SUBNORMALITY OF BERGMAN-LIKE WEIGHTED SHIFTS

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ABSTRACT. For \( a, b, c, d \geq 0 \) with \( ad - bc > 0 \), we consider the unilateral weighted shift \( S(a, b, c, d) \) with weights \( \alpha_n := \sqrt{\frac{an + b}{cn + d}} \) (\( n \geq 0 \)). Using Schur product techniques, we prove that \( S(a, b, c, d) \) is always subnormal; more generally, we establish that for every \( p \geq 1 \), all \( p \)-subshifts of \( S(a, b, c, d) \) are subnormal. As a consequence, we show that all Bergman-like weighted shifts are subnormal.

1. Introduction

Let \( \mathcal{H} \) be a complex Hilbert space and let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of bounded linear operators on \( \mathcal{H} \). We say that \( T \in \mathcal{B}(\mathcal{H}) \) is normal if \( T^*T = TT^* \), subnormal if \( T = N|_{\mathcal{H}} \), where \( N \) is normal and \( N(\mathcal{H}) \subseteq \mathcal{H} \), and hyponormal if \( T^*T \geq TT^* \). For \( k \geq 1 \) \( T \) is \( k \)-hyponormal if \( (I, T, \ldots, T^k) \) is (jointly) hyponormal. Additionally, \( T \) is weakly \( k \)-hyponormal if \( p(T) \) is hyponormal for every polynomial \( p \) of degree at most \( k \). Thus \( k \)-hyponormal \( \Rightarrow \) weakly \( k \)-hyponormal, and “hyponormal”, “1-hyponormal” and “weakly 1-hyponormal” are identical notions ([1]). In particular, each subnormal operator is polynomially hyponormal (i.e., weakly \( k \)-hyponormal for every \( k \geq 1 \)). The converse implication, whether \( T \) polynomially hyponormal \( \Rightarrow \) \( T \) subnormal, was settled in the negative in [11]; indeed, it was shown that there exists a polynomially hyponormal operator which is not \( 2 \)-hyponormal. Previously, S. McCullough and V. Paulsen had established [14] that one can find a non-subnormal polynomially hyponormal operator if and only if one can find a unilateral weighted shift with the same property. Thus, although the existence proof in [11] is abstract, by combining the results in [11] and [14] we now know that there exists a polynomially hyponormal unilateral weighted shift which is not subnormal. The following diagram gives a simple representation of the above mentioned relations:

\[
\begin{array}{cccccc}
\text{subnormal} & \iff & \text{\( \infty \)-hyponormal} & \iff & \cdots & \iff & \text{3-hyponormal} & \iff & \text{2-hyponormal} \\
\downarrow & \iff & \downarrow & \iff & \downarrow & \iff & \downarrow & \iff & \text{hypo.}
\end{array}
\]

\[
\begin{array}{cccccc}
\text{polyn. hypo.} & \iff & \text{weakly \( \infty \)-hypo.} & \iff & \cdots & \iff & \text{weakly 3-hypo.} & \iff & \text{weakly 2-hypo.}
\end{array}
\]

For \( \alpha \equiv \{\alpha_n\}_{n=0}^{\infty} \) a bounded sequence of positive real numbers (called weights), let \( W_\alpha : \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+) \) be the associated unilateral weighted shift, defined by \( W_\alpha e_n := \alpha_n e_{n+1} \) (all \( n \geq 0 \)), where \( \{e_n\}_{n=0}^{\infty} \) is the canonical orthonormal basis in \( \ell^2(\mathbb{Z}_+) \). The moments of \( \alpha \) are given as

\[
\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cl}
1 & \text{if } k = 0 \\
\frac{1}{\alpha_0 \cdot \cdots \cdot \alpha_{k-1}} & \text{if } k > 0
\end{array} \right.
\]

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It is easy to see that $W_{\alpha}$ is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [4, III.8.16]): $W_{\alpha}$ is subnormal if and only if there exists a probability measure $\xi$ supported in $[0,[|W_{\alpha}|]^2]$ (called the Berger measure of $W_{\alpha}$) such that $\gamma_n(\alpha) := \alpha_0^2 \cdots \alpha_{n-1}^2 = \int t^n d\xi(t) \quad (n \geq 1)$. If $W_{\alpha}$ is subnormal, and if for $h \geq 1$ we let $M_h := \sqrt{\{e_n : n \geq h\}}$ denote the invariant subspace obtained by removing the first $h$ vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_{\alpha}[M_h]$ is $\frac{1}{\gamma_n} t^h d\xi(t)$. We will often write $\text{shift}(\alpha_0, \alpha_1, \alpha_2, \cdots)$ to denote the weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$. We also denote by $U_+ := \text{shift}(1,1,1,\cdots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := \text{shift}(a,1,1,\cdots)$.

2. Main Results

For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their Schur product, i.e., $(A \circ B)_{ij} := A_{ij} B_{ij} \quad (1 \leq i, j \leq n)$. The following result is well known: If $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ ([16]).

We are now ready to introduce the class of Bergman-like weighted shifts.

**Definition 2.1. ([12])** For $\ell \geq 1$ and $n \geq 0$, let $\alpha^{(\ell)}_n := \sqrt{\frac{\ell}{n+2}}$ and let $B^{(\ell)}_+ := \text{shift}(\alpha^{(\ell)}_0, \alpha^{(\ell)}_1, \alpha^{(\ell)}_2, \cdots)$. In particular, $B^{(1)}_+ \equiv B_+ := \text{shift}(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots)$. Each $B^{(\ell)}_+$ is called a Bergman-like weighted shift.

**Remark 2.2.** (i) $B_+$ is subnormal with Berger measure $d\xi(s) := ds$ on $[0,1]$.
(ii) $B^{(2)}_+$ is subnormal with Berger measure $d\xi(s) := \frac{ds}{\pi\sqrt{2s-s^2}}$ on $[0,1]$.

**Lemma 2.3. ([6])** Let $W_{\alpha}e_i = \alpha_i e_{i+1}$ ($i \geq 0$) be a hyponormal weighted shift, and let $k \geq 1$. The following statements are equivalent:
(i) $W_{\alpha}$ is $k$-hyponormal.
(ii) The matrix

\[
([W_{\alpha}^{*j}, W_{\alpha}^i]e_{n+j}, e_{n+i})_{i,j=1}^k
\]

is positive semi-definite for all $n \geq -1$.
(iii) The matrix

\[
(\gamma_n \gamma_{n+i+j} - \gamma_{n+i} \gamma_{n+j})_{i,j=1}^k
\]

is positive semi-definite for all $n \geq 0$, where as usual $\gamma_0 = 1$, $\gamma_n = \alpha_0^2 \cdots \alpha_{n-1}^2 \quad (n \geq 1)$.
(iv) The Hankel matrix

\[
H(k;n) := (\gamma_{n+i+j-2})_{i,j=1}^{k+1}
\]

is positive semi-definite for all $n \geq 0$.

Symbolic manipulation easily implies the following result.

**Theorem 2.4.** All Bergman-like shifts $B^{(\ell)}_+$ (all $\ell \geq 1$) are $4$-hyponormal.

**Proof.** By Lemma 2.3, to check $k$-hyponormality it suffices to prove that the determinant of the Hankel matrix $H(k;n)$ in Lemma 2.3(iv) is positive for all $n \geq 0$. 

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For $k = 2$, and all $n \geq 0$, we have
\[
\det H(2; n) = \gamma_n^3 \det \begin{pmatrix}
1 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\alpha_n^2 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\alpha_n^2 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\alpha_n^2 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\end{pmatrix}
\]
\[
= \gamma_n^3 \frac{2(\ell+1)((n+2)\ell-1)^2((n+3)\ell-1)}{(n+2)(n+3)(n+4)(n+5)} \geq 0.
\]

When $k = 3$,
\[
\det H(3; n) = \gamma_n^4 \frac{12(\ell+1)^2((n+2)\ell-1)^3((n+3)\ell-1)^2((n+4)\ell-1)}{(n+2)^2(n+3)(n+4)(n+5)(n+6)(n+7)} > 0
\]
Finally, if $k = 4$, we see that
\[
\det H(4; n) = \gamma_n^5 \frac{288(\ell+1)^3((n+2)\ell-1)^4((n+3)\ell-1)^3((n+4)\ell-1)^2((n+5)\ell-1)}{(n+2)^2(n+3)(n+4)(n+5)(n+6)(n+7)(n+8)(n+9)} > 0
\]
It follows that $H(4; n) \geq 0$ for all $n \geq 0$, as desired.

For $k \geq 1$, we observe that
\[
\det H(k; n) = \gamma_k^{k+1} \alpha_n^{2k} \alpha_{n+1}^{2k-2} \cdots \alpha_n^{2} \alpha_{n+1}^{2}
\]
\[
\det \left( \begin{array}{cccc}
\alpha_n^2 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\alpha_n^2 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\alpha_n^2 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\alpha_n^2 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 & \alpha_n^2 \alpha_{n+1}^2 \\
\end{array} \right)
\]
\[
= \gamma_n^k \frac{288(\ell+1)^3((n+2)\ell-1)^4((n+3)\ell-1)^3((n+4)\ell-1)^2((n+5)\ell-1)}{(n+2)^2(n+3)(n+4)(n+5)(n+6)(n+7)(n+8)(n+9)} > 0.
\]
Thus, to check the positivity of $\det H(k; n)$ is generally quite complicated. Also, it appears that $\det H(k; n)$ is related to the determinant of the Hilbert matrix (after performing column operations and substituting $\alpha_n^2$ by $\ell - \frac{1}{n+2}$). We conclude that a new idea is needed, to bypass the use of nested Determinants [2], which we now present.

We introduce a new class of weighted shifts that includes the class of Bergman-like weighted shifts.

**Definition 2.5.** Let $a, b, c, d \geq 0$ satisfy $ad - bc > 0$. Let $S(a, b, c, d) := shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$, where $\alpha_n := \sqrt{n + \frac{b^2}{cn+1}} \ (n \geq 0)$.

**Remark 2.6.** Note that for a Bergman-like weighted shift $B^{(\ell)}_+$, we have
\[
\alpha_n = \sqrt{\ell - \frac{1}{n+2}} = \sqrt{\frac{\ell n + 2\ell - 1}{n+2}} \ (n \geq 0).
\]

Therefore, $B^{(\ell)}_+ = S(\ell, 2\ell - 1, 1, 2)$ and $ad - bc = 1$.

**Theorem 2.7.** Let $a, b, c, d \geq 0$ satisfy $ad - bc > 0$. Then $S(a, b, c, d)$ is subnormal.

**Proof.** Recall that for $n \geq 0$, $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$. Then the moments of $\alpha$ are $\gamma_0 = 1$ and $\gamma_n = \alpha_n^2 \cdots \alpha_{n-1}^2 \ (n \geq 1)$. By the Bram-Halmos characterization of subnormality [10, Proposition 1.9]) and Lemma 2.3 ($i) \iff iv$), we only need to show that the Hankel matrix $(\gamma_{n+i+j-2})_{i,j=1}^{k+1}$ is positive semi-definite for all $n \geq 0$ and $k \geq 1$. For $n \geq 0$ and $k \geq 1$, let $\beta_n^k := \frac{\gamma_{n+k}}{\gamma_n}$ and $L(k; n) := \left( \beta_n^k \right)_{i,j=1}^{k+1}$. Since
\(H(k;n) = \gamma_n L(k;n)\), it suffices to show that \(L(k;n)\) is positive semi-definite for all \(n \geq 0\) and \(k \geq 1\). We prove this by induction on \(k \geq 1\). For \(k = 1\), \(L(1;n) = \left( \begin{array}{c}
\frac{1}{\alpha_n^2} \\
\alpha_n^2 \alpha_{n+1}^2 \end{array} \right)\). Since

\[
\det L(1;n) = \frac{(an + b)(ad - bc)}{(c(n + 1) + d)(cn + d)^2} > 0,
\]

it follows that \(L(1;n)\) is positive semi-definite. For \(k > 1\), let

\[
Q(k;n) := \left( \begin{array}{cccc}
1 & -\alpha_n^2 & & \\
0 & 1 & -\alpha_{n+1}^2 & \\
& & \ddots & \\
& & & 1 - \alpha_{n+k-1}^2
\end{array} \right).
\]

Then \(Q(k;n)^T L(k;n)Q(k;n) = 1 \oplus [L(k-1;n) \circ B(k;n)]\), where \(B(k;n) := (b_{ij})_{i,j=1}^n\), with

\[
b_{ij} := \alpha_{n+i+j-2}^2 (\alpha_{n+i+j-1}^2 - \alpha_{n+j-1}^2) - \alpha_{n+i-1}^2 (\alpha_{n+i+j-2}^2 - \alpha_{n+j-1}^2).
\]

Note that the \((i, j)\) entry of \(B(k;n)\) corresponds to the \((i + 1, j + 1)\) entry of \(Q(k;n)^T L(k;n)Q(k;n)\). Since by induction hypothesis we know that \(L(k-1;n)\) is positive semi-definite, it remains to show that \(B(k;n)\) is positive semi-definite for all \(k, n \geq 1\). By direct computation we have

\[
b_{ij} = [(c(n + i - 1) + d)(c(n + j - 1) + d)(c(n + i + j - 2) + d)(c(n + i + j - 1) + d)]^{-1}
\]

\[
\cdot [(ad - bc)(a - b)(c - d) + (bc + ad - 2ac)n + acn^2]
\]

\[
+ (acn + bc - ac)(i + j) + (ac - bc + ad)ij].
\]

Therefore, we can write

\[
B(k;n) = (ad - bc)D((c_{ij}) \circ \left( \frac{1}{c(n + i + j - 2) + d} \right) \circ \left( \frac{1}{c(n + i + j - 1) + d} \right))D,
\]

where \(D\) is the diagonal matrix with diagonal entry \((\frac{1}{c(n+i-1)+d})\) and

\[
c_{ij} := (a - b)(c - d) + (bc + ad - 2ac)n + acn^2 + (acn + bc - ac)(i + j) + (ac - bc + ad)ij.
\]

Now observe that \((\frac{1}{c(n+i+j-2)+d})_{i,j=1}^{k+1} \geq 0\) (by [15, Example 18.A2]), since \(c(n + i + j - 2) + d = x_i + x_j\), where \(x_i := c(\frac{n}{2} + i - 1) + \frac{d}{2}\) is positive and increasing in \(i\). Similarly, \((\frac{1}{c(n+i+j-1)+d})_{i,j=1}^{k+1} \geq 0\).

We will now show that \(C := (c_{ij})_{i,j=1}^{k+1}\) is positive semi-definite with positive diagonal. Let

\[
P := \left( \begin{array}{cccc}
1 & -1 & & \\
0 & 1 & -1 & \\
& & \ddots & \\
& & & 1 - 1
\end{array} \right).
\]

Then \(P^TCP\) has \(bd + bcn + adn + acn^2\) at the \((1, 1)\) position, \(acn + ad\) at \((1, j)\) and \((i, 1)\) positions \((i, j > 1)\), and \((c + d) - bc\) elsewhere. Therefore, \(C\) is a positive semi-definite matrix of rank 2. For \(i \geq 1\), the \(i\)-th diagonal entry of \(C\) is

\[
c_{ii} = (ac - bc + ad)x^2 + 2c(a(n - 1) + b)i + (a(n - 1) + b)(c(n - 1) + d) > 0.
\]
By Schur’s Theorem [15, Theorem 9.1.5], \( C \circ M \) is positive semi-definite for every positive semi-definite matrix \( M \). By using this in (2.1), we conclude that \( B(k; n) \geq 0 \), as desired.

**Corollary 2.8.** All Bergman-like shifts \( B_{\ell}^{(t)} \) are subnormal.

**Definition 2.9.** Suppose \( \alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots) \) and \( p \) is a positive integer. A subsequence \( \beta = (\beta_0, \beta_1, \beta_2, \cdots) \) is called a \( p \)-subsequence of \( \alpha \) if there exists \( 0 \leq r < p \) such that \( \beta_n = \alpha_{pn+r} \). The operator shift \((\beta_0, \beta_1, \beta_2, \cdots)\) is called a \( p \)-subshift of shift \((\alpha_0, \alpha_1, \alpha_2, \cdots)\).

**Example 2.10.** (i) The only 1-subsequence of \( \alpha \) is \( \alpha \) itself.
(ii) The 2-subsequences of \( \alpha \) are \( \alpha_{\text{even}} := \{ \alpha_{2n} : n \geq 0 \} \) and \( \alpha_{\text{odd}} := \{ \alpha_{2n+1} : n \geq 0 \} \).

The following examples show that a 2-subshift of a subnormal weighted shift may not be subnormal. To this end, we consider recursively generated weighted shifts [8], [9]. We briefly recall some key facts about these shifts, specifically the case when there are two coefficients of recursion. In [18], J. Stampflii proved that given three positive numbers \( \sqrt{a} \leq \sqrt{b} \leq \sqrt{c} \), it is always possible to find a subnormal weighted shift, denoted \( W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})} \), whose first three weights are \( \sqrt{a}, \sqrt{b}, \sqrt{c} \). In this case, the coefficients of recursion (cf. [8, Example 3.12], [9, Section 3], [7, Section 1, p. 81]) are given by

\[
\varphi_0 = -\frac{ab(c-b)}{b-a} \quad \text{and} \quad \varphi_1 = \frac{b(c-a)}{b-a},
\]

(2.2)
the atoms \( t_0 \) and \( t_1 \) are the roots of the equation

\[
t^2 - (\varphi_0 + \varphi_1 t) = 0,
\]

(2.3)
and the densities \( \rho_0 \) and \( \rho_1 \) uniquely solve the \( 2 \times 2 \) system of equations

\[
\begin{cases}
\rho_0 + \rho_1 = 1 \\
\rho_0 t_0 + \rho_1 t_1 = \alpha_2
\end{cases}
\]

(2.4)
Thus, we get \( \mu = \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1} \), which is the Berger measure of \( W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})} \).

**Example 2.11.** For \( a = \frac{1}{4}, b = \frac{1}{3}, c = \frac{1}{2} \), the Berger measure of \( W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})} \) is

\[
\mu = \frac{2 + \sqrt{3}}{4} \delta_{\frac{1}{2}(1-\frac{1}{\sqrt{3}})} + \frac{2 - \sqrt{3}}{4} \delta_{\frac{1}{2}(1+\frac{1}{\sqrt{3}})}.
\]

Thus, \( W_\alpha \) is subnormal, but \( W_{\alpha_{\text{even}}} \) is not subnormal.

**Proof.** We have

\[
\det H(2; 0) = \det \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{32} & \frac{1}{128} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{32} & \frac{1}{128} & \frac{1}{512} \\ \frac{1}{8} & \frac{1}{32} & \frac{1}{128} & \frac{1}{512} & \frac{1}{2048} \\ \frac{1}{32} & \frac{1}{128} & \frac{1}{512} & \frac{1}{2048} & \frac{1}{8192} \\ \frac{1}{128} & \frac{1}{512} & \frac{1}{2048} & \frac{1}{8192} & \frac{1}{32768} \end{pmatrix} = -\frac{1}{3584} < 0.
\]

Therefore, \( W_{\alpha_{\text{even}}} \) is not 2-hyponormal which implies \( W_{\alpha_{\text{even}}} \) is not subnormal.

**Example 2.12.** Let

\[
\alpha \equiv \alpha_n := \begin{cases} \sqrt{n}, & \text{if } n = 0 \\ \sqrt{2n+1}, & \text{if } n \geq 1 \end{cases}
\]

Then \( W_\alpha \) is subnormal, but \( W_{\alpha_{\text{even}}} \) is not subnormal.
Theorem 2.14. Suppose \( \alpha_n \) are subnormal. Thus, \( W \) is not subnormal; hence, \( W_\beta \) is not subnormal.

Proof. \( W_\alpha \) is subnormal: Consider the 3-atomic measure \( \xi := \frac{1}{3} \delta_0 + \frac{1}{3} \delta_1 + \frac{1}{3} \delta_1 \). For \( n \geq 1 \),

\[
\gamma_n = \alpha_0^2 \alpha_1^2 \cdots \alpha_{n-2}^2 \alpha_{n-1}^2 \equiv \frac{1}{2} \left( 2 + \frac{1}{2} \right) \frac{2^2 + \frac{1}{2}}{2^2 + 1} \cdots \frac{2^n - 1}{2^n + 1} \cdot \frac{2^n - 1}{2^n + 1} = \frac{1}{2} \frac{2^n}{3} \cdot (1 + 2^{-n}) = \frac{1}{3} (2^n + 1) = \frac{1}{3} \cdot (\frac{1}{2})^n + \frac{1}{3} = \int s^n \xi(s),
\]

which shows that \( \xi \) is the Berger measure of \( W_\alpha \). Therefore, \( W_\alpha \) is subnormal.

\( W_\beta \) is not subnormal: Let

\[
\tilde{\gamma}_n = \beta_0^2 \beta_1^2 \cdots \beta_{n-2}^2 \beta_{n-1}^2 \equiv \alpha_0^2 \alpha_1^2 \cdots \alpha_{2n-4}^2 \alpha_{2n-2}^2 \equiv \frac{1}{2} \frac{2^n}{3} \frac{1}{3} \left( 2 + \frac{1}{2} \right) \frac{2^2 + \frac{1}{2}}{2^2 + 1} \cdots \frac{2^n - 1}{2^n + 1} \cdot \frac{2^n - 1}{2^n + 1} = \frac{1}{2} \frac{2^n}{3} \cdot (1 + 2^{-n}) = \frac{1}{3} (2^n + 1) = \frac{1}{3} \cdot (\frac{1}{2})^n + \frac{1}{3} = \int s^n \xi(s),
\]

and consider \( \tilde{H}(k; n) := (\tilde{\gamma}_{n+i+j-2}^{k+1})_{i,j=1}^{k+1} \) \( (n \geq 0) \). For \( k = 2 \), we have

\[
\det \tilde{H}(2; n) = \frac{3^n}{n} \det \left( \begin{array}{cccc}
\alpha_2^2 & \alpha_2^2 & \alpha_2^2 & \alpha_2^2 \\
\alpha_2^2 & \alpha_2^2 & \alpha_2^2 & \alpha_2^2 \\
\alpha_2^2 & \alpha_2^2 & \alpha_2^2 & \alpha_2^2 \\
\alpha_2^2 & \alpha_2^2 & \alpha_2^2 & \alpha_2^2 \\
\end{array} \right) = \frac{3^n}{n} \frac{1}{(1 + 4n + 2^2 (1 + 2^{n+1}) (1 + 2^{n+1}) s)} < 0.
\]

Thus, \( W_\beta \) is not 2-hyponormal; hence, \( W_\beta \) is not subnormal. \( \square \)

Theorem 2.13. Suppose \( a, b, c, d \geq 0 \) satisfy \( ad - bc > 0 \). Then for \( p \geq 1 \), all \( p \)-subshifts of \( S(a, b, c, d) \) are subnormal.

Proof. Suppose \( \beta_n = \alpha_{pn+r} \) for some \( 0 \leq r < p \). Since

\[
\begin{align*}
\frac{a(pn + r) + b}{c(pn + r) + d} &= \frac{(ap)n + (ar + b)}{(cp)n + (cr + d)} \\
\end{align*}
\]

and

\[
(ap)(cr + d) - (ar + b)(cp) = p(ad - bc) > 0,
\]

it follows that \( \text{shift}(\beta) = S(ap, ar + b, cp, cr + d) \) is also subnormal. \( \square \)

Theorem 2.14. The 2-subsequences \( \{\alpha_{2n} : n \geq 0\} \) and \( \{\alpha_{2n+1} : n \geq 0\} \) of \( B_+ \) are subnormal with \( dp(s) = \frac{ds}{\pi \sqrt{s-s^2}} \) and \( dv(s) = \frac{ds}{2 \sqrt{1-s}} \) respectively.

Proof. Case 1: Let \( W_{\alpha_{2n}} := \text{shift}(\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{7}{8}}, \ldots) \) and consider the “\( \gamma \)” numbers of \( W_{\alpha_{2n}} \), that is, \( \gamma_n = \frac{(2n-1)!!}{n!} \) \( (n \geq 1) \). Using Berger’s Theorem, we want to find the Berger measure of \( W_{\alpha_{2n}} \). Let
\( d\mu_{\alpha_{2n}}(s) := \frac{ds}{\pi \sqrt{s-2}}, s \neq 0, 1. \) Then
\[
\int_{0}^{1} d\mu(s) = \int_{0}^{1} \frac{ds}{\pi \sqrt{\frac{1}{4} - (s-\frac{1}{2})^2}} = \frac{1}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dy}{\sqrt{\frac{1}{4} - y^2}} \quad \text{(by letting } y := s - \frac{1}{2})
\]
\[
= \frac{2}{\pi} [\sin^{-1} y]_{-\frac{1}{2}}^{\frac{1}{2}} = 1.
\]
Thus, \( \mu \) is a probability measure. Let \( s := \sin^2 x \left(= \frac{1-\cos 2x}{2}\right) \). Then
\[
ds = 2 \sin x \cos x \,dx = \sqrt{1 - (1-2s)^2} \,dx.
\]
Thus
\[
\int_{0}^{1} s^n d\mu(s) = \int_{0}^{1} s^n \frac{ds}{\pi \sqrt{s-2}} = \frac{2}{\pi} \int_{0}^{1} s^n \frac{ds}{\sqrt{1 - (1-2s)^2}}
\]
\[
= \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \sin^2 n \,dx = \frac{(2n-1)!!}{2n!} = \gamma_n.
\]
Therefore, \( W_{\alpha_{2n}} \) is subnormal with \( d\mu(s) = \frac{ds}{\pi \sqrt{s-2}} \).

**Case 2:** Let \( W_{\alpha_{2n+1}} := \text{shift}(\sqrt{\frac{2}{3}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{6}{7}}, \sqrt{\frac{8}{9}}, \cdots) \) and consider the “\( \gamma \)” numbers of \( W_{\alpha_{2n+1}} \), that is, \( \gamma_n = \frac{2n!}{(2n+1)!!} \) (all \( n \geq 1 \)). Let \( d\nu(s) := \frac{ds}{2\sqrt{1-s}} \) (\( s \neq 1 \)). Then \( \int_{0}^{1} d\nu(s) = 1 \). Thus, \( \nu \) is also probability measure. Let \( s := \sin^2 x \), then \( ds = 2 \sin x \cos x \,dx \) and \( \cos x = \sqrt{1-s} \). Thus
\[
\int_{0}^{1} s^n \,d\nu(s) = \int_{0}^{1} s^n \frac{ds}{2\sqrt{1-s}} = \int_{0}^{\frac{\pi}{2}} \sin^2 n \,dx \frac{2 \sin x \cos x \,dx}{2 \cos x} = \int_{0}^{\frac{\pi}{2}} \sin^{2n+1} x \,dx
\]
\[
= \frac{2n!}{(2n+1)!!} = \gamma_n.
\]
Therefore, \( W_{\alpha_{2n+1}} \) is also subnormal with \( d\nu(s) = \frac{ds}{2\sqrt{1-s}} \). \( \square \)

We conclude this section with a problem of independent interest.

**Problem 2.15.** Recall that \( B_+^{(\ell)} = S(\ell, 2\ell - 1, 1, 2) \), so Theorem 2.13 guarantees that \( B_+^{(\ell)} \) and all of its \( p \)-subshifts are subnormal. For \( \ell \geq 2 \) and \( p \geq 1 \), find the Berger measure of \( B_+^{(\ell)} \) and the Berger measure of its \( p \)-subshifts.

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