path distance at most 2 (cf. Figure 1).

Theorem 1.4. *([10]) (Six-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \Leftrightarrow (([T_j^*, T_i] e_{\mathbf{k}+\varepsilon_j}, e_{\mathbf{k}+\varepsilon_i}))_{i,j=1}^2 \geq 0 \text{ (all } \mathbf{k} \in \mathbf{Z}^2_+)$$

$$\Leftrightarrow H(\mathbf{k}) := \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geq 0 \text{ (all } \mathbf{k} \in \mathbf{Z}^2_+).$$

Unlike the single variable case, in which there is a clear separation between hyponormality and subnormality (cf. [12], [11], [14]), much less is known about the

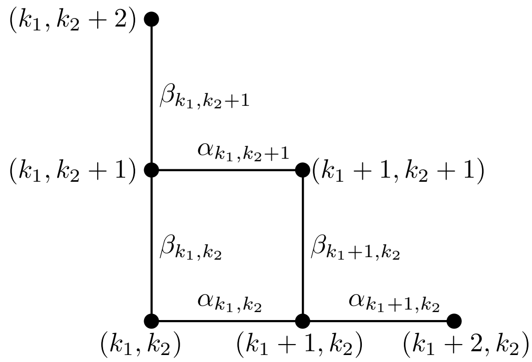

FIGURE 1. Weight diagram used in the Six-point Test

multivariable case. In [19] we constructed three conceptually different families of counterexamples to the following conjecture.

Conjecture 1.5. *([16]) Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a pair of commuting subnormal operators on* $\mathcal{H}$*. Then* $\mathbf{T}$ *is subnormal if and only if* $\mathbf{T}$ *is hyponormal.*

In this paper, we will first establish a new necessary condition for the existence of a (joint) Berger measure μ: for each $j \geq 0$, $\xi_{j+1} \ll \xi_j$ and, similarly, for each $i \geq 0$, $\eta_{i+1} \ll \eta_i$, where ξ_j (resp. η_i) is the Berger measure of the j-th horizontal slice of T_1 (resp. the i-th vertical slice of T_2). We do this by properly identifying these 1-variable measures as the marginal components of a family of 2-variable measures associated with μ. We then obtain several families of counterexamples to Conjecture 1.5, by showing that the new necessary condition is not sufficient. These new counterexamples are structurally different from those presented in [19], in that they emphasize the measure-theoretical aspects of Problem 1.3. (We mention here that M. Dritschel and S. McCullough, working independently, have been able to obtain a separate counterexample to Conjecture 1.5 [22].)

On occasion, we will write $shift(\alpha_0, \alpha_1, ...)$ to denote the weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$. We also denote by $U_+ := shift(1, 1, ...)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := shift\{a, 1, 1, ...)$. Observe that the Berger measures of U_+ and S_a are δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_p denotes the point-mass probability measure with support the singleton $\{p\}$. Finally, we let B_+ denote the Bergman shift, whose Berger measure is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are given by the formula $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

Acknowledgment. All the examples in this paper were obtained using calculations with the software tool *Mathematica* [31].

2. A Brief Account of Some Basic Results in the Theory of Disintegration of Measures

Recall that a unilateral weighted shift W_α is subnormal if and only if there exists a probability measure $\xi \equiv \xi_\alpha$ supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdot ... \cdot \alpha_{k-1}^2 = \int t^k \, d\xi(t) \quad (k \geq 1)$. For instance, when $\alpha_1 = \alpha_2 = ... = 1$ (i.e., $W_\alpha \equiv shift(\alpha_0, 1, 1, ...)$), we have $\xi_\alpha = (1-\alpha_0^2)\delta_0 + \alpha_0^2\delta_1$. The proof of the following lemma is straightforward.

Lemma 2.1. *Given two* 1*-variable weight sequences* α *and* β*, the* 2*-variable weighted shift* $(W_\alpha \bigotimes I, I \bigotimes W_\beta)$ *is always subnormal, with Berger measure* $\mu := \xi_\alpha \times \xi_\beta$*.*

In the sequel we always assume that X and Y are compact metric spaces; in case reference is made to weighted shifts, we assume that $X, Y \subseteq \mathbb{R}_+$.

Definition 2.2. *Given a measure* μ *on* $X \times Y$*, the marginal measure* μ^X *is given by* $\mu^X := \mu \circ \pi_X^{-1}$*, where* $\pi_X : X \times Y \to X$ *is the canonical projection onto* X*. Thus,* $\mu^X(E) = \mu(E \times Y)$*, for every* $E \subseteq X$*. Observe that if* μ *is a probability measure, then so is* μ^X*.*

Lemma 2.3. *(cf.* [19, Lemma 3.6]*) Let* μ *be the Berger measure of a* 2*-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$*, and let* ξ *be the Berger measure of* $shift(\alpha_{00}, \alpha_{10}, ...)$*. Then* $\xi = \mu^X$*. As a consequence,* $\iint f(s) \, d\mu(s,t) = \int f(s) \, d\mu^X(s)$ *for all* $f \in C(X)$*.*

Corollary 2.4. *(cf.* [19, Corollary 3.7]*) Let* μ *be the Berger measure of a* 2*-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$*. For* $j \geq 0$*, let* $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$*. Then the Berger measure of* $shift(\alpha_{0j}, \alpha_{1j}, ...)$ *is* $\xi_j \equiv \mu_j^X$*. In particular, the Berger measure of* $shift(\alpha_{00}, \alpha_{10}, ...)$ *is* μ^X*.*

Example 2.5. Let $\mu := \xi \times \eta$ be a probability product measure on $X \times Y$. Then $\mu^X = \xi$.

Recall that given two positive regular Borel measures μ and ω, μ is said to be absolutely continuous with respect to ω (in symbols, $\mu \ll \omega$) if for every Borel set E, $\omega(E) = 0 \Rightarrow \mu(E) = 0$. It follows at once that $\mu \ll \omega \Rightarrow supp\mu \subseteq supp\omega$.

Lemma 2.6. *Let* μ *and* ω *be two measures on* $X \times Y$*, and assume that* $\mu \ll \omega$*. Then* $\mu^X \ll \omega^X$*.*

Proof. Straightforward from Definition 2.2. □

Let μ be the Berger measure of a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$. Although Corollary 2.4 indicates how to obtain the Berger measure ξ_j of the horizontal j-th slice of T_1 in terms of μ, the description is not completely satisfactory, in that it may not be easy to find the marginal measures μ_j $(j \geq 0)$. We will now employ disintegration of measure techniques to give a precise description of ξ_j. First, we need to review some basic concepts and general results about disintegration of measures; most of the discussion in the rest of this section is is taken from [7, VII.2, pp. 317-319].

Let X and Z be compact metric spaces and let μ be a positive regular Borel measure on Z. For $\phi : Z \to X$ a Borel mapping, let ν be the Borel measure $\mu \circ \phi^{-1}$ on X; that is,

$$\nu(A) := \mu(\phi^{-1}(A)) \tag{2.1}$$

for all Borel sets A contained in X. Let $\mathcal{L}^1(\mu) := \{f : f \text{ is Borel function on } Z \text{ such that } \int |f|\, d\mu < \infty\}$, and let $L^1(\mu) := \{[f] : f \in \mathcal{L}^1(\mu)\}$, where $[f] := \{g \in \mathcal{L}^1(\mu) : \int |f - g|\, d\mu = 0\}$. The map

$$\psi \to \int_Z (\psi \circ \phi) f d\mu \tag{2.2}$$

defines a bounded linear functional on $L^\infty(\nu)$. If attention is restricted to characteristic functions χ_A in $L^\infty(\nu)$,

$$A \to \int_Z (\chi_A \circ \phi) f d\mu = \int_{\phi^{-1}(A)} f d\mu \tag{2.3}$$

is a countably additive measure defined on Borel sets in X, that is absolutely continuous with respect to ν. Hence there is a unique element $E(f)$ in $L^1(\nu)$ such that

$$\int_Z (\chi_A \circ \phi) f d\mu = \int_X \chi_A E(f) d\nu \tag{2.4}$$

for all Borel subsets A of X. By an an approximation argument one can show that

$$\int_Z (\psi \circ \phi) f d\mu = \int_X \psi E(f) d\nu \tag{2.5}$$

for all ψ in $L^\infty(\nu)$. This defines a map

$$E : \mathcal{L}^1(\mu) \to L^1(\nu) \tag{2.6}$$

called the *expectation operator.*

Notation 2.7. The space of all Borel measures on Z will be denoted by $M(Z)$.

Definition 2.8. *A disintegration of the measure μ with respect to ϕ is a function $x \mapsto \Phi_x$ from X to $M(Z)$, such that*
(i) for each x in X, Φ_x is a probability measure;
(ii) if $f \in \mathcal{L}^1(\mu)$, $E(f)(x) = \int_Z f d\Phi_x$ a.e. $[\nu]$.

Remark 2.9. In equation (2.5) we can take $f \equiv \chi_A$ for any Borel subset A of Z and $\psi \equiv 1$; the definition of disintegration then implies that

$$\begin{aligned} \int_X \Phi_x(A) d\nu(x) &= \int_X (\int_Z \chi_A d\Phi_x) d\nu(x) \\ &= \int_X E(\chi_A) d\nu = \int_Z (1 \circ \phi) \chi_A d\mu \\ &= \int_{\phi^{-1}(Z)} \chi_A d\mu = \int_X \chi_A d\mu = \mu(A) \end{aligned}$$

Thus disintegration does indeed "disintegrate" the measure into the pieces Φ_x.

Example 2.10. Suppose we take X and Z to be the same space, let μ be any positive measure on Z, and let $\phi(x) := x$ for all x. Here $\nu = \mu$ and the disintegration is obtained by letting $\Phi_x := \delta_x$, because

$$\begin{aligned} \int_X (\psi \circ \phi) f d\mu &= \int_X \psi E(f) d\nu \ \ (\text{all } \psi \in L^\infty(\nu)) \\ &\Rightarrow \int_X f d\mu = \int_X E(f) d\nu \ \ (\text{if } \psi \equiv 1) \\ &\Rightarrow f = E(f) \ \ \text{a.e. } [\mu] \ \ (\text{if } \nu = \mu) \\ &\Rightarrow f(x) = E(f)(x) \ \ \text{a.e. } [\mu] \\ &\Rightarrow \int f d\delta_x = \int f d\Phi_x \ \ (\text{all } f \in \pounds^1(\mu)) \\ &\Rightarrow \delta_x = \Phi_x. \end{aligned}$$

Example 2.11. Suppose X is given, ξ is any measure on X, Y is any compact metric space, η is a probability measure on Y, and put

$$Z := X \times Y, \ \phi(x, y) \equiv \pi_X(x, y) := x, \ \text{and } \mu := \xi \times \eta.$$

Then $\xi = \mu \circ \phi^{-1}$. Here $E(f)(x) = \int f(x, y) d\eta(y)$ and the disintegration arises by letting

$$\Phi_x(A) := \eta(\pi_Y(A \cap (\{x\} \times Y))).$$

Proposition 2.12. *([7, Proposition VII.2.10]) Let $x \mapsto \Phi_x$ be a disintegration of μ with respect to ϕ. Then supp$\Phi_x = \phi^{-1}(x)$ a.e. $[\nu]$.*

We now list the main theorem on existence and uniqueness of disintegration of measures.

Theorem 2.13. *([7, Theorem VII.2.11]) Given a regular Borel measure μ on a compact metric space Z, and a Borel function ϕ from Z into a compact metric space X, there is a disintegration $x \mapsto \Phi_x$ of μ with respect to ϕ. If $x \mapsto \Phi_x'$ is another disintegration of μ with respect to ϕ, then $\Phi_x = \Phi_x'$ a.e. $[\nu]$.*

3. A new Necessary Condition for the Existence of a Lifting

Given the notation and terminology of Section 2, and given the Berger measure μ of a subnormal 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, we are now ready to calculate explicitly the measures ξ_j $(j \geq 0)$. Fix $j \geq 0$ and observe that the moments of ξ_j are given by

$$\int_X s^i \, d\xi_j(s) = \alpha_{0j}^2 \cdot ...\alpha_{i-1,j}^2 = \frac{\gamma_{ij}}{\gamma_{0j}} = \frac{1}{\gamma_{0j}} \int\int_R s^i t^j \, d\mu(s,t) \ \ (\text{all } i \geq 0), \tag{3.1}$$

where $R := X \times Y \equiv [0, a_1] \times [0, a_2]$. Since μ is regular and Borel, we now use Theorem 2.13 to disintegrate μ with respect to $\phi \equiv \pi_X$ and obtain

$$\mu(A) = \int_X \Phi_x(A) \, d\mu^X(x),$$

where as above $\mu^X = \mu \circ \pi_X^{-1}$. Now recall that $supp\Phi_x = \phi^{-1}(x) = \{x\} \times Y$, so with a slight abuse of notation we shall regard Φ_x as a measure on Y and write $d\Phi_x(t)$ for $d\Phi_x(x,t)$. Therefore, for all $i \geq 0$ we have

$$\begin{aligned} \int\int_R s^i t^j \, d\mu(s,t) &= \int_X (\int\int_R s^i t^j \, d\Phi_x(s,t)) \, d\mu^X(x) \\ &= \int_X (\int\int_{\{x\}\times Y} x^i t^j \, d\Phi_x(x,t)) \, d\mu^X(x) \\ &= \int_X (\int_Y s^i t^j \, d\Phi_s(t)) \, d\mu^X(s). \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) we obtain

$$\begin{aligned} \int_X s^i \, d\xi_j(s) &= \frac{1}{\gamma_{0j}} \int\int_R s^i t^j \, d\mu(s,t) \\ &= \frac{1}{\gamma_{0j}} \int_X (\int_Y s^i t^j \, d\Phi_s(t)) \, d\mu^X(s) \\ &= \int_X s^i \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t)\} \, d\mu^X(s) \ \ (\text{all } i \geq 0). \end{aligned}$$

Thus,

$$d\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t)\} \, d\mu^X(s).$$

We now observe that ξ_j is indeed μ_j^X. First recall that $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$, so the above disintegration of μ with respect to π^X yields $d\mu_j(s,t) = \frac{1}{\gamma_{0j}} t^j d\Phi_s(t) d\mu^X(s)$.

For a Borel set $E \subseteq X$, it follows that

$$\begin{aligned}\mu_j^X(E) &= \mu_j(E \times Y) = \frac{1}{\gamma_{0j}} \int\int_{E\times Y} t^j \, d\Phi_s(t) d\mu^X(s) \\ &= \int_E \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t))\} d\mu^X(s) \\ &= \int_E d\xi_j(s) = \xi_j(E).\end{aligned}$$

We summarize this in the following

Theorem 3.1. *Let μ be the Berger measure of a subnormal 2-variable weighted shift, and for $j \geq 0$ let ξ_j be the Berger measure of the associated j-th horizontal 1-variable weighted shift $W_{\alpha^{(j)}}$. Then $\xi_j = \mu_j^X$ (cf. Definition 2.2), where $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$; more precisely,*

$$d\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t)\} \, d\mu^X(s),$$

where $d\mu(s,t) \equiv d\Phi_s(t)\, d\mu^X(s)$ is the disintegration of μ by vertical slices. A similar result holds for the Berger measure η_i of the associated i-th vertical 1-variable weighted shifts $W_{\beta^{(i)}}$ $(i \geq 0)$.

We next need an elementary result.

Lemma 3.2. *Let μ and ν be two regular Borel measures on R, and assume that $\mu \ll \nu$. Then $\mu^X \ll \nu^X$ and $\mu^Y \ll \nu^Y$.*

Proof. Let E be a Borel subset of X, and assume that $\nu^X(E) = 0$, i.e., $\nu(E \times Y) = 0$. It follows that $\mu(E \times Y) = 0$, that is, $\mu^X(E) = 0$. Thus, $\mu^X \ll \nu^X$. □

We are finally ready to establish our new necessary condition for the existence of a lifting for two commuting subnormal weighted shifts, that is, the sequences of Berger measures of horizontal and vertical slices form non-increasing families relative to the order given by absolute continuity.

Theorem 3.3. *Let μ, ξ_j and η_i be as in Theorem 3.1. For every $i, j \geq 0$ we have*

$$\xi_{j+1} \ll \xi_j \tag{3.3}$$

and

$$\eta_{i+1} \ll \eta_i. \tag{3.4}$$

Proof. Straightforward from Theorem 3.1 and Lemma 3.2. □

4. The New Necessary Condition is Not Sufficient

We shall now construct examples of commuting pairs of subnormal weighted shifts, satisfying conditions (3.3) and (3.4), but without admitting a lifting to a commuting pair of normal operators. We begin with a review of the theory of subnormal backward extensions.

Definition 4.1. *Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \le \nu$ if $\mu(E) \le \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \le \nu$ if and only if $\int f d\mu \le \int f d\nu$ for all $f \in C(\mathbb{R}_+)$ such that $f \ge 0$ on $\mathbb{R}_+$.*

Definition 4.2. *Let μ be a probability measure on $\mathbb{R}_+ \times \mathbb{R}_+$, and assume that $\frac{1}{t} \in L^1(\mu)$. The* extremal measure *μ_{ext} (which is also a probability measure) on $\mathbb{R}_+ \times \mathbb{R}_+$ is given by $d\mu_{ext}(s,t) := (1-\delta_0(t)) \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t)$.*

Example 4.3. ([19, Example 3.4]) Let B_+ be the Bergman shift on $\ell^2(\mathbb{Z}_+)$ and let $\mathcal{M} \equiv \mathcal{M}_1 := \bigvee\{e_1, e_2, ...\}$. The shift $B_+|_{\mathcal{M}}$ is subnormal, with Berger measure $d\mu(t) := 2tdt$ on $[0,1]$. Then $d\mu_{ext}(t) = dt$, so the extremal measure μ_{ext} is the Berger measure of B_+.

Lemma 4.4. *([19, Lemma 3.9]) Let μ and ω be two measures on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, and assume that $\mu \le \omega$. Then $\mu^X \le \omega^X$.*

Proposition 4.5. *([19, Proposition 3.10]) (Subnormal backward extension of a 2-variable weighted shift) Consider the following 2-variable weighted shift (see Figure 2), and let $\mathcal{M}$ be the subspace associated to indices $\mathbf{k}$ with $k_2 \ge 1$. Assume that $\mathbf{T}_{\mathcal{M}}$ is subnormal with Berger measure $\mu_{\mathcal{M}}$ and that $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with associated measure ξ_0. Then $\mathbf{T}$ is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \le (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \le \xi_0$.
Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \xi_0$. In the case when $\mathbf{T}$ is subnormal, the Berger measure μ of $\mathbf{T}$ is given by

$$\begin{aligned} d\mu(s,t) &= \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &\quad + (d\xi_0(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)) d\delta_0(t). \end{aligned}$$

An important consequence of Proposition 4.5 is that, under certain rigidity conditions, the existence of a lifting follows from commutativity and the subnormality of T_1 and T_2, without necessarily appealing to the joint hyponormality of $\mathbf{T}$. The next result generalizes [19, Theorem 5.2].

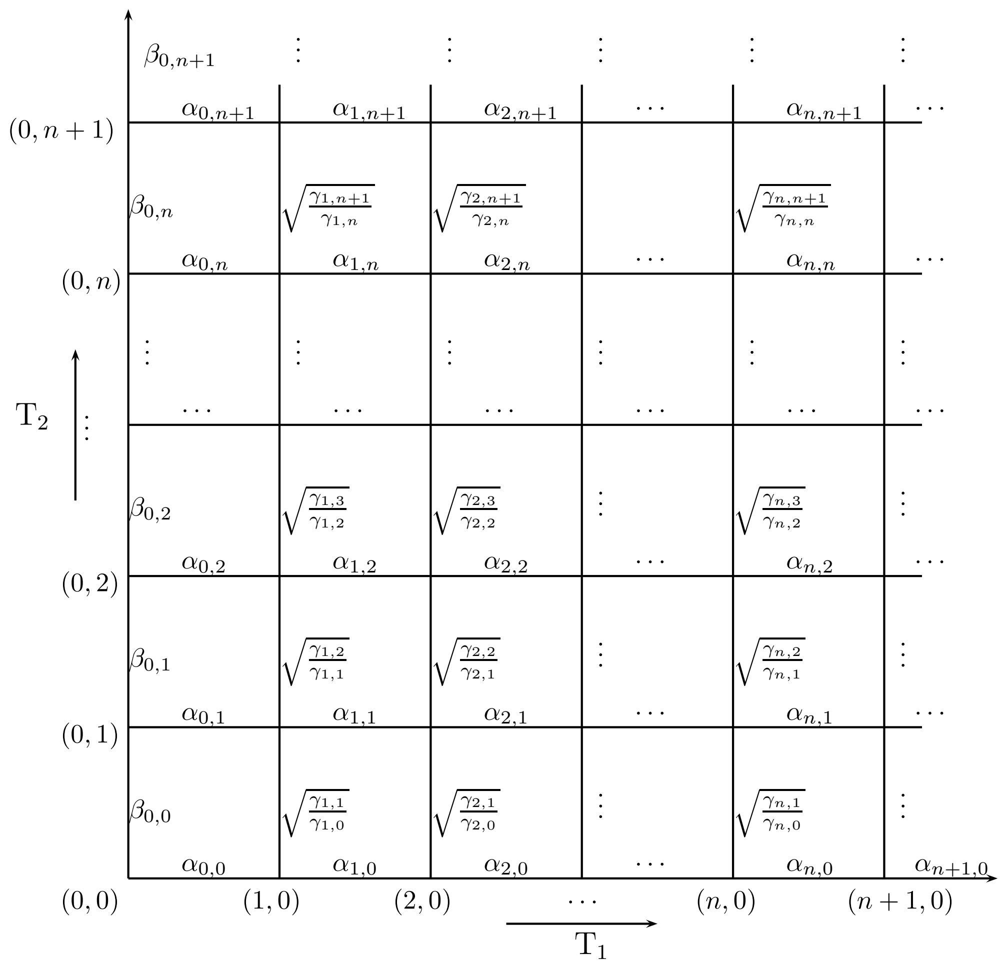

Figure 2. Weight diagram of the 2-variable weighted shift in Proposition 4.5

Corollary 4.6. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *and* $\mathcal{M}$ *be as in Proposition 4.5, and assume that* $\mathbf{T}_{\mathcal{M}}$ *is subnormal with Berger measure* $\delta_1 \times \eta$ *(cf. Figure 3 below). Assume further that* T_1 *and* T_2 *are contractions, that* $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with associated measure* ξ_0*, and that* T_2 *is subnormal. Then* $\mathbf{T}$ *is subnormal.*

Proof. For $n \geq 0$, let $\gamma_n \equiv \gamma_n(\xi_0)$ be the n-th moment of ξ_0, i.e., $\gamma_n := \int s^n \, d\xi_0(s)$. First observe that $T_1T_2 = T_2T_1$ and the fact that $\alpha_{n1} = 1$ (all $n \geq 0$) imply that $\beta_{n0} = \frac{\beta_{00}}{\sqrt{\gamma_n}}$ $(n \geq 0)$ and $\beta_{n,j} = \beta_{0,j}$ (all $j \geq 1$). Since T_2 is subnormal, each of the vertical

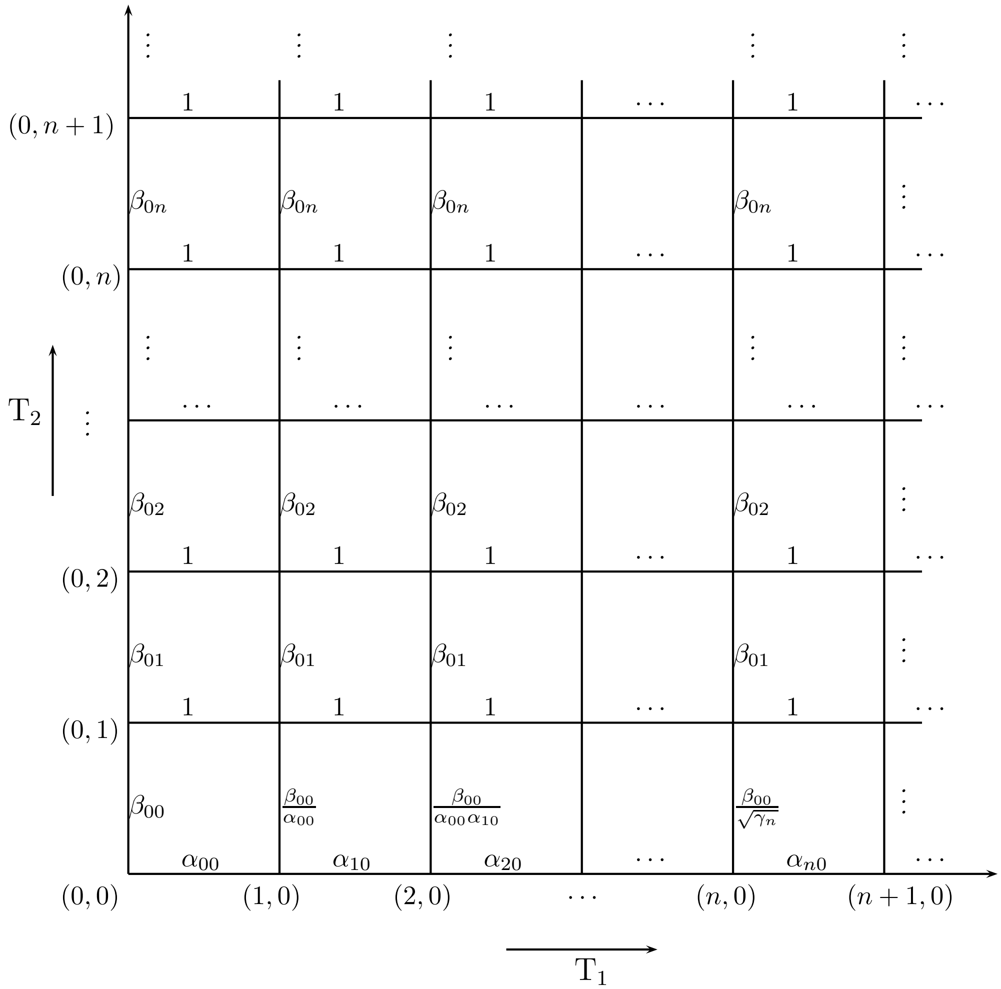

FIGURE 3. Weight diagram of the 2-variable weighted shift in Corollary 4.6.

shifts $\beta_{n0}, \beta_{n1}, ...$ must be subnormal, that is, for each n, $shift(\frac{\beta_{00}}{\sqrt{\gamma_n}}, \beta_{01}, \beta_{02}, ...)$ is subnormal. By Proposition 1.1, we must have $(\frac{\beta_{00}}{\sqrt{\gamma_n}})^2 \left\|\frac{1}{t}\right\|_{L^1(\eta)} \leq 1$, where η is the Berger measure of $shift(\beta_{01}, \beta_{02}, ...)$. Thus,

$$\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\eta)} \leq \gamma_n \text{ (all } n \geq 0). \tag{4.1}$$

It follows that $\{\gamma_n\}$ is bounded below, and since T_1 is a contraction, [19, Lemma 5.1] implies that ξ_0 must have an atom at 1. Thus, $\xi_0 = (1-\rho)\nu + \rho\delta_1$, where $\rho > 0$. If we now recompute the moments γ_n, and apply the Lebesgue Dominated Convergence

Theorem to the sequence of functions $\{s^n\}_{n=0}^{\infty}$, we see that $\rho = \lim_n \gamma_n$. It follows that (4.1) is equivalent to

$$\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta)} \leq \lim_n \gamma_n = \rho. \tag{4.2}$$

We conclude that the subnormality of T_2 implies

$$\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta)} \leq \rho. \tag{4.3}$$

Let us consider now the conditions in Proposition 4.5. Since $\mu_{\mathcal{M}} = \delta_1 \times \eta$, $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ if and only if $\frac{1}{t} \in L^1(\eta)$, and it follows that conditions (i) and (ii) are already satisfied. Moreover, it is straightforward to verify that $(\mu_{\mathcal{M}})_{ext}^X = \delta_1$. Thus, condition (iii) becomes $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta)} \delta_1 \leq (1-\rho)\nu + \rho\delta_1$ or, equivalently, $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\eta)} \leq \rho$. Since this is precisely the condition (4.3) above (implied by the subnormality of T_2), we have shown that $\mathbf{T}$ is subnormal. □

One might wonder what happens if one insists on having all measures ξ_j and η_i 1-atomic. The next result shows that the lifting is not guaranteed in that case either; as a matter of fact, it is even possible for the pair to fail to be hyponormal.

Proposition 4.7. *Consider the* 2*-variable weighted shift given by the weight diagram in Figure 4, where* $\alpha < 1$. *Then* $\mathbf{T} \equiv (T_1, T_2)$ *is commuting, each of* T_1 *and* T_2 *is subnormal, and all horizontal and vertical marginal measures* ξ_j *and* η_i $(i, j \geq 0)$ *are* 1*-atomic, but* $\mathbf{T}$ *is not hyponormal.*

Proof. By Theorem 1.4, a necessary condition for (joint) hyponormality is

$$\begin{aligned} H(\mathbf{0}) &\equiv \begin{pmatrix} \alpha_{10}^2 - \alpha_{00}^2 & \alpha_{01}\beta_{10} - \alpha_{00}\beta_{00} \\ \alpha_{01}\beta_{10} - \alpha_{00}\beta_{00} & \beta_{01}^2 - \beta_{00}^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha^2 - 1 \\ \alpha^2 - 1 & 0 \end{pmatrix} \geq 0, \end{aligned}$$

which cannot occur with $0 < \alpha < 1$. Therefore $\mathbf{T}$ is not hyponormal. □

Remark 4.8. (i) We first observe that for arbitrary $j \geq 0$, $\xi_{j+1} \not\ll \xi_j$; in fact, $supp\xi_{j+1} \bigcap supp\xi_j = \varnothing$. Thus, Theorem 3.3 says that $\mathbf{T}$ could never be subnormal. Proposition 4.7 says that $\mathbf{T}$ can't even be hyponormal.
(ii) The 2-variable weighted shift given by Figure 4 has a peculiar spectral picture, unlike that found for single variable unilateral weighted shifts [28], or for subnormal 2-variable weighted shifts [18]. In fact, the Taylor spectrum of $\mathbf{T}$ is

$$\sigma_T(\mathbf{T}) = [(|z_1| \leq 1) \times \{0\}] \bigcup [\{0\} \times (|z_2| \leq 1)]$$

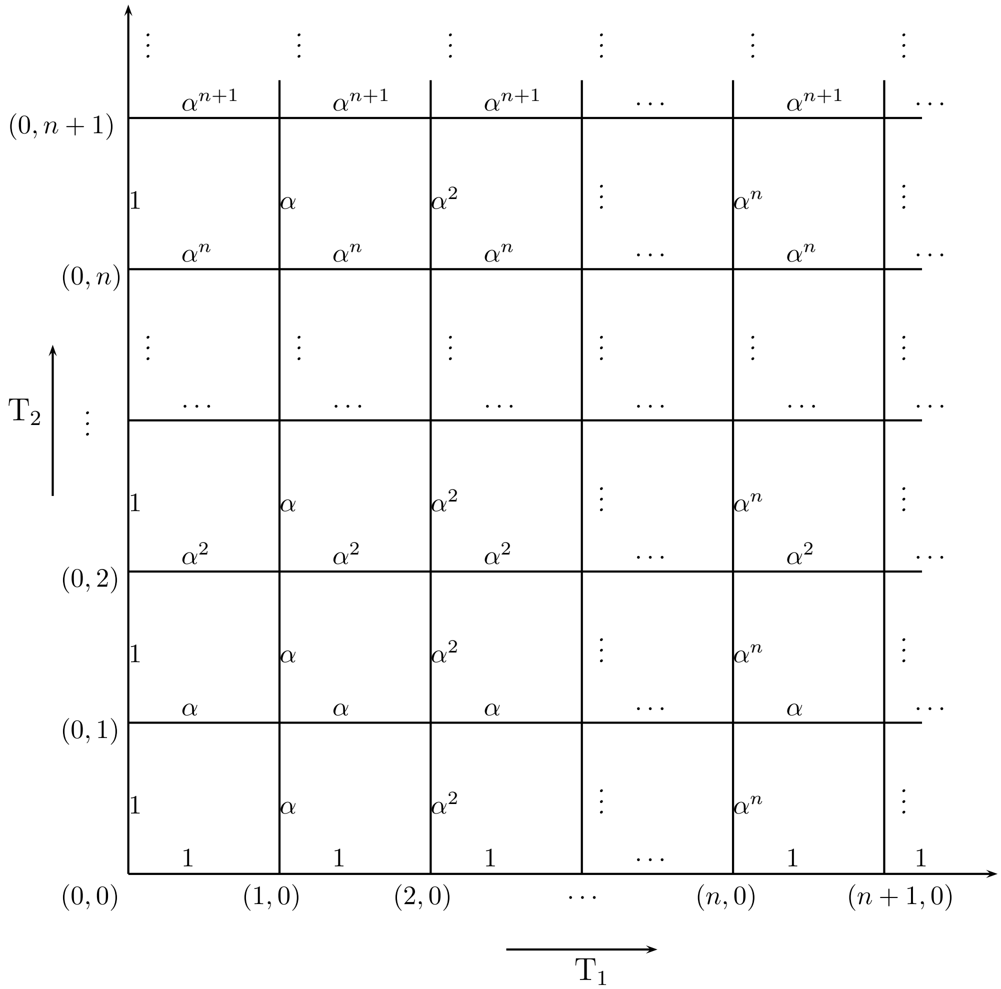

FIGURE 4. Weight diagram the of 2-variable weighted shift in Proposition 4.7

and the Taylor essential spectrum of $\mathbf{T}$ is

$$\sigma_{Te}(\mathbf{T}) = \{(0,0)\} \bigcup \{[\bigcup_{k=0}^{\infty}(|z_1| = \alpha^k)] \times \{0\}\} \bigcup \{\{0\} \times [\bigcup_{\ell=0}^{\infty}(|z_2| = \alpha^{\ell})]\}.$$

Quite surprisingly, the Taylor essential spectrum of $\mathbf{T}$ is disconnected. We study this example and, more generally, the spectral properties of 2-variable weighted shifts in [20].

For $\mathbf{T} \equiv (T_1, T_2)$ a 2-variable weighted shift, and if $\mathcal{M}$ denotes the subspace associated to indices $\mathbf{k}$ with $k_2 \geq 1$, we now consider the case when $T|_{\mathcal{M}} \cong (S_a \bigotimes I, I \bigotimes U_+)$,

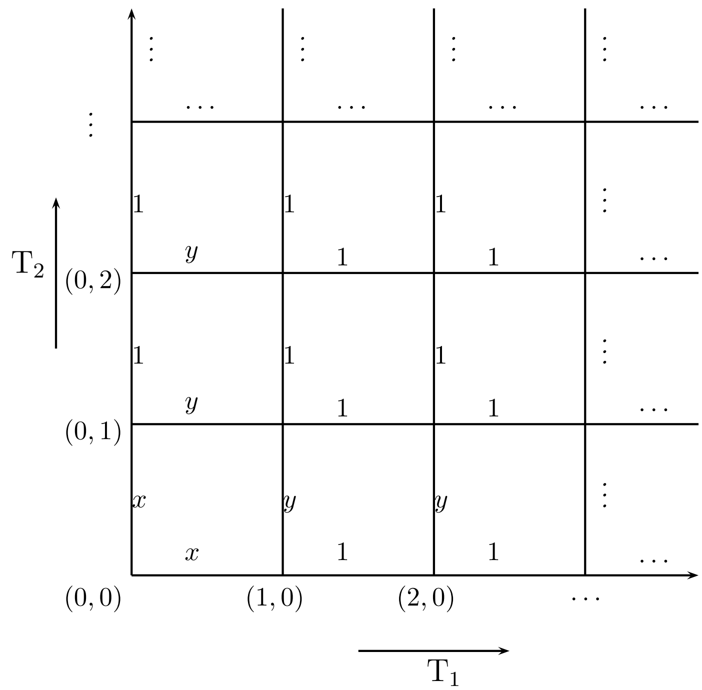

FIGURE 5. Weight diagram of 2-variable weighted shift in Proposition 4.9

where as usual $S_a \equiv shift(a, 1, 1, ...)$. For such shifts, we study hyponormality and subnormality. Notice that $\mu_{\mathcal{M}} = \xi_a \times \delta_1$, where $\xi_a \equiv (1-a^2)\delta_0 + a^2\delta_1$ is the Berger measure of S_a.

We first consider the case when all marginal measures are 2-atomic.

Proposition 4.9. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be the* 2*-variable weighted shift in Figure 5. Then*
(i) $\mathbf{T}$ *is hyponormal* $\Leftrightarrow 1 - 2x^2 + y^2 \geq 0$
(ii) $\mathbf{T}$ *is subnormal* $\Leftrightarrow 1 - 2x^2 + x^2y^2 \geq 0$.
As a consequence, for $(x, y) \in \mathbb{R}_+^2$ *such that* $1 - 2x^2 + x^2y^2 < 0 \leq 1 - 2x^2 + y^2$, $\mathbf{T}$ *is hyponormal but not subnormal (cf. Figure 6 below).*

Proof. The statement in (i) is a straightforward application of Theorem 1.4, while (ii) follows from Proposition 4.5, once we observe that $(\mu_{\mathcal{M}})_{ext}^X = (1-y^2)\delta_0 + y^2\delta_1$. □

We now allow the 0-th horizontal Berger measure ξ_0 to be 3-atomic. The following example shows that it is still possible for T_1 and T_2 to be commuting subnormals,

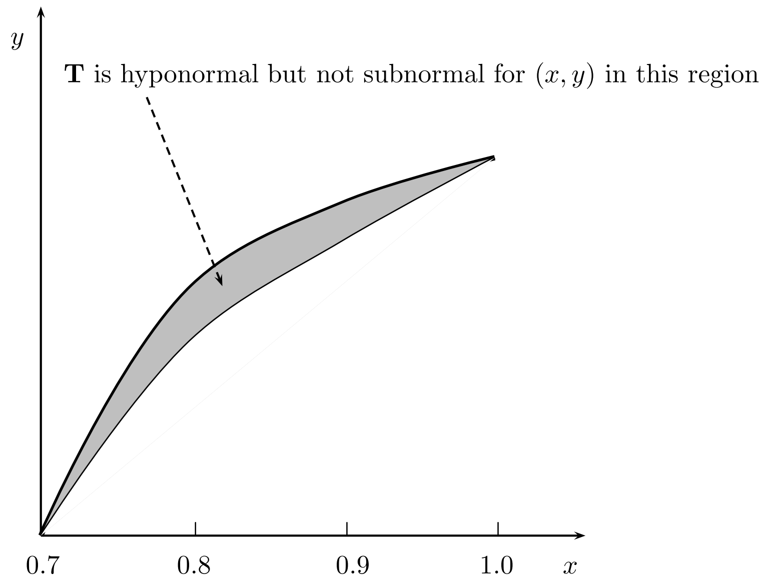

FIGURE 6. Graph of the regions of hyponormality and subnormality in Proposition 4.9

for the pair $\mathbf{T} \equiv (T_1, T_2)$ to be hyponormal, for the marginal measures to satisfy the conditions

$$\cdots \ll \mu_{\alpha_2} \ll \mu_{\alpha_1} \ll \mu_{\alpha_0}$$

and

$$\cdots \ll \mu_{\beta_2} \ll \mu_{\beta_1} \ll \mu_{\beta_0},$$

and for $\mathbf{T}$ not to be subnormal. We first need some preliminary facts.

Proposition 4.10. *Let*

$$\alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n+\frac{1}{2}}{2^n+1}}, & \text{if } n \geq 1 \end{cases}. \tag{4.4}$$

Then W_α is subnormal.

Proof. Consider the 3-atomic measure $\xi := \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{\frac{1}{2}} + \frac{1}{3}\delta_1$. For $n \geq 1$,

$$\begin{aligned}
\gamma_n &\equiv \alpha_0^2\alpha_1^2\alpha_2^2 \cdots \alpha_{n-2}^2\alpha_{n-1}^2 \\
&= \frac{1}{2} \cdot \frac{2+\frac{1}{2}}{2+1} \cdot \frac{2^2+\frac{1}{2}}{2^2+1} \cdot \cdots \cdot \frac{2^{n-2}+\frac{1}{2}}{2^{n-2}+1} \cdot \frac{2^{n-1}+\frac{1}{2}}{2^{n-1}+1} \\
&= \frac{1}{2} \cdot \frac{\frac{2^2+1}{2^2}}{\frac{2+1}{2}} \cdot \frac{\frac{2^3+1}{2^3}}{\frac{2^2+1}{2^2}} \cdot \cdots \cdot \frac{\frac{2^{n-1}+1}{2^{n-1}}}{\frac{2^{n-2}+1}{2^{n-2}}} \cdot \frac{\frac{2^n+1}{2^n}}{\frac{2^{n-1}+1}{2^{n-1}}} \\
&= \frac{1}{2} \cdot \frac{2}{3} \cdot (1+2^{-n}) = \frac{1}{3}(2^{-n}+1) = \frac{1}{3} \cdot (\frac{1}{2})^n + \frac{1}{3} = \int s^n d\xi(s), \qquad (4.5)
\end{aligned}$$

which shows that ξ is the Berger measure of W_α. Therefore, W_α is subnormal. □

Lemma 4.11. *Let*

$$\widehat{\alpha_n} := \begin{cases} \sqrt{2}, & \text{if } n = 0 \\ \sqrt{\frac{2^n+1}{2^n+\frac{1}{2}}}, & \text{if } n \geq 1 \end{cases},$$

then $\prod_{n=0}^{\infty} \widehat{\alpha_n} = \sqrt{3}$. *(Observe that* $\widehat{\alpha_n} = \frac{1}{\alpha_n}$*, for* α_n *given by (4.4).)*

Proof. Observe that

$$\prod_{n=0}^{k} (\widehat{\alpha_n})^2 = 2\prod_{n=1}^{k} \frac{2(2^n+1)}{2^{n+1}+1} = 2^{k+1} \cdot \frac{3}{2^{k+1}+1},$$

which converges to 3 as $k \to \infty$. □

Proposition 4.12. *Consider the weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ *in Figure 7, where* $y < 1$ *and* $ay \leq \frac{1}{\sqrt{3}}$*. Let* $W_\alpha := shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$*, with* α_n *as in (4.4), i.e.,*

$$\alpha_n := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n+\frac{1}{2}}{2^n+1}}, & \text{if } n \geq 1 \end{cases}.$$

Then $\mathbf{T}$ *is subnormal if and only if* $y \leq \min\{\frac{1}{\sqrt{3(1-a^2)}}, \frac{1}{\sqrt{3}a}\}$.

Proof. Observe that the subnormality of T_2 requires $y < 1$ and $\frac{ay}{\sqrt{\gamma_n}} \leq 1$, that is, $a^2y^2 \leq \gamma_n = \frac{1}{3}(2^{-n}+1)$ (all $n \geq 1$) (by (4.5)), which leads to the condition $ay \leq \frac{1}{\sqrt{3}}$. We now refer to Proposition 4.5. First observe that

$$\begin{aligned}
d(\mu_{\mathcal{M}})_{ext}(s,t) &\equiv (1-\delta_0(t))\frac{1}{t\left\|\frac{1}{t}\right\|_\mu} d\mu(s,t) \\
&= \frac{1}{t\left\|\frac{1}{t}\right\|_\mu}[(1-a^2)d\delta_0(s) + a^2 d\delta_1(s)]d\delta_1(t) \\
&= \frac{1}{\left\|\frac{1}{t}\right\|_\mu}[(1-a^2)d\delta_0(s) + a^2 d\delta_1(s)]d\delta_1(t),
\end{aligned}$$

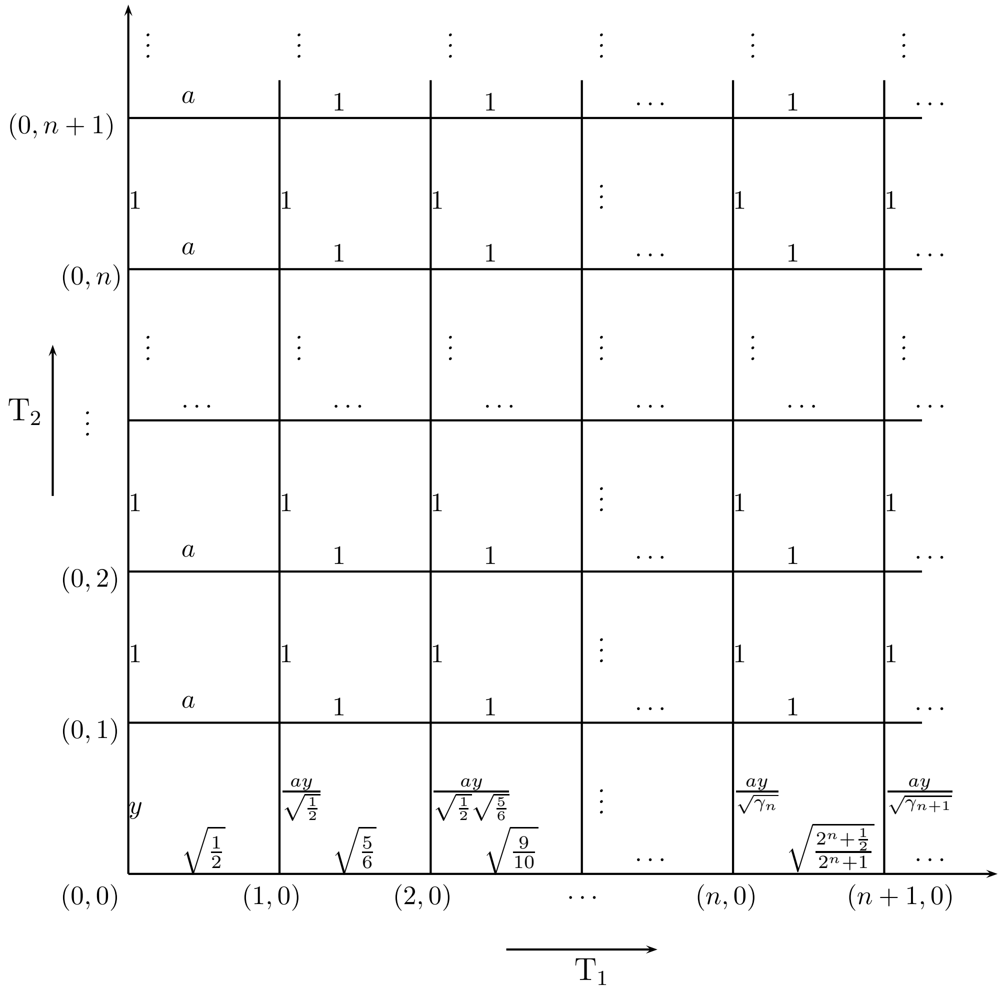

FIGURE 7. Weight diagram of 2-variable weighted shift in Proposition 4.12

Since $\xi_0(s) \equiv \frac{1}{3}\delta_0(s) + \frac{1}{3}\delta_{\frac{1}{2}}(s) + \frac{1}{3}\delta_1(s)$, Proposition 4.5 then shows that

$$
\begin{aligned}
(T_1, T_2) \text{ is subnormal} \quad & \Leftrightarrow y^2 \left\| \tfrac{1}{t} \right\|_{\mu} d(\mu_{\mathcal{M}})^X_{ext}(s,t) \leq d\xi_0(s) \\
& \Leftrightarrow y^2((1-a^2)\delta_0(s) + a^2\delta_1(s)) \leq \tfrac{1}{3}\delta_0(s) + \tfrac{1}{3}\delta_{\frac{1}{2}}(s) + \tfrac{1}{3}\delta_1(s) \\
& \Leftrightarrow y \leq \min\{\tfrac{1}{\sqrt{3(1-a^2)}}, \tfrac{1}{\sqrt{3}a}\}.
\end{aligned}
$$

□

For the next result, we will need the following lemma.

Lemma 4.13. *([19, Theorem 5.2 and Remark 5.3]) Let* $\mathbf{T}$ *be a commuting* 2*-variable weighted shift such that*
(i) $\mathbf{T}|_{\vee\{e_{(i,j)}:i\geq 0,j\geq 1\}} \cong (U_+ \bigotimes I, I \bigotimes U_+)$*; and*
(ii) $\mathbf{T}|_{\vee\{e_{(i,0)}:i\geq 0\}}$ *is subnormal.*
Then $\mathbf{T}$ *is subnormal* $\Leftrightarrow$ $\mathbf{T}$ *is hyponormal* $\Leftrightarrow$ T_2 *is subnormal.*

Proposition 4.14. *The 2-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ *defined in Proposition 4.12 is hyponormal if and only if* $y \leq \min\{\sqrt{\frac{2}{5-12a^2+12a^4}}, \frac{1}{\sqrt{3}a}\}$.

Proof. Since the restriction of $\mathbf{T}$ to $\vee\{e_{(i,j)} : i \geq 0, j \geq 1\}$ is clearly subnormal (being unitarily equivalent to $(S_a \otimes I, I \otimes U^+)$), it suffices to apply the Six-point Test (Theorem 1.4) to $\mathbf{k} = (n, 0)$ $(n \geq 0)$. Moreover, the restriction of $\mathbf{T}$ to $\vee\{e_{(i,j)} : i \geq 1, j \geq 0\}$ satisfies the hypothesis in Lemma 4.13. Thus, to verify the Six-point Test we need to check the case $\mathbf{k} = (0, 0)$ and verify that $\beta_{n,0} \leq 1$ (all $n \geq 1$), that is, $\frac{ay}{\sqrt{\gamma_n}} \leq 1$, or $\gamma_n \geq a^2y^2$ (all $n \geq 1$).

Case 1: $\mathbf{k} = (0, 0)$. Here we have

$$\begin{aligned} H(\mathbf{0}) &\equiv \begin{pmatrix} \frac{5}{6} - \frac{1}{2} & \sqrt{2}a^2y - \sqrt{\frac{1}{2}}y \\ \sqrt{2}a^2y - \sqrt{\frac{1}{2}}y & 1 - y^2 \end{pmatrix} \geq 0 \\ &\Leftrightarrow \frac{1}{3} \geq (\frac{5}{6} - 2a^2 + 2a^4)y^2 \\ &\Leftrightarrow y \leq \sqrt{\frac{2}{5 - 12a^2 + 12a^4}}. \end{aligned}$$

Case 2: We require that $y^2 \leq \frac{1}{a^2}\gamma_n$ (all $n \geq 1$). Since $\gamma_n = \frac{1}{3}(2^{-n} + 1)$ (by (4.5)), we must have $y^2 \leq \frac{2^{-n}+1}{3a^2}$ (all $n \geq 1$), that is $y \leq \frac{1}{\sqrt{3}a}$.

Therefore, $\mathbf{T}$ is hyponormal if and only if $y \leq \min\{\sqrt{\frac{2}{5-12a^2+12a^4}}, \frac{1}{\sqrt{3}a}\}$, as desired. □

Theorem 4.15. *The 2-variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ *defined in Proposition 4.12 is hyponormal and not subnormal if and only if*
(4.6)

$$m_{sub}(a) := \min\{\frac{1}{\sqrt{3(1-a^2)}}, \frac{1}{\sqrt{3}a}\} < y \leq \min\{\sqrt{\frac{2}{5-12a^2+12a^4}}, \frac{1}{\sqrt{3}a}\} =: m_{hyp}(a).$$

Remark 4.16. Observe that for $0 < a < 1$, the left-hand side of (4.6) in Theorem 4.15 is strictly less than the right-hand side, so there is indeed a nonempty range of values for y that guarantees that $\mathbf{T}$ is hyponormal but not subnormal. For, if we let $f(a) := \frac{1}{\sqrt{3(1-a^2)}}$, $g(a) := \sqrt{\frac{2}{5-12a^2+12a^4}}$ and $h(a) := \frac{1}{\sqrt{3}a}$, we can analyze the relative sizes of f, g and h to detect the set of points

$$\{a \in (0,1\} : m_{sub}(a) \equiv \min\{f(a), h(a)\} < \min\{g(a), h(a)\} \equiv m_{hyp}(a)\}$$

(cf. Figure 8 below).

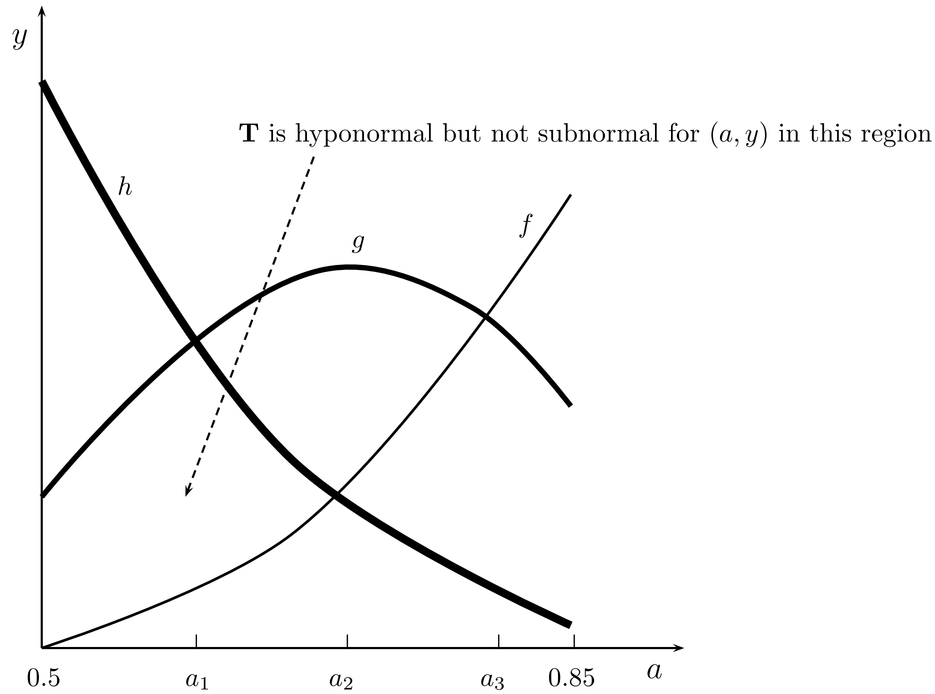

FIGURE 8. Graphs of f, g and h on the interval $[0.5, 0.85]$, showing $a_1, a_2 = \frac{\sqrt{2}}{2}$ and a_3.

Here $a_1 = \frac{\sqrt{3-\sqrt{\frac{7}{3}}}}{2} \cong 0.607$, $a_2 = \frac{\sqrt{2}}{2} \cong 0.707$ and $a_3 = \frac{\sqrt{1+\sqrt{\frac{7}{3}}}}{2} \cong 0.795$ are the a-coordinates of the points of intersection between the graphs of g and h, f and h, and f and g, respectively. Thus,

$$m_{sub}(a) = \begin{cases} f(a) & \text{if } 0 < a \leq a_2 \\ h(a) & \text{if } a_2 < a < 1 \end{cases}$$

and

$$m_{hyp}(a) = \begin{cases} g(a) & \text{if } 0 < a \leq a_1 \\ h(a) & \text{if } a_1 < a < 1 \end{cases} .$$

It easily follows that $m_{sub}(a) < m_{hyp}(a)$ precisely when $0 < a < a_2$. For this a-interval one can always build a 2-hyponormal weighted shift $\mathbf{T} \equiv \mathbf{T}(a, y)$ with the required properties.

Recall now that in Proposition 4.7 we showed that hyponormality may be absent even if all horizontal and vertical marginal measures are 1-atomic. We now show that allowing the support of the horizontal marginal measures to grow from one level to the one immediately below does not guarantee a 2-variable measure either.

Proposition 4.17. *For $n \geq 3$, there exists a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ such that*
(i) $T_1T_2 = T_2T_1$;
(ii) each of T_1 and T_2 is subnormal;
(iii) the i-th row of weights corresponds to a subnormal weighted shift with $(n-i)$-atomic Berger measure $(0 \leq i \leq n)$;
(iv) the marginal measures ξ_j and η_i satisfy the conditions (3.3) and (3.4) in Theorem 3.3;
(v) $\mathbf{T}$ is not subnormal.

Proof. Without loss of generality, let us assume that $n = 3$, and consider a 2-variable weight diagram such that
(i) $W_{\beta_j} := shift(\beta_{i0}, \beta_{i1}, \cdots)$ is subnormal with Berger measure η_i (all $i \geq 0$);
(ii) $W_{\alpha^{(0)}} := shift(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with 3-atomic Berger measure ξ_0;
(iii) $W_{\alpha^{(1)}} := shift(\alpha_{01}, \alpha_{11}, \cdots)$ is subnormal with 2–atomic Berger measure ξ_1;
(iv) $W_{\alpha^{(j)}} := shift(\alpha_{0j}, \alpha_{1j}, \cdots)$ is subnormal with 1–atomic Berger measure ξ_j (all $j \geq 2$);
(v) the marginal measures ξ_j and η_i satisfy (3.3) and (3.4).
Assume now that $\mathbf{T}$ is indeed subnormal, with Berger measure μ. Since ... $\ll \xi_4 \ll \xi_3 \ll \xi_2$, and ξ_2 is 1-atomic, we must have $supp \xi_j = supp \xi_2$ (all $j \geq 3$) (by the comment right before the statement of Lemma 2.6). Without loss of generality, we can thus assume that $\alpha_{ij} = 1$, for all $i \geq 0, j \geq 2$. Let $\mathcal{M}_2 := \vee\{e_{k,\ell} : k \geq 0, \ell \geq 2\}$ and $\mathcal{M}_2^Y := \vee\{e_{0,\ell} : \ell \geq 2\}$. It follows that

$$\mu_{\mathcal{M}_2} = \delta_1 \times (\eta_0)_{\mathcal{M}_2^Y}. \tag{4.7}$$

By Proposition 1.1, we must have $\beta_{01}^2 \left\| \frac{1}{t} \right\|_{L^1((\eta_0)_{\mathcal{M}_2^Y})} = 1$ (otherwise $shift(\beta_{00}, \beta_{01}, ...)$ would not be subnormal). Since

$$\left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}_2})} = \left\| \frac{1}{t} \right\|_{L^1((\eta_0)_{\mathcal{M}_2^Y})} \quad \text{(by (4.7))},$$

we must have $\beta_{01}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}_2})} = 1$, so Proposition 4.5 now implies that $(\mu_{\mathcal{M}_2})_{ext}^X = \xi_1$. On the other hand, (4.7) clearly implies that $(\mu_{\mathcal{M}_2})_{ext}^X = \delta_1$. Thus, ξ_1 is 1-atomic, a contradiction. □

In the following result, we strengthen considerably the necessary conditions (3.3) and (3.4), that is, we require that the Berger measures of all horizontal and vertical

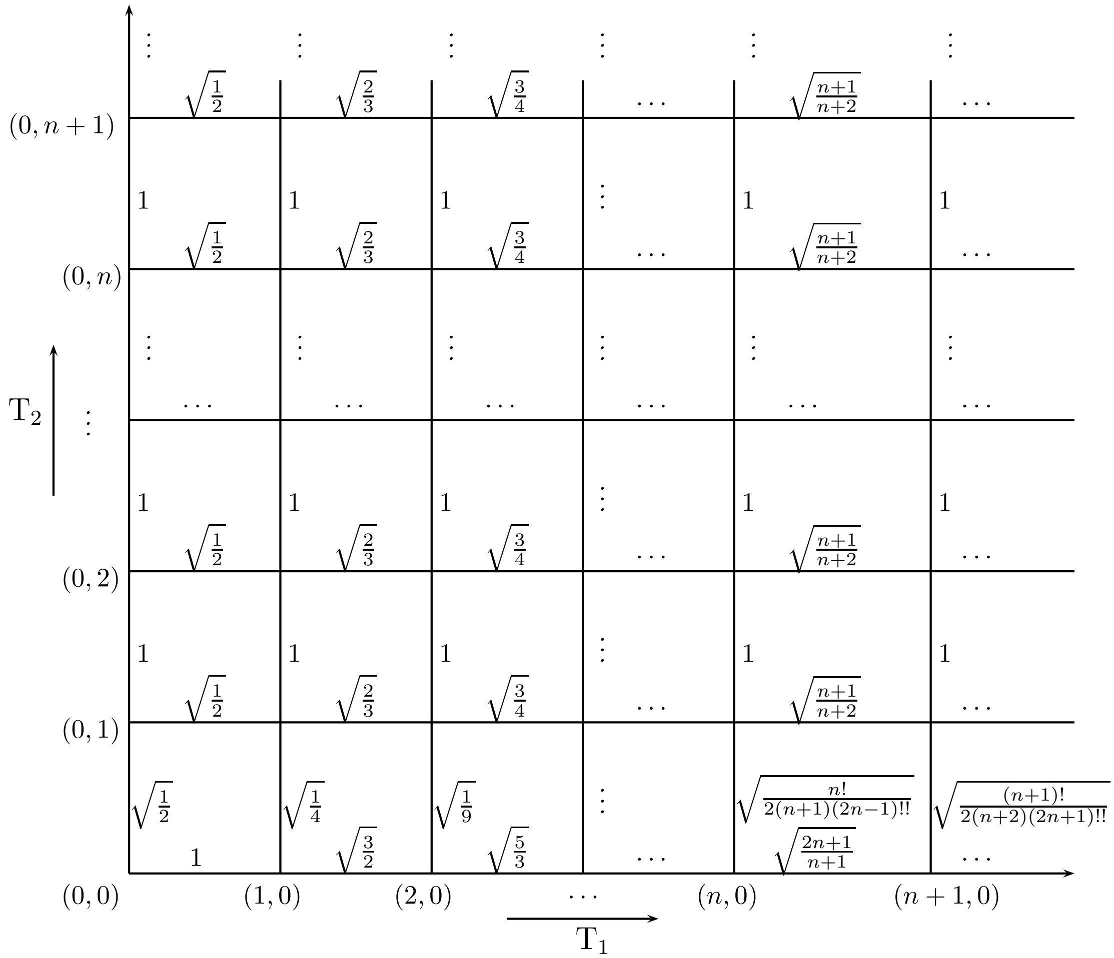

FIGURE 9. Weight diagram of the 2-variable weighted shift in Proposition 4.18

slices be mutually absolutely continuous, but that still does not suffice to yield (joint) subnormality.

Proposition 4.18. *Consider the following 2-variable weighted shift (see Figure 9).*
Then $\mathbf{T} \equiv (T_1, T_2)$ *is commuting, hyponormal, with each of* T_1 *and* T_2 *subnormal, and* $\xi_{j+1} \approx \xi_j$ $(j \geq 0)$ *and* $supp\eta_i = \{0,1\}$ $(i \geq 0)$*, but* $\mathbf{T}$ *is not subnormal.*

Proof. We show in [20] that the Berger measure of $shift(\alpha_{00}, \alpha_{10}, ...)$ is $d\xi_0(s) = \frac{sds}{\pi\sqrt{2s-s^2}}$, and of course $d\xi_j(s) = ds$ for all $j \geq 1$. Thus, $\xi_{j+1} \approx \xi_j$ for all $j \geq 0$. Moreover, $\eta_i = \delta_1$ for all $i \geq 0$. Thus, T_1 and T_2 are subnormal. We also show in

[20] that the spectral picture of $\mathbf{T}$ fails to fulfill the description in [18] of the spectral picture corresponding to a subnormal 2-variable weighted shift. It then follows that $\mathbf{T}$ cannot be subnormal.

We now use the Six-point Test (Theorem 1.4) to show that $\mathbf{T}$ is hyponormal. In this case

$$H(\mathbf{0}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{1}{2}} \\ -\frac{1}{2}\sqrt{\frac{1}{2}} & \frac{1}{2} \end{pmatrix} \geq 0,$$

and for $n \geq 1$, $H((n,0)) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where

$$\begin{aligned} a &:= \frac{2n+3}{n+2} - \frac{2n+1}{n+1} \\ b &:= \sqrt{\frac{n+1}{n+2}}\sqrt{\frac{(n+1)!}{2(n+2)(2n+1)!!}} - \sqrt{\frac{2n+1}{n+1}}\sqrt{\frac{n!}{2(n+1)(2n-1)!!}} \\ c &:= 1 - \frac{n!}{2(n+1)(2n-1)!!}. \end{aligned}$$

After simplification, we have

$$\det H((n,0)) = \frac{2(2+5n+2n^2)(2n-1)!! - (3+8n+5n^2+n^3)n!}{2(n+1)(n+2)^2(2n+1)!!} \geq 0 \ \ (\text{all } n \geq 1).$$

It follow that $\mathbf{T}$ is hyponormal. □

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