SUBNORMALITY OF BERGMAN-LIKE WEIGHTED SHIFTS

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Abstract. For $a, b, c, d \geq 0$ with $ad - bc > 0$, we consider the unilateral weighted shift $S(a, b, c, d)$ with weights $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$ $(n \geq 0)$. Using Schur product techniques, we prove that $S(a, b, c, d)$ is always subnormal; more generally, we establish that for every $p \geq 1$, all p-subshifts of $S(a, b, c, d)$ are subnormal. As a consequence, we show that all Bergman-like weighted shifts are subnormal.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$. For $k \geq 1$ T is *k-hyponormal* if $(I, T, \cdots, T^k)$ is (jointly) hyponormal. Additionally, T is *weakly k-hyponormal* if $p(T)$ is hyponormal for every polynomial p of degree at most k. Thus k-hyponormal $\Rightarrow$ weakly k-hyponormal, and "hyponormal", "1-hyponormal" and "weakly 1-hyponormal" are identical notions ([1]). On the other hand, results in ([10]), ([6]) and ([14]) show that weakly 2-hyponormal operators (also called *quadratically hyponormal* operators) are not necessarily 2-hyponormal. The Bram-Halmos characterization of subnormality ([4, III.1.9]) can be paraphrased as follow: T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([10, Proposition 1.9]). In particular, each subnormal operator is *polynomially hyponormal* (i.e., weakly k-hyponormal for every $k \geq 1$). The converse implication, whether T polynomially hyponormal $\Rightarrow$ T subnormal, was settled in the negative in [11]; indeed, it was shown that there exists a polynomially hyponormal operator which is not 2-hyponormal. Previously, S. McCullough and V. Paulsen had established [14] that one can find a non-subnormal polynomially hyponormal operator if and only if one can find a unilateral weighted shift with the same property. Thus, although the existence proof in [11] is abstract, by combining the results in [11] and [14] we now know that there exists a polynomially hyponormal unilateral weighted shift which is not subnormal. The following diagram gives a simple representation of the above mentioned relations:

$$\begin{array}{cccccc}
\text{subnormal} & \Leftrightarrow \infty\text{-hyponormal} & \Rightarrow \cdots & \Rightarrow 3\text{-hyponormal} & \Rightarrow 2\text{-hyponormal} & \searrow \\
\not\Uparrow\Downarrow & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & \not\Uparrow\Downarrow & \text{hypo.} \\
\text{polyn. hypo.} & \Leftrightarrow \text{weakly } \infty\text{-hypo.} & \Rightarrow \cdots & \Rightarrow \text{weakly 3-hypo.} & \Rightarrow \text{weakly 2-hypo.} & \nearrow
\end{array}$$

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{cases}.$$

1991 *Mathematics Subject Classification.* Primary 47B20, 47B37, 47A13; Secondary 44A60, 47-04, 47A20.
Key words and phrases. Bergman-like weighted shifts, Schur product techniques, p-period subsequences, p-subshifts.
The first named author was partially supported by NSF Grants DMS-0099357 and DMS-0400741.

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \le \alpha_1 \le \cdots$.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [4, III.8.16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$ (called the *Berger measure* of W_α) such that $\gamma_n(\alpha) := \alpha_0^2 \cdot \ldots \cdot \alpha_{n-1}^2 = \int t^n d\xi(t)$ $(n \ge 1)$. If W_α is subnormal, and if for $h \ge 1$ we let $\mathcal{M}_h := \bigvee\{e_n : n \ge h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$.

We will often write $shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$ to denote the weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. We also denote by $U_+ := shift(1,1,1,\cdots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := shift(a,1,1,\cdots)$.

2. Main Results

For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, i.e., $(A \circ B)_{ij} := A_{ij} B_{ij}$ $(1 \le i, j \le n)$. The following result is well known: If $A \ge 0$ and $B \ge 0$, then $A \circ B \ge 0$ ([16]).

We are now ready to introduce the class of Bergman-like weighted shifts.

Definition 2.1. *([12]) For $\ell \ge 1$ and $n \ge 0$, let $\alpha_n^{(\ell)} := \sqrt{\ell - \frac{1}{n+2}}$ and let $B_+^{(\ell)} := shift(\alpha_0^{(\ell)}, \alpha_1^{(\ell)}, \alpha_2^{(\ell)}, \cdots)$. In particular, $B_+^{(1)} \equiv B_+ := shift(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots)$. Each $B_+^{(\ell)}$ is called a Bergman-like weighted shift.*

Remark 2.2. (i) B_+ is subnormal with Berger measure $d\xi(s) := ds$ on $[0,1]$.
(ii) ([12]) $B_+^{(2)}$ is subnormal with Berger measure $d\xi(s) := \frac{s ds}{\pi\sqrt{2s - s^2}}$ on $[0,1]$.

Lemma 2.3. *([6]) Let $W_\alpha e_i = \alpha_i e_{i+1}$ $(i \ge 0)$ be a hyponormal weighted shift, and let $k \ge 1$. The following statements are equivalent:*
(i) W_α is k-hyponormal.
(ii) The matrix

$$(([W_\alpha^{*j}, W_\alpha^i] e_{n+j}, e_{n+i}))_{i,j=1}^k$$

is positive semi-definite for all $n \ge -1$.
(iii) The matrix

$$(\gamma_n \gamma_{n+i+j} - \gamma_{n+i} \gamma_{n+j})_{i,j=1}^k$$

is positive semi-definite for all $n \ge 0$, where as usual $\gamma_0 = 1$, $\gamma_n = \alpha_0^2 \cdot \ldots \cdot \alpha_{n-1}^2$ $(n \ge 1)$.
(iv) The Hankel matrix

$$H(k;n) := (\gamma_{n+i+j-2})_{i,j=1}^{k+1}$$

is positive semi-definite for all $n \ge 0$.

Symbolic manipulation easily implies the following result.

Theorem 2.4. *All Bergman-like shifts $B_+^{(\ell)}$ (all $\ell \ge 1$) are 4-hyponormal.*

Proof. By Lemma 2.3, to check k-hyponormality it suffices to prove that the determinant of the Hankel matrix $H(k;n)$ in Lemma 2.3(iv) is positive for all $n \ge 0$.

For $k=2$, and all $n \geq 0$, we have

$$\det H(2;n) = \gamma_n^3 \det \begin{pmatrix} 1 & \alpha_n^2 & \alpha_n^2\alpha_{n+1}^2 \\ \alpha_n^2 & \alpha_n^2\alpha_{n+1}^2 & \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 \\ \alpha_n^2\alpha_{n+1}^2 & \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 & \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2\alpha_{n+3}^2 \end{pmatrix}$$
$$= \gamma_n^3 \tfrac{2(\ell+1)((n+2)\ell-1)^2((n+3)\ell-1)}{(n+2)^3(n+3)^3(n+4)^2(n+5)} > 0.$$

When $k=3$,

$$\det H(3;n) = \gamma_n^4 \tfrac{12(\ell+1)^2(2\ell+1)((n+2)\ell-1)^3((n+3)\ell-1)^2((n+4)\ell-1)}{(n+2)^4(n+3)^4(n+4)^4(n+5)^3(n+6)^2(n+7)} > 0$$

Finally, if $k=4$, we see that

$$\det H(4;n) = \gamma_n^5 \tfrac{288(\ell+1)^3(2\ell+1)^2(3\ell+1)((n+2)\ell-1)^4((n+3)\ell-1)^3((n+4)\ell-1)^2((n+5)\ell-1)}{(n+2)^5(n+3)^5(n+4)^5(n+5)^5(n+6)^4(n+7)^3(n+8)^2(n+9)} > 0.$$

It follows that $H(4;n) \geq 0$ for all $n \geq 0$, as desired. $\square$

For $k \geq 1$, we observe that

$$\det H(k;n) = \gamma_n^{k+1}\alpha_n^{2k}\alpha_{n+1}^{2k-2}\cdot \ldots \cdot \alpha_{n+k-1}^2 \cdot$$
$$\det \begin{pmatrix} \alpha_{n+1}^2-\alpha_n^2 & \alpha_{n+2}^2-\alpha_{n+1}^2 & \cdots & \alpha_{n+k}^2-\alpha_{n+k-1}^2 \\ \alpha_{n+1}^2(\alpha_{n+2}^2-\alpha_n^2) & \alpha_{n+2}^2(\alpha_{n+3}^2-\alpha_{n+1}^2) & \cdots & \alpha_{n+k}^2(\alpha_{n+k+1}^2-\alpha_{n+k-1}^2) \\ \vdots & \vdots & \ddots & \vdots \\ (\Pi_{i=n+1}^{n+k-1}a_i^2)(\alpha_{n+k}^2-\alpha_n^2) & (\Pi_{i=n+2}^{n+k}a_i^2)(\alpha_{n+k+1}^2-\alpha_{n+1}^2) & \cdots & (\Pi_{i=n+k}^{n+2k-2}a_i^2)(\alpha_{n+2k-1}^2-\alpha_{n+k-1}^2) \end{pmatrix}.$$

Thus, to check the positivity of $\det H(k;n)$ is generally quite complicated. Also, it appears that $\det H(k;n)$ is related to the determinant of the Hilbert matrix (after performing column operations and substituting α_n^2 by $\ell - \frac{1}{n+2}$). We conclude that a new idea is needed, to bypass the use of nested Determinants [2], which we now present.

We introduce a new class of weighted shifts that includes the class of Bergman-like weighted shifts.

Definition 2.5. *Let $a,b,c,d \geq 0$ satisfy $ad-bc>0$. Let $S(a,b,c,d) := shift(\alpha_0,\alpha_1,\alpha_2,\cdots)$, where $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$ $(n \geq 0)$.*

Remark 2.6. Note that for a Bergman-like weighted shift $B_+^{(\ell)}$, we have

$$\alpha_n = \sqrt{\ell - \frac{1}{n+2}} = \sqrt{\frac{\ell n + (2\ell-1)}{n+2}} \ \ (n \geq 0).$$

Therefore, $B_+^{(\ell)} = S(\ell, 2\ell-1, 1, 2)$ and $ad-bc=1$.

Theorem 2.7. *Let $a,b,c,d \geq 0$ satisfy $ad-bc>0$. Then $S(a,b,c,d)$ is subnormal.*

Proof. Recall that for $n \geq 0$, $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$. Then the moments of α are $\gamma_0 = 1$ and $\gamma_n = \alpha_0^2 \cdot \ldots \cdot \alpha_{n-1}^2$ $(n \geq 1)$. By the Bram-Halmos characterization of subnormality [10, Proposition 1.9]) and Lemma 2.3 ($(i) \Leftrightarrow (iv)$), we only need to show that the Hankel matrix $(\gamma_{n+i+j-2})_{i,j=1}^{k+1}$ is positive semi-definite for all $n \geq 0$ and $k \geq 1$. For $n \geq 0$ and $k \geq 1$, let $\beta_k^n := \frac{\gamma_{n+k}}{\gamma_n}$ and $L(k;n) := \left(\beta_{i+j-2}^n\right)_{i,j=1}^{k+1}$. Since

$H(k;n)=\gamma_n L(k;n)$, it suffices to show that $L(k;n)$ is positive semi-definite for all $n\geq 0$ and $k\geq 1$. We prove this by induction on $k\geq 1$. For $k=1$, $L(1;n)=\begin{pmatrix} 1 & \alpha_n^2 \\ \alpha_n^2 & \alpha_n^2\alpha_{n+1}^2 \end{pmatrix}$. Since

$$\det L(1;n)=\frac{(an+b)(ad-bc)}{(c(n+1)+d)(cn+d)^2}>0,$$

it follows that $L(1;n)$ is positive semi-definite. For $k>1$, let

$$Q(k;n):=\begin{pmatrix} 1 & -\alpha_n^2 & & & \\ & 1 & -\alpha_{n+1}^2 & & \\ & & \ddots & \ddots & \\ & & & 1 & -\alpha_{n+k-1}^2 \\ & & & & 1 \end{pmatrix}.$$

Then $Q(k;n)^T L(k;n) Q(k;n)=1\oplus[L(k-1;n)\circ B(k;n)]$, where $B(k;n):=(b_{ij})_{i,j=1}^k$, with

$$b_{ij}:=\alpha_{n+i+j-2}^2(\alpha_{n+i+j-1}^2-\alpha_{n+j-1}^2)-\alpha_{n+i-1}^2(\alpha_{n+i+j-2}^2-\alpha_{n+j-1}^2).$$

Note that the (i,j) entry of $B(k;n)$ corresponds to the $(i+1,j+1)$ entry of $Q(k;n)^T L(k;n) Q(k;n)$. Since by induction hypothesis we know that $L(k-1;n)$ is positive semi-definite, it remains to show that $B(k;n)$ is positive semi-definite for all $k,n\geq 1$. By direct computation we have

$$\begin{aligned} b_{ij} \;=\; & [(c(n+i-1)+d)(c(n+j-1)+d)(c(n+i+j-2)+d)(c(n+i+j-1)+d)]^{-1} \\ & \cdot[(ad-bc)((a-b)(c-d)+(bc+ad-2ac)n+acn^2 \\ & +(acn+bc-ac)(i+j)+(ac-bc+ad)ij)]. \end{aligned}$$

Therefore, we can write

$$B(k;n)=(ad-bc)D((c_{ij})\circ(\frac{1}{c(n+i+j-2)+d})\circ(\frac{1}{c(n+i+j-1)+d}))D, \tag{2.1}$$

where D is the diagonal matrix with diagonal entry $(\frac{1}{c(n+i-1)+d})$ and

$$c_{ij}:=(a-b)(c-d)+(bc+ad-2ac)n+acn^2+(acn+bc-ac)(i+j)+(ac-bc+ad)ij\,.$$

Now observe that $(\frac{1}{c(n+i+j-2)+d})_{i,j=1}^{k+1}\geq 0$ (by [15, Example 18.A2]), since $c(n+i+j-2)+d=x_i+x_j$, where $x_i:=c(\frac{n}{2}+i-1)+\frac{d}{2}$ is positive and increasing in i. Similarly, $(\frac{1}{c(n+i+j-1)+d})_{i,j=1}^{k+1}\geq 0$.

We will now show that $C:=(c_{ij})_{i,j=1}^{k+1}$ is positive semi-definite with positive diagonal. Let

$$P:=\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}.$$

Then P^TCP has $bd+bcn+adn+acn^2$ at the $(1,1)$ position, $acn+ad$ at $(1,j)$ and $(i,1)$ positions $(i,j>1)$, and $a(c+d)-bc$ elsewhere. Therefore, C is a positive semi-definite matrix of rank 2. For $i\geq 1$, the i-th diagonal entry of C is

$$c_{ii}=(ac-bc+ad)i^2+2c(a(n-1)+b)i+(a(n-1)+b)(c(n-1)+d)>0.$$

By Schur's Theorem [15, Theorem 9.I.5], $C \circ M$ is positive semi-definite for every positive semi-definite matrix M. By using this in (2.1), we conclude that $B(k;n) \geq 0$, as desired. $\square$

Corollary 2.8. *All Bergman-like shifts $B_+^{(\ell)}$ are subnormal.*

Definition 2.9. *Suppose $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots)$ and p is a positive integer. A subsequence $\beta = (\beta_0, \beta_1, \beta_2, \cdots)$ is called a p-subsequence of α if there exists $0 \leq r < p$ such that $\beta_n = \alpha_{pn+r}$. The operator $shift(\beta_0, \beta_1, \beta_2, \cdots)$ is called a p-subshift of $shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$.*

Example 2.10. (i) The only 1-subsequence of α is α itself.
(ii) The 2-subsequences of α are $\alpha_{even} := \{\alpha_{2n} : n \geq 0\}$ and $\alpha_{odd} := \{\alpha_{2n+1} : n \geq 0\}$.

The following examples show that a 2-subshift of a subnormal weighted shift may not be subnormal. To this end, we consider *recursively generated weighted shifts* [8], [9]. We briefly recall some key facts about these shifts, specifically the case when there are two coefficients of recursion. In [18], J. Stampfli proved that given three positive numbers $\sqrt{a} < \sqrt{b} < \sqrt{c}$, it is always possible to find a subnormal weighted shift, denoted $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, whose first three weights are $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$. In this case, the coefficients of recursion (cf. [8, Example 3.12], [9, Section 3], [7, Section 1, p. 81]) are given by

$$\varphi_0 = -\frac{ab(c-b)}{b-a} \text{ and } \varphi_1 = \frac{b(c-a)}{b-a}, \tag{2.2}$$

the atoms t_0 and t_1 are the roots of the equation

$$t^2 - (\varphi_0 + \varphi_1 t) = 0, \tag{2.3}$$

and the densities ρ_0 and ρ_1 uniquely solve the 2×2 system of equations

$$\left\{ \begin{array}{rcl} \rho_0 + \rho_1 & = & 1 \\ \rho_0 t_0 + \rho_1 t_1 & = & \alpha_0^2. \end{array} \right. \tag{2.4}$$

Thus, we get $\mu = \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1}$ which is the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$.

Example 2.11. For $a = \frac{1}{4}, b = \frac{1}{3}, c = \frac{1}{2}$, the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is

$$\mu = \frac{2+\sqrt{3}}{4}\delta_{\frac{1}{2}(1-\frac{1}{\sqrt{3}})} + \frac{2-\sqrt{3}}{4}\delta_{\frac{1}{2}(1+\frac{1}{\sqrt{3}})}.$$

Thus, W_α is subnormal, but $W_{\alpha_{even}}$ is not subnormal.

Proof. We have

$$\det H(2;0) = \det \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{3}{32} \\ \frac{1}{8} & \frac{3}{32} & \frac{33}{448} \end{pmatrix} = -\frac{1}{3584} < 0.$$

Therefore, $W_{\alpha_{even}}$ is not 2-hyponormal which implies $W_{\alpha_{even}}$ is not subnormal. $\square$

Example 2.12. Let

$$\alpha \equiv \alpha_n := \left\{ \begin{array}{ll} \sqrt{\frac{1}{2}}, & \text{if } n = 0 \\ \sqrt{\frac{2^n + \frac{1}{2}}{2^n + 1}}, & \text{if } n \geq 1 \end{array} \right.,$$

Then W_α is subnormal, but $W_{\alpha_{even}}$ is not subnormal.

Proof. W_α **is subnormal:** Consider the 3-atomic measure $\xi := \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{\frac{1}{2}} + \frac{1}{3}\delta_1$. For $n \geq 1$,

$$\begin{aligned}
\gamma_n &\equiv \alpha_0^2\alpha_1^2\alpha_2^2\cdots\alpha_{n-2}^2\alpha_{n-1}^2 \\
&= \frac{1}{2}\cdot\frac{2+\frac{1}{2}}{2+1}\cdot\frac{2^2+\frac{1}{2}}{2^2+1}\cdot\ldots\cdot\frac{2^{n-2}+\frac{1}{2}}{2^{n-2}+1}\cdot\frac{2^{n-1}+\frac{1}{2}}{2^{n-1}+1} \\
&= \frac{1}{2}\cdot\frac{\frac{2^2+1}{2^2}}{\frac{2+1}{2}}\cdot\frac{\frac{2^3+1}{2^3}}{\frac{2^2+1}{2^2}}\cdot\ldots\cdot\frac{\frac{2^{n-1}+1}{2^{n-1}}}{\frac{2^{n-2}+1}{2^{n-2}}}\cdot\frac{\frac{2^n+1}{2^n}}{\frac{2^{n-1}+1}{2^{n-1}}} \\
&= \frac{1}{2}\cdot\frac{2}{3}\cdot(1+2^{-n}) = \frac{1}{3}(2^{-n}+1) = \frac{1}{3}\cdot(\frac{1}{2})^n + \frac{1}{3} = \int s^n d\xi(s),
\end{aligned} \tag{2.5}$$

which shows that ξ is the Berger measure of W_α. Therefore, W_α is subnormal.
W_{β_n} **is not subnormal:** Let

$$\begin{aligned}
\tilde{\gamma}_n &\equiv \beta_0^2\beta_1^2\beta_2^2\cdots\beta_{n-2}^2\beta_{n-1}^2 \\
&\equiv \alpha_0^2\alpha_2^2\alpha_4^2\cdots\alpha_{2n-4}^2\alpha_{2n-2}^2,
\end{aligned} \tag{2.6}$$

and consider $\tilde{H}(k;n) := (\tilde{\gamma}_{n+i+j-2})_{i,j=1}^{k+1}$ $(n \geq 0)$. For $k = 2$, we have

$$\begin{aligned}
\det \tilde{H}(2;n) &= \tilde{\gamma}_n^3 \det\begin{pmatrix} 1 & \alpha_{2n}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2 \\ \alpha_{2n}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2\alpha_{2(n+2)}^2 \\ \alpha_{2n}^2\alpha_{2(n+1)}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2\alpha_{2(n+2)}^2 & \alpha_{2n}^2\alpha_{2(n+1)}^2\alpha_{2(n+2)}^2\alpha_{2(n+3)}^2 \end{pmatrix} \\
&= \tilde{\gamma}_n^3 \frac{-135\cdot 2^{6n-1}(1+2^{2n+1})^2(1+2^{2n+3})}{(1+4^{n+2})^2(1+4^{n+3})(1+5\cdot 4^n+4^{2n+1})^3} < 0.
\end{aligned}$$

Thus, W_β is not 2-hyponormal; hence, W_β is not subnormal. □

Theorem 2.13. *Suppose $a, b, c, d \geq 0$ satisfy $ad - bc > 0$. Then for $p \geq 1$, all p-subshifts of $S(a, b, c, d)$ are subnormal.*

Proof. Suppose $\beta_n = \alpha_{pn+r}$ for some $0 \leq r < p$. Since

$$\frac{a(pn+r)+b}{c(pn+r)+d} = \frac{(ap)n+(ar+b)}{(cp)n+(cr+d)}$$

and

$$(ap)(cr+d) - (ar+b)(cp) = p(ad-bc) > 0,$$

it follows that $shift(\beta) = S(ap, ar+b, cp, cr+d)$ is also subnormal. □

Theorem 2.14. *The 2-subsequences $\{\alpha_{2n} : n \geq 0\}$ and $\{\alpha_{2n+1} : n \geq 0\}$ of B_+ are subnormal with $d\mu(s) = \frac{ds}{\pi\sqrt{s-s^2}}$ and $d\nu(s) = \frac{ds}{2\sqrt{1-s}}$ respectively.*

Proof. **Case 1 :** Let $W_{\alpha_{2n}} := shift(\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{7}{8}}, \cdots)$ and consider the "γ" numbers of $W_{\alpha_{2n}}$, that is, $\gamma_n = \frac{(2n-1)!!}{n!}$ (all $n \geq 1$). Using Berger's Theorem, we want to find the Berger measure of $W_{\alpha_{2n}}$. Let

$d\mu_{\alpha_{2n}}(s) := \frac{ds}{\pi\sqrt{s-s^2}}, s \neq 0, 1$. Then

$$\begin{aligned}\int_0^1 d\mu(s) &= \int_0^1 \frac{ds}{\pi\sqrt{\frac{1}{4}-(s-\frac{1}{2})^2}} = \frac{1}{\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dy}{\sqrt{\frac{1}{4}-y^2}} .. (\text{by letting } y := s - \frac{1}{2}) \\ &= \frac{2}{\pi}[\sin^{-1} y]_{-\frac{1}{2}}^{\frac{1}{2}} = 1.\end{aligned}$$

Thus, μ is a probability measure. Let $s := \sin^2 x$ $(= \frac{1-\cos 2x}{2})$. Then

$$ds = 2\sin x \cos x dx = \sqrt{1-(1-2s)^2}dx.$$

Thus

$$\begin{aligned}\int_0^1 s^n d\mu(s) &= \int_0^1 s^n \frac{ds}{\pi\sqrt{s-s^2}} = \frac{2}{\pi}\int_0^1 s^n \frac{ds}{\sqrt{1-(1-2s)^2}} \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)!!}{2n!} = \gamma_n.\end{aligned}$$

Therefore, $W_{\alpha_{2n}}$ is subnormal with $d\mu(s) = \frac{ds}{\pi\sqrt{s-s^2}}$.

Case 2 : Let $W_{\alpha_{2n+1}} := shift(\sqrt{\frac{2}{3}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{6}{7}}, \sqrt{\frac{8}{9}}, \cdots)$ and consider the "γ" numbers of $W_{\alpha_{2n+1}}$, that is, $\gamma_n = \frac{2n!}{(2n+1)!!}$ (all $n \geq 1$). Let $d\nu(s) := \frac{ds}{2\sqrt{1-s}}$ $(s \neq 1)$. Then $\int_0^1 d\nu(s) = 1$. Thus, ν is also probability measure. Let $s := \sin^2 x$, then $ds = 2\sin x \cos x dx$ and $\cos x = \sqrt{1-s}$. Thus

$$\begin{aligned}\int_0^1 s^n d\nu(s) &= \int_0^1 s^n \frac{ds}{2\sqrt{1-s}} = \int_0^{\frac{\pi}{2}} \sin^{2n} x \frac{2\sin x \cos x dx}{2\cos x} = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx \\ &= \frac{2n!}{(2n+1)!!} = \gamma_n.\end{aligned}$$

Therefore, $W_{\alpha_{2n+1}}$ is also subnormal with $d\nu(s) = \frac{ds}{2\sqrt{1-s}}$. $\square$

We conclude this section with a problem of independent interest.

Problem 2.15. *Recall that $B_+^{(\ell)} = S(\ell, 2\ell-1, 1, 2)$, so Theorem 2.13 guarantees that $B_+^{(\ell)}$ and all of its p-subshifts are subnormal. For $\ell \geq 2$ and $p \geq 1$, find the Berger measure of $B_+^{(\ell)}$ and the Berger measure of its p-subshifts.*

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