k-HYPONORMALITY OF MULTIVARIABLE WEIGHTED SHIFTS

RAÚL E. CURTO, SANG HOON LEE, AND JASANG YOON

Abstract. We characterize joint k-hyponormality for 2-variable weighted shifts. Using this characterization we construct a family of examples which establishes and illustrates the gap between k-hyponormality and $(k+1)$-hyponormality for each $k \geq 1$. As a consequence, we obtain an abstract solution to the Lifting Problem for Commuting Subnormals.

1. Notation and Preliminaries

The Lifting Problem for Commuting Subnormals asks for necessary and sufficient conditions for a pair of subnormal operators on Hilbert space to admit commuting normal extensions. It is well known that the commutativity of the pair is necessary but not sufficient ([Abr], [Lu1], [Lu2], [Lu3]), and it has recently been shown that the joint hyponormality of the pair is necessary but not sufficient [CuYo1]. In this paper we provide an abstract answer to the Lifting Problem, by stating and proving a multivariable analogue of the Bram-Halmos criterion for subnormality, and then showing concretely that no matter how k-hyponormal a pair might be, it may still fail to be subnormal. To do this, we obtain a matricial characterization of k-hyponormality for multivariable weighted shifts, which extends that found in [Cu1] for joint hyponormality.

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. For $S, T \in \mathcal{B}(\mathcal{H})$ let $[S,T] := ST - TS$. We say that an n -tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \tag{1.1}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [Ath] , [CMX]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal. Moreover, the restriction of a hyponormal n-tuple to an invariant subspace is again hyponormal. The Bram-Halmos criterion states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if the k -tuple $(T, T^2, \cdots, T^k)$ is hyponormal for all $k \geq 1$.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

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It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \le \alpha_1 \le \cdots$. Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. We define the 2-*variable weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \ \ (\text{all } \mathbf{k}). \tag{1.2}$$

In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ with $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, we have $T_1 \cong I \bigotimes W_\alpha$ and $T_2 \cong W_\beta \bigotimes I$, so $\mathbf{T}$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [Con, III.8.16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int t^k \, d\xi(t)$ $(k \ge 1)$. If W_α is subnormal, and if for $h \ge 1$ we let $\mathcal{M}_h := \bigvee \{e_n : n \ge h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$.

We also recall the notion of moment of order $\mathbf{k}$ for a pair (α, β) satisfying (1.2). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \left\{ \begin{array}{cc} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \ge 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \ge 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \ge 1 \text{ and } k_2 \ge 1 \end{array} \right\}.$$

We remark that, due to the commutativity condition (1.2), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2). Moreover, $\mathbf{T}$ is subnormal if and only if there is a regular Borel probability measure μ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ $(a_i := \|T_i\|^2)$ such that $\gamma_{\mathbf{k}} = \iint_R \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) := \iint_R t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2)$ (all $\mathbf{k} \in \mathbb{Z}_+^2$) [JeLu].

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2. Main Results

We recall some useful notation. For $n \ge 0$, let $m := \frac{(n+1)(n+2)}{2}$. For $A \in M_m(\mathbb{R})$, we denote the successive rows and columns according to the following lexicographic ordering: $1, x, y, x^2, yx, y^2, \cdots, x^n, yx^{n-1}, \cdots, y^{n-1}x, y^n$ [CuFi2]. For $0 \le i + j \le n, 0 \le l + k \le n$, we denote the entry of $A \in M_m(\mathbb{R})$ in row $y^j x^i$ and column $y^k x^l$ by $A_{(i,j)(l,k)}$. In the notation $0 \le i + j \le n$ it will always be understood that $i, j \ge 0$. For $0 \le i + j \le n, 0 \le l + k \le n$, $(a_{(i,j)(l,k)})_{\substack{0 \le i+j \le n \\ 0 \le l+k \le n}}$ denotes an $m \times m$

matrix and $(a_{(i,j)(l,k)})_{\substack{1\le i+j\le n\\ 1\le l+k\le n}}$ denotes the associated $(m-1)\times(m-1)$ matrix obtained by deleting the first row and column.

For a subnormal 2-variable weighted shift $\mathbf{T}\equiv(T_1,T_2)$, it is clear that each component T_i must be subnormal. For instance, $T_1\cong\bigoplus_{j=0}^{\infty}W_{\alpha^{(j)}}$, where $\alpha_i^{(j)}:=\alpha_{(i,j)}$, so that $W_{\alpha^{(j)}}$ has associated Berger measure $d\nu_j(t_1):=\frac{1}{\gamma_{(0,j)}}\int_{[0,a_2]}t_2^j d\Phi_{t_1}(t_2)$, where $d\mu(t_1,t_2)\equiv d\Phi_{t_1}(t_2)d\eta(t_1)$ is the canonical disintegration of the Berger measure μ by vertical slices [CuYo2]. On the other hand, if we only know that each of T_1, T_2 is subnormal, and that they commute, the following problem is natural.

Problem 2.1. *(Lifting Problem for Commuting Subnormals) Find necessary and sufficient conditions on T_1 and T_2 to guarantee the subnormality of $\mathbf{T}\equiv(T_1,T_2)$.*

It is well known that the above mentioned necessary conditions do not suffice (cf.[Cu1]). In terms of the *marginal* measures for T_1 and T_2, the problem can be phrased as a reconstruction-of-measure problem, that is, under what conditions on the single variable measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$ associated with T_1 and T_2, respectively, does there exist a 2-variable measure μ correctly interpolating all the powers $t_1^{k_1}t_2^{k_2}$ $(k_1,k_2\ge 0)$? We also recall that a pair $\mathbf{S}=(S_1,S_2)$ of commuting subnormal operators is called *polynomially subnormal* if $p(\mathbf{S})$ is subnormal for all 2-variable polynomials $p\in\mathbb{C}[z_1,z_2]$. In [Fra], it was shown that a polynomial subnormal tuple is a subnormal tuple. Using this fact, we can give an abstract answer to Problem 2.1. First we need a definition.

Definition 2.2. A commuting pair $\mathbf{T}\equiv(T_1,T_2)$ is called *k-hyponormal* if $\mathbf{T}(k):=(T_1,T_2,T_1^2,T_2T_1,T_2^2,\cdots,T_1^k,T_2T_1^{k-1},\cdots,T_2^k)$ is hyponormal, or equivalently

$$([(T_2^qT_1^p)^*,T_2^nT_1^m])_{\substack{1\le m+n\le k\\ 1\le p+q\le k}}\ge 0.$$

Clearly, subnormal $\Rightarrow$ $(k+1)$-hyponormal $\Rightarrow$ k-hyponormal for every $k\ge 1$, and of course 1-hyponormality agrees with the usual definition of joint hyponormality.

We now present our multivariable version of the Bram-Halmos criterion for subnormality. When combined with Theorem 2.5 below, Theorem 2.3 provides an abstract answer to Problem 2.1, by showing that no matter how k-hyponormal the pair $\mathbf{T}$ might be, it may still fail to be subnormal.

Theorem 2.3. *Let $\mathbf{T}\equiv(T_1,T_2)$ be a commuting pair of operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent.*
(i) $\mathbf{T}$ is subnormal.
(ii) $\mathbf{T}(k)$ is subnormal for all $k\in\mathbb{Z}_+$.
(iii) $\mathbf{T}$ is k-hyponormal for all $k\in\mathbb{Z}_+$.

In the single variable case, there are useful criteria for k-hyponormality ([Cu2], [CLL]); for 2-variable weighted shifts, a simple criterion for joint hyponormality was given in([Cu1]). We now present a new characterization of k-hyponormality for 2-variable weighted shifts; this generalizes a result in ([Cu1]).

Theorem 2.4. *Let $\mathbf{T}\equiv(T_1,T_2)$ be a 2-variable weighted shift with weight sequences $\alpha\equiv\{\alpha_{\mathbf{k}}\}$ and $\beta\equiv\{\beta_{\mathbf{k}}\}$. The following statements are equivalent.*
(a) $\mathbf{T}$ is k-hyponormal.
(b) $((T_2^nT_1^m)^*[(T_2^qT_1^p)^*,T_2^nT_1^m](T_2^qT_1^p))_{\substack{1\le m+n\le k\\ 1\le p+q\le k}}\ge 0$.
(c) $(\langle[(T_2^qT_1^p)^*,T_2^nT_1^m]e_{\mathbf{u}+(m,n)},e_{\mathbf{u}+(p,q)}\rangle)_{\substack{1\le m+n\le k\\ 1\le p+q\le k}}\ge 0$ for all $\mathbf{u}\in\mathbb{Z}_+^2$.
(d) $(\gamma_{\mathbf{u}}\gamma_{\mathbf{u}+(m,n)+(p,q)}-\gamma_{\mathbf{u}+(m,n)}\gamma_{\mathbf{u}+(p,q)})_{\substack{1\le m+n\le k\\ 1\le p+q\le k}}\ge 0$ for all $\mathbf{u}\in\mathbb{Z}_+^2$.

(e) $M_{\mathbf{u}}(k) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{\substack{0\le m+n\le k\\ 0\le p+q\le k}} \geq 0$ *for all* $\mathbf{u} \in \mathbb{Z}_+^2$. *(For a subnormal pair* $\mathbf{T}$*, the matrix* $M_{\mathbf{u}}(k)$ *is the truncation of the moment matrix associated to the Berger measure of* $\mathbf{T}$*.)*

As an application of Theorem 2.4, we build in Section 4 a two-parameter family of 2-variable weighted shifts (see Figure 1 below), and we identify the precise parameter ranges that separate hyponormality from 2-hyponormality, 2-hyponormality from 3-hyponormality, etc., and k-hyponormality from subnormality. We believe these are the first examples in the literature of commuting pairs of subnormal operators which are k-hyponormal but not $(k+1)$-hyponormal. We record this in the following result. First, we need some notation. For $0 < y \leq 1$, let $x \equiv \{x_n\}_{n=0}^{\infty}$ where

$$x_n := \begin{cases} y\sqrt{\frac{3}{4}}, & \text{if } n = 0 \\ \frac{\sqrt{(n+1)(n+3)}}{(n+2)}, & \text{if } n \geq 1. \end{cases}$$

(We shall see later (Proposition 4.2) that $W_x \equiv shift(x_0, x_1, \cdots)$ is subnormal.)

Theorem 2.5. *For* $0 < a \leq \frac{1}{\sqrt{2}}$*, the* 2*-variable weighted shift* $\mathbf{T}$ *given by Figure 1 is*

(i) hyponormal $\Leftrightarrow 0 < y \leq \frac{\sqrt{32-48a^4}}{\sqrt{59-72a^2}}$;

(ii) k-hyponormal $\Leftrightarrow 0 < y \leq \sqrt{\frac{\frac{(k+1)^2}{2k(k+2)}-a^2}{a^4-\frac{5}{2}a^2+\frac{(k+1)^2}{2k(k+2)}+\frac{2k^2+4k+3}{4(k+1)^2}}}$ $(k \geq 2)$;

(iii) subnormal $\Leftrightarrow 0 < y \leq \sqrt{\frac{1}{2-a^2}}$.

In particular, $\mathbf{T}$ *is hyponormal and not subnormal if and only if* $\sqrt{\frac{1}{2-a^2}} < y \leq \frac{\sqrt{32-48a^4}}{\sqrt{59-72a^2}}$.

Remark 2.6. (i) Even for 1-variable weighted shifts, it is generally difficult to provide concrete parameterizations that separate k-hyponormality from $(k+1)$-hyponormality (cf. [CLL, Example 8]). That we can accomplish the same separation for 2-variable weighted shifts is an indication that the condition in Theorem 2.4(e) is sharp.
(ii) In [CMX], the authors conjectured that if $\mathbf{T} \equiv (T_1, T_2)$ is a pair of commuting subnormal operators, then $\mathbf{T}$ is subnormal if and only if $\mathbf{T}$ is hyponormal. In [CuYo1], three different families of examples were given of such pairs $\mathbf{T}$ for which hyponormality does not imply subnormality. Thus, any of those examples can be used to disprove the conjecture in [CMX]. Theorem 2.5 gives a new family of examples, with explicit parameter values to distinguish between k-hyponormality and $(k+1)$-hyponormality, and a fortiori between hyponormality and subnormality.

3. Proofs of Theorems 2.3 and 2.4

Proof of Theorem 2.3. *(i)* $\Rightarrow$ *(ii)*: Suppose $\mathbf{T} \equiv (T_1, T_2)$ is subnormal, that is, $\mathbf{T}$ admits a normal extension $\mathbf{N} \equiv (N_1, N_2)$ acting on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. The tuple $\mathbf{N}(k) := (N_1, N_2, N_1^2, N_2N_1, N_2^2, \cdots, N_1^k, N_2N_1^{k-1}, \cdots, N_2^k)$ is also normal, so its restriction to $\mathcal{H}$, $\mathbf{T}(k)$, is subnormal.

(ii) $\Rightarrow$ *(iii)*: This is trivial.

(iii) $\Rightarrow$ *(i)*: Suppose $\mathbf{T}(k)$ is hyponormal for all $k \in \mathbb{Z}_+$, and let $p \in \mathbb{C}[z_1, z_2]$. It follows that $p(T_1, T_2)$ (as a single operator on $\mathcal{H}$) is k-hyponormal for every $k \geq 1$. By the Bram-Halmos criterion for single operators we then see that $p(T_1, T_2)$ is subnormal. Finally, the main result in [Fra] implies that $\mathbf{T}$ is subnormal. $\square$

We now give the proof of Theorem 2.4, which we restate for the reader's convenience.

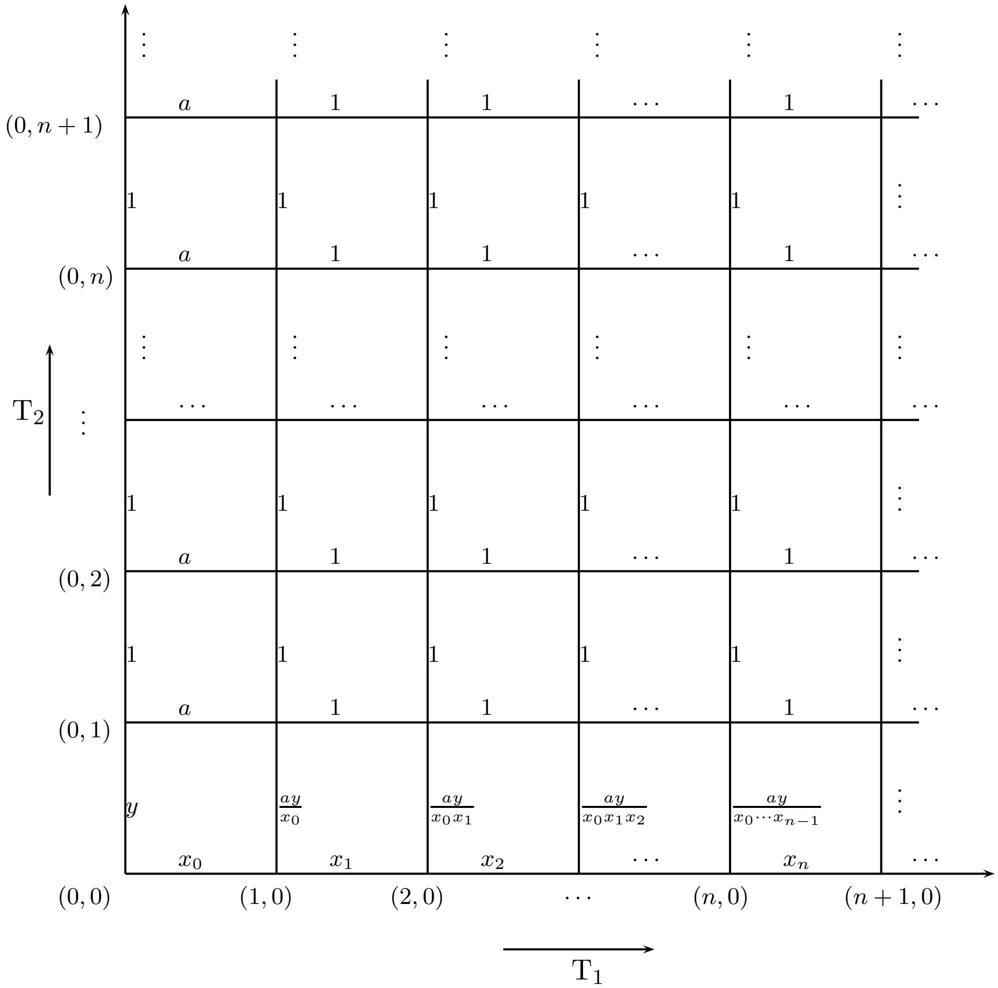

FIGURE 1. Weight diagram of the 2-variable weighted shift in Theorem 2.5

Theorem 3.1. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift with weight sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}$. The following statements are equivalent.*
(a) $\mathbf{T}$ is k-hyponormal.
(b) $((T_2^n T_1^m)^* [(T_2^q T_1^p)^*, T_2^n T_1^m](T_2^q T_1^p))_{\substack{1 \le m+n \le k \\ 1 \le p+q \le k}} \ge 0$.
(c) $(\langle [(T_2^q T_1^p)^*, T_2^n T_1^m] e_{\mathbf{u}+(m,n)}, e_{\mathbf{u}+(p,q)} \rangle)_{\substack{1 \le m+n \le k \\ 1 \le p+q \le k}} \ge 0$ for all $\mathbf{u} \in \mathbb{Z}_+^2$.
(d) $(\gamma_{\mathbf{u}} \gamma_{\mathbf{u}+(m,n)+(p,q)} - \gamma_{\mathbf{u}+(m,n)} \gamma_{\mathbf{u}+(p,q)})_{\substack{1 \le m+n \le k \\ 1 \le p+q \le k}} \ge 0$ for all $\mathbf{u} \in \mathbb{Z}_+^2$.
(e) $M_{\mathbf{u}}(k) := (\gamma_{\mathbf{u}+(m,n)+(p,q)})_{\substack{0 \le m+n \le k \\ 0 \le p+q \le k}} \ge 0$ for all $\mathbf{u} \in \mathbb{Z}_+^2$.

Proof. $(a) \Leftrightarrow (b)$: Let

$$A := ([(T_2^q T_1^p)^*, T_2^n T_1^m])_{\substack{1 \le m+n \le k \\ 1 \le p+q \le k}}$$

and

$$B := ((T_2^n T_1^m)^*[(T_2^q T_1^p)^*, T_2^n T_1^m](T_2^q T_1^p))_{\substack{1\le m+n\le k\\ 1\le p+q\le k}}.$$

Note that

$$B = \begin{pmatrix} T_1^* & & & & & \\ & T_2^* & & & & \\ & & \ddots & & & \\ & & & (T_1^k)^* & & \\ & & & & \ddots & \\ & & & & & (T_2^k)^* \end{pmatrix} A \begin{pmatrix} T_1 & & & & & \\ & T_2 & & & & \\ & & \ddots & & & \\ & & & T_1^k & & \\ & & & & \ddots & \\ & & & & & T_2^k \end{pmatrix}.$$

Thus it is easily seen that $(a) \Longrightarrow (b)$. For the converse implication, let

$$\mathcal{M} := T_1(\mathcal{H}) \oplus T_2(\mathcal{H}) \oplus T_1^2(\mathcal{H}) \oplus T_2T_1(\mathcal{H}) \oplus T_2^2(\mathcal{H}) \oplus \cdots \oplus T_1^k(\mathcal{H}) \oplus \cdots \oplus T_2^k(\mathcal{H})$$

where $\mathcal{H} := \ell^2(\mathbb{Z}_+^2)$. For

$$x := T_1(h_{(1,0)}) \oplus T_2(h_{(0,1)}) \oplus T_1^2(h_{(2,0)}) \oplus T_2T_1(h_{(1,1)}) \oplus T_2^2(h_{(0,2)}) \oplus \cdots$$
$$\oplus T_1^k(h_{(k,0)}) \oplus \cdots \oplus T_2^k(h_{(0,k)})$$

in $\mathcal{M}$ (here $h := h_{(1,0)} \oplus h_{(0,1)} \oplus \cdots \oplus h_{(0,k)}$), we have

$$\langle Ax, x\rangle = \sum_{\substack{1\le m+n\le k\\ 1\le p+q\le k}} \left\langle [(T_2^q T_1^p)^*, T_2^n T_1^m](T_2^q T_1^p) h_{(p,q)}, T_2^n T_1^m h_{(m,n)} \right\rangle = \langle Bh, h\rangle.$$

Thus, $\langle Bh, h\rangle \ge 0$ implies that the compression of A to $\mathcal{M}$ is positive. Since T_1 and T_2 are weighted shifts, we have $[(T_2^q T_1^p)^*, T_2^n T_1^m](T_2^q T_1^p)(\mathcal{H}) \subseteq T_2^n T_1^m(\mathcal{H})$, and it follows that $\mathcal{M}$ is invariant for A; since A is selfadjoint, it then follows that $\mathcal{M}$ reduces A. Therefore, the proof will be completed once we show that the compression of A to $\mathcal{M}^\perp$ is also positive. Let $\{e_{(p,q)}\}_{p,q\in\mathbb{Z}_+}$ be the canonical orthonormal basis of $\mathcal{H}$. Note that

$$\mathcal{M}^\perp = L_{(1,0)} \oplus L_{(0,1)} \oplus \cdots \oplus L_{(k,0)} \oplus \cdots \oplus L_{(0,k)},$$

where $L_{(i,j)}$ is the span of $\{e_{(p,q)}\}_{\substack{0\le p\le (i-1)\\ q\in\mathbb{Z}_+}}$ and $\{e_{(r,s)}\}_{\substack{r\in\mathbb{Z}_+\\ 0\le s\le (j-1)}}$. Let

$$e_{(\mathbf{u},\mathbf{m})} := 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^j x^i}{e_{\mathbf{u}}} \oplus 0 \oplus \cdots \oplus 0 \in \mathcal{H}^m$$

where $\mathbf{m} = (i,j)$. (The notation $0 \oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^j x^i}{\mathbf{v}} \oplus 0 \oplus \cdots \oplus 0$ indicates that the vector $\mathbf{v}$ appears in the $y^j x^i$-th summand.) Observe that $\{e_{(p,q)(r,s)}\}_{\substack{1\le r+s\le k\\ 1\le p\le (r-1)\\ q\in\mathbb{Z}_+}} \cup \{e_{(p,q)(r,s)}\}_{\substack{1\le r+s\le k\\ 1\le p\le (s-1)\\ p\in\mathbb{Z}_+}}$ is an orthonormal basis for $\mathcal{M}^\perp$. Now, for $1 \le i+j \le k$, we define the subspace $\mathcal{K}_{(i,j)}$ of $\mathcal{M}^\perp$ as follows:

$$\mathcal{K}_{(i,j)} := \begin{cases} \bigvee_n \{e_{((0,n),(i,j))}, e_{((1,n),(i+1,j))}, \cdots, e_{((k-(i+j),n),(k-j,j))}\}, \text{ if } i > j \\ \bigvee_n \{e_{((n,0),(i,j))}, e_{((n,1),(i,j+1))}, \cdots, e_{((n,k-(i+j)),(i,k-i))}\}, \text{ if } i < j \\ \bigvee_n \{e_{((0,n),(i,i))}, e_{((n,0),(i,i))}, \cdots, e_{((k-2i,n),(k-i,i))}, e_{((n,k-2i),(i,k-i))}\}, \text{ if } i = j. \end{cases}$$

It is easily seen that $\mathcal{K}_{(i,j)} \perp \mathcal{K}_{(s,t)}$ for $(i,j) \ne (s,t)$ and $\mathcal{M}^\perp = \mathcal{K}_{(1,0)} \oplus \mathcal{K}_{(0,1)} \oplus \ldots \oplus \mathcal{K}_{(k,0)} \oplus \mathcal{K}_{(0,k)}$. Note that $\mathcal{K}_{(i,j)}$ is invariant for A and hence $\mathcal{K}_{(i,j)}$ reduces A. We must then show that the compression of A to $\mathcal{K}_{(i,j)}$ is also positive. Note that each vector h in $\mathcal{K}_{(i,j)} (i > j)$ has the form

$$0 \oplus \cdots \oplus 0 \oplus \overset{y^j x^i}{c(i,j) e_{(0,\ell)}} \oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^j x^{i+1}}{c(i+1,j) T_1 e_{(0,\ell)}} \oplus 0 \oplus \cdots \oplus 0$$
$$\oplus \overset{y^j x^{k-j}}{c(k-j,j) T_1^{k-(i+j)} e_{(0,\ell)}} \oplus 0 \oplus \cdots \oplus 0$$

for some scalars $c(i,j), c(i+1,j), \cdots, c(k-j,j)$ and $\ell \in \mathbb{Z}_+$. Thus,

$$\left\{ \begin{array}{l} \langle Ah, h \rangle = \sum_{\substack{i \le m \le k-j,\ n=j \\ i \le p \le k-j,\ q=j}} \left\langle [(T_2^q T_1^p)^*, T_2^n T_1^m] c(p,q) T_1^{p-i} e_{(0,\ell)}, c(m,n) T_1^{m-i} e_{(0,\ell)} \right\rangle \\ = \sum_{\substack{i \le m \le k-j \\ i \le p \le k-j}} c(p,j) \overline{c(m,j)} \left\langle T_1^{*(m-i)} [(T_2^j T_1^p)^*, T_2^j T_1^m] T_1^{p-i} e_{(0,\ell)}, e_{(0,\ell)} \right\rangle \\ = \sum_{\substack{i \le m \le k-j \\ i \le p \le k-j}} c(p,j) \overline{c(m,j)} \left\langle (T_2^j T_1^{p+m-i})^* T_2^j T_1^{p+m-j} e_{(0,\ell)}, e_{(0,\ell)} \right\rangle \\ = \left\langle M \begin{pmatrix} c(i,j) e_{(0,\ell)} \\ c(i+1,j) e_{(0,\ell)} \\ \vdots \\ c(k-j,j) e_{(0,\ell)} \end{pmatrix}, \begin{pmatrix} c(i,j) e_{(0,\ell)} \\ c(i+1,j) e_{(0,\ell)} \\ \vdots \\ c(k-j,j) e_{(0,\ell)} \end{pmatrix} \right\rangle, \end{array} \right\} \tag{3.1}$$

where

$$M := \begin{pmatrix} (T_2^j T_1^i)^* (T_2^j T_1^i) & (T_2^j T_1^{i+1})^* (T_2^j T_1^{i+1}) & \cdots & (T_2^j T_1^{k-j})^* (T_2^j T_1^{k-j}) \\ (T_2^j T_1^{i+1})^* (T_2^j T_1^{i+1}) & (T_2^j T_1^{i+2})^* (T_2^j T_1^{i+2}) & \cdots & (T_2^j T_1^{k-j+1})^* (T_2^j T_1^{k-j+1}) \\ \vdots & \vdots & \ddots & \vdots \\ (T_2^j T_1^{k-j})^* (T_2^j T_1^{k-j}) & \cdots & \cdots & (T_2^j T_1^{2(k-j)-i})^* (T_2^j T_1^{2(k-j)-i}) \end{pmatrix}.$$

In (3.1) above, the third equality (from line 2 to line 3) follows from the fact that

$$\left\langle T_2^j T_1^m (T_2^j T_1^p)^* T_1^{p-i} e_{(0,\ell)}, e_{(0,\ell)} \right\rangle = 0.$$

Now,

$$M = \begin{pmatrix} (T_2^j T_1^i)^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (T_2^j T_1^i)^* \end{pmatrix} M' \begin{pmatrix} T_2^j T_1^i & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & T_2^j T_1^i \end{pmatrix}$$

where

$$M' := \begin{pmatrix} I & T_1^* T_1 & \cdots & (T_1^{k-(i+j)})^* T_1^{k-(i+j)} \\ T_1^* T_1 & (T_1^2)^* T_1^2 & \cdots & (T_1^{k-(i+j)+1})^* T_1^{k-(i+j)+1} \\ \vdots & \vdots & \ddots & \vdots \\ (T_1^{k-(i+j)})^* T_1^{k-(i+j)} & (T_1^{k-(i+j)+1})^* T_1^{k-(i+j)+1} & \cdots & (T_1^{2(k-(i+j))})^* T_1^{2(k-(i+j))} \end{pmatrix}.$$

Now, by Smul'jan's Theorem [Smu], $M' \equiv (M'_{uv})_{u,v=0}^{k-(i+j)} \ge 0$ if and only if $Q \equiv (Q_{uv})_{u,v=1}^{k-(i+j)} \ge 0$, where

$$Q_{uv} := (T_1^{u+v})^* T_1^{u+v} - (T_1^u)^* T_1^u (T_1^v)^* T_1^v.$$

Now, observe that $Q_{uv} = (T_1^u)^*[(T_1^v)^*, T_1^u]T_1^v$, so that

$$Q = \begin{pmatrix} T_1^* & & \\ & \ddots & \\ & & (T_1^k)^* \end{pmatrix} \begin{pmatrix} [T_1^*, T_1] & \cdots & [(T_1^{k-(i+j)})^*, T_1] \\ \vdots & \ddots & \vdots \\ [(T_1)^*, T_1^{k-(i+j)}] & \cdots & [(T_1^{k-(i+j)})^*, T_1^{k-(i+j)}] \end{pmatrix} \begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_1^k \end{pmatrix}$$

It is now easy to see that Q is a submatrix of B. Thus, if $B \geq 0$, then $M' \geq 0$ and hence $\langle Ah, h\rangle \geq 0$ for all $h \in \mathcal{K}_{(i,j)}$ with $i > j$. On the other hand, if $i < j$, then each vector h in $\mathcal{K}_{(i,j)}$ has the form

$$\begin{aligned} &0 \oplus \cdots \oplus 0 \oplus \overset{y^j x^i}{c(i,j)e_{(\ell,0)}} \oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^{j+1}x^i}{c(i,j+1)T_2 e_{(\ell,0)}} \oplus 0 \oplus \cdots \\ &\oplus 0 \oplus \overset{y^{k-i}x^i}{c(i,k-i)T_2^{k-(i+j)}} e_{(\ell,0)} \oplus 0 \oplus \cdots \oplus 0, \end{aligned}$$

for some scalars $c(i,j), c(i,j+1), \cdots, c(i,k-i)$ and $\ell \in \mathbb{Z}_+$. An analogous argument shows that $\langle Ah, h\rangle \geq 0$ for all $h \in \mathcal{K}_{(i,j)}$ with $i < j$. Finally, if $i = j$, then each vector h in $\mathcal{K}_{(i,i)}$ has the form

$$\begin{aligned} &0 \oplus \cdots \oplus 0 \oplus \overset{y^i x^i}{c(i,i)e_{(s,t)}} \oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^i x^{i+1}}{c(i+1,i)T_1 e_{(s,t)}} \\ &\oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^{i+1}x^i}{c(i,i+1)T_2 e_{(s,t)}} \oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^i x^{k-i}}{c(k-i,i)T_1^{k-2i}} e_{(s,t)} \\ &\oplus 0 \oplus \cdots \oplus 0 \oplus \overset{y^{k-i}x^i}{c(i,k-i)T_2^{k-2i}} e_{(s,t)} \oplus 0 \oplus \cdots \oplus 0 \end{aligned}$$

for some scalars $c(i,i), c(i+1,i), c(i,i+1) \cdots, c(i,k-i)$ and $s,t \in \mathbb{Z}_+$ with $st = 0$. Define

$$\sigma_{(p,q)} := \begin{cases} 1 & \text{if } p \geq q \\ 2 & \text{if } p < q, \end{cases}$$

and let $M(p,q) := \max\{p,q\}$. We then have

$$\begin{aligned} \langle Ah, h\rangle &= \sum_{\substack{2i \leq m+n \leq k,\ n=i \text{ or } m=i \\ 2i \leq p+q \leq k,\ p=i \text{ or } q=i}} c(p,q)\overline{c(m,n)} \left\langle [(T_2^q T_1^p)^*, T_2^n T_1^m] T_{\sigma_{(p,q)}}^{M(p,q)-i} e_{(s,t)}, T_{\sigma_{(m,n)}}^{M(m,n)-i} e_{(s,t)} \right\rangle \\ &= \sum_{\substack{2i \leq m+n \leq k,\ n=i \text{ or } m=i \\ 2i \leq p+q \leq k,\ p=i \text{ or } q=i}} c(p,q)\overline{c(m,n)} \left\langle (T_{\sigma_{(m,n)}}^{M(m,n)-i})^* [(T_2^q T_1^p)^*, T_2^n T_1^m] T_{\sigma_{(p,q)}}^{M(p,q)-i} e_{(s,t)}, e_{(s,t)} \right\rangle. \end{aligned}$$

Since $(T_2^q T_1^p)^* T_{\sigma_{(p,q)}}^{M(p,q)-i} e_{(s,t)} = 0$, we have

$$\begin{aligned} \langle Ah, h\rangle &= \sum_{\substack{2i \leq m+n \leq k,\ n=i \text{ or } m=i \\ 2i \leq p+q \leq k,\ p=i \text{ or } q=i}} c(p,q)\overline{c(m,n)} \left\langle (T_{\sigma_{(m,n)}}^{M(m,n)-i})^* (T_2^q T_1^p)^* T_2^n T_1^m T_{\sigma_{(p,q)}}^{M(p,q)-i} e_{(s,t)}, e_{(s,t)} \right\rangle \\ &= \left\langle M \begin{pmatrix} c(i,i)e_{(s,t)} \\ c(i+1,i)e_{(s,t)} \\ \vdots \\ c(i,k-i)e_{(s,t)} \end{pmatrix}, \begin{pmatrix} c(i,i)e_{(s,t)} \\ c(i+1,i)e_{(s,t)} \\ \vdots \\ c(i,k-i)e_{(s,t)} \end{pmatrix} \right\rangle, \end{aligned}$$

where

$$M := \begin{pmatrix} (T_2^iT_1^i)^*(T_2^iT_1^i) & (T_2^iT_1^{i+1})^*(T_2^iT_1^{i+1}) & \cdots & (T_2^{k-i}T_1^i)^*(T_2^{k-i}T_1^i) \\ (T_2^iT_1^{i+1})^*(T_2^iT_1^{i+1}) & (T_2^iT_1^{i+2})^*(T_2^iT_1^{i+2}) & \cdots & (T_2^{k-i}T_1^{i+1})^*(T_2^{k-i}T_1^{i+1}) \\ \vdots & \vdots & \ddots & \vdots \\ (T_2^{k-i}T_1^i)^*(T_2^{k-i}T_1^i) & \cdots & \cdots & (T_2^{k-i+1}T_1^i)^*(T_2^{k-i+1}T_1^i) \end{pmatrix}.$$

However,

$$M = \begin{pmatrix} (T_2^iT_1^i)^* & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (T_2^iT_1^i)^* \end{pmatrix} M' \begin{pmatrix} T_2^iT_1^i & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & T_2^iT_1^i \end{pmatrix},$$

where

$$M' := \begin{pmatrix} I & T_1^*T_1 & \cdots & (T_2^{k-2i})^*T_2^{k-2i} \\ T_1^*T_1 & (T_1^2)^*T_1^2 & \cdots & (T_2^{k-2i}T_1)^*T_2^{k-2i}T_1 \\ \vdots & \vdots & \ddots & \vdots \\ (T_2^{k-2i})^*T_2^{k-2i} & \cdots & \cdots & (T_2^{2(k-2i)})^*T_2^{2(k-2i)} \end{pmatrix}.$$

If $B \geq 0$, then $M' \geq 0$ and hence $\langle Ah, h \rangle \geq 0$ for all $h \in \mathcal{K}_{(i,i)}$.

$(b) \Leftrightarrow (c)$: Note that $(T_2^nT_1^m)^*[(T_2^qT_1^p)^*, T_2^nT_1^m](T_2^qT_1^p)$ is a diagonal operator. Thus, $B \geq 0$ if and only if $\langle Be_{\mathbf{u}}, e_{\mathbf{u}} \rangle \geq 0$ for all $\mathbf{u} \in \mathbb{Z}_+^2$ if and only if (c) holds.

$(b) \Leftrightarrow (d)$: Since $(T_2^mT_1^n)^*[(T_2^qT_1^p)^*, T_2^mT_1^n](T_2^qT_1^p)$ is a diagonal operator whose $\mathbf{u}$-th diagonal entry is

$$\frac{\gamma_{\mathbf{u}+(p,q)+(m,n)}}{\gamma_{\mathbf{u}}} - \frac{\gamma_{\mathbf{u}+(p,q)}\gamma_{\mathbf{u}+(m,n)}}{\gamma_{\mathbf{u}}^2},$$

we easily see that $(b) \Leftrightarrow (d)$.

$(d) \Leftrightarrow (e)$: This is a straightforward application of Choleski's algorithm [Atk]. □

4. Applications

Unlike the single variable case, in which there is a clear separation between hyponormality and subnormality (cf. [CuFi1], [Cu3], [CuLe],[CLL]), much less is known about the multivariable case. We will now construct an example which exhibits the gap between k-hyponormality and $(k+1)$-hyponormality for each $k \geq 1$, and gives another counterexample to the following conjecture, recently answered in the negative ([CuYo1]).

Conjecture 4.1. *([CMX]) Let $\mathbf{T} \equiv (T_1, T_2)$ be a pair of commuting subnormal operators on $\mathcal{H}$. Then $\mathbf{T}$ is subnormal if and only if $\mathbf{T}$ is hyponormal.*

We begin with:

Proposition 4.2. *For $0 < y \leq 1$, let $x \equiv \{x_n\}_{n=0}^\infty$ where*

$$x_n := \begin{cases} y\sqrt{\frac{3}{4}}, & \text{if } n = 0 \\ \frac{\sqrt{(n+1)(n+3)}}{(n+2)}, & \text{if } n \geq 1. \end{cases}$$

Then $W_x \equiv shift(x_0, x_1, \cdots)$ is subnormal.

Proof. We need to find a regular Borel probability measure μ_x such that $\gamma_n = \int s^n d\mu_x(s)$ $(n \geq 0)$. On the interval $[0,1]$, consider $d\mu_x := (1-y^2)d\delta_0(s) + \frac{y^2}{2}ds + \frac{y^2}{2}d\delta_1(s)$. Then $\gamma_0 = 1$ and for $n \geq 1$,

$$\begin{aligned}\gamma_n &\equiv x_0^2 x_1^2 x_2^2 \cdots x_{n-1}^2 \\ &= y^2 \frac{3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \cdots \cdot \frac{n(n+2)}{(n+1)^2} \\ &= \frac{(n+2)y^2}{2(n+1)} = \frac{y^2}{2} \cdot \frac{1}{n+1} + \frac{y^2}{2} = \int s^n d\mu_x(s).\end{aligned}$$

It follows that W_x is subnormal, with Berger measure μ_x. $\square$

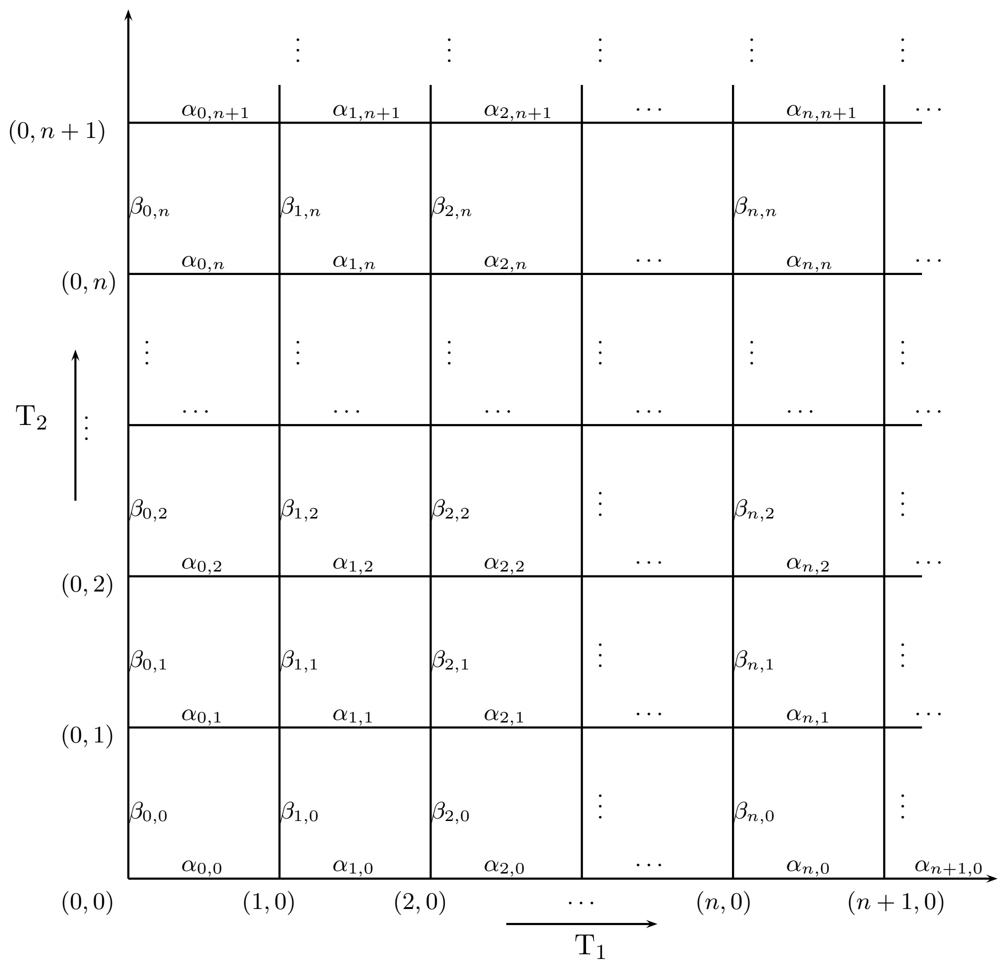

FIGURE 2. Weight diagram of the 2-variable weighted shift in Lemma 4.3

To recall the following result, we need some notation and terminology from [CuYo1]. Given a probability measure μ on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$, and assuming that $\frac{1}{t} \in L^1(\mu)$, the *extremal measure* μ_{ext}

(which is also a probability measure) on $\mathbb{R}_+ \times \mathbb{R}_+$ is given by $d\mu_{ext}(s,t) := (1-\delta_0(t)) \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t)$. On the other hand, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. We observe that if μ is a probability measure, then so is μ^X.

Lemma 4.3. [CuYo1, Proposition 3.10] *(Subnormal backward extension of a 2-variable weighted shift) Consider the 2-variable weighted shift* $\mathbf{T}$ *whose weight sequence is given by Figure 2, and let* $\mathcal{M}$ *be the subspace associated with indices* $\mathbf{k}$ *with* $k_2 \geq 1$. *Assume that* $\mathbf{T}|_{\mathcal{M}}$ *is subnormal with Berger measure* $\mu_{\mathcal{M}}$ *and that the weighted shift* W_0 *with weight sequence* $(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with Berger measure* ν. *Then* $\mathbf{T}$ *is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu$.
Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, *then* $(\mu_{\mathcal{M}})_{ext}^X = \nu$. *In the case when* $\mathbf{T}$ *is subnormal, the Berger measure* μ *of* $\mathbf{T}$ *is given by*

$$d\mu(s,t) = \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + [d\nu(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)] d\delta_0(t).$$

We are now ready to present our example of a nonsubnormal, hyponormal commuting pair of subnormal weighted shifts. At the same time we will exhibit concretely the gap between k-hyponormality and $(k+1)$-hyponormality for each $k \geq 1$. For $0 < a \leq \frac{1}{\sqrt{2}}$, consider the 2-variable weighted shift given by Figure 1, where $x \equiv \{x_n\}_{n=0}^{\infty}$ is as in Proposition 4.2.

Proposition 4.4. *The 2-variable weighted shift* $\mathbf{T}$ *given by Figure 1 is subnormal if and only if* $0 < y \leq \sqrt{\frac{1}{2-a^2}}$.

Proof. Let $\mathcal{M}$ be the subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+^2)$ except for $e_{(0,0)}, e_{(1,0)}, \cdots, e_{(n,0)}, \cdots$. Then from Figure 1, it is obvious that $\mathbf{T}|_{\mathcal{M}} \cong (I \otimes S_a, U_+ \otimes I)$. (Recall that S_a is the subnormal weighted shift which has weight sequence $(a, 1, 1, \cdots)$ and Berger measure $(1-a^2)\delta_0 + a^2\delta_1$, and $U_+ \equiv S_1$ is the (unweighted) unilateral shift.) Thus, $\mathbf{T}|_{\mathcal{M}}$ is subnormal with Berger measure

$$\mu_{\mathcal{M}} := [(1-a^2)\delta_0 + a^2\delta_1] \times \delta_1.$$

By Lemma 4.3,

$$\begin{aligned} \mathbf{T} \text{ is subnormal } &\Leftrightarrow y^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \mu_x \\ &\Leftrightarrow y^2[(1-a^2)\delta_0 + a^2\delta_1] \leq (1-y^2)\delta_0 + \frac{y^2}{2}\lambda + \frac{y^2}{2}\delta_1 \\ &\quad \text{(here } \lambda \text{ denotes Lebesgue measure on } [0,1]) \\ &\Leftrightarrow y \leq \sqrt{\frac{1}{2-a^2}} \text{ and } a \leq \frac{1}{\sqrt{2}} \\ &\Leftrightarrow y \leq \sqrt{\frac{1}{2-a^2}} \text{ (recall that } a \leq \frac{1}{\sqrt{2}} \text{ is being assumed).} \end{aligned}$$

□

Proposition 4.5. *The 2-variable weighted shift* $\mathbf{T}$ *given by Figure 1 is hyponormal if and only if* $0 < y \leq \frac{\sqrt{32-48a^4}}{\sqrt{59-72a^2}}$.

Proof. By Theorem 2.4(d), to show the joint hyponormality of $\mathbf{T}$ it is enough to check that

$$H_{\mathbf{k}} := \begin{pmatrix} \gamma_{\mathbf{k}}\gamma_{\mathbf{k}+(2,0)} - \gamma^2_{\mathbf{k}+(1,0)} & \gamma_{\mathbf{k}}\gamma_{\mathbf{k}+(1,1)} - \gamma_{\mathbf{k}+(1,0)}\gamma_{\mathbf{k}+(0,1)} \\ \gamma_{\mathbf{k}}\gamma_{\mathbf{k}+(1,1)} - \gamma_{\mathbf{k}+(1,0)}\gamma_{\mathbf{k}+(0,1)} & \gamma_{\mathbf{k}}\gamma_{\mathbf{k}+(0,2)} - \gamma^2_{\mathbf{k}+(0,1)} \end{pmatrix} \geq 0$$

for all $\mathbf{k} \in \mathbb{Z}^2_+$. Since $\mathbf{T}|_{\mathcal{M}}$ is subnormal (as noted in Proposition 4.4), it is also hyponormal, so it remains to show that $H_{\mathbf{k}} \geq 0$ for $\mathbf{k} = (0,0), (1,0), (2,0), \cdots, (n,0), \cdots$. A straightforward calculation shows that

$$\begin{aligned} H_{(n,0)} &= \begin{pmatrix} \gamma_{(n,0)}\gamma_{(n+2,0)} - \gamma^2_{(n+1,0)} & \gamma_{(n,0)}\gamma_{(n+1,1)} - \gamma_{(n+1,0)}\gamma_{(n,1)} \\ \gamma_{(n,0)}\gamma_{(n+1,1)} - \gamma_{(n+1,0)}\gamma_{(n,1)} & \gamma_{(n,0)}\gamma_{(n,2)} - \gamma^2_{(n,1)} \end{pmatrix} \\ &= \gamma_{(n,0)} \begin{pmatrix} \gamma_{(n+2,0)} - x_n^2\gamma_{(n+1,0)} & \gamma_{(n+1,1)} - x_n^2\gamma_{(n,1)} \\ \gamma_{(n+1,1)} - x_n^2\gamma_{(n,1)} & \gamma_{(n,2)} - \frac{\gamma^2_{(n,1)}}{\gamma_{(n,0)}} \end{pmatrix} \\ &= \gamma_{(n,0)} \begin{pmatrix} (x_0\cdots x_{n+1})^2 - x_n^2(x_0\cdots x_n)^2 & a^2y^2 - x_n^2a^2y^2 \\ a^2y^2 - x_n^2a^2y^2 & a^2y^2 - \frac{(a^2y^2)^2}{(x_0\cdots x_{n-1})^2} \end{pmatrix} \\ &= y^2\gamma_{(n,0)} \begin{pmatrix} \frac{n+4}{2(n+3)} - \frac{n+3}{2(n+2)}\cdot\frac{(n+1)(n+3)}{(n+2)^2} & a^2(1 - \frac{(n+1)(n+3)}{(n+2)^2}) \\ a^2(1 - \frac{(n+1)(n+3)}{(n+2)^2}) & a^2(1 - \frac{(n+2)}{2(n+1)}a^2) \end{pmatrix} \\ &= y^2\gamma_{(n,0)} \begin{pmatrix} \frac{2n+5}{2(n+2)^3(n+3)} & \frac{1}{(n+2)^2}a^2 \\ \frac{1}{(n+2)^2}a^2 & a^2(1 - \frac{(n+2)}{2(n+1)}a^2) \end{pmatrix}, \end{aligned}$$

which is positive because $0 < a \leq \frac{1}{\sqrt{2}}$ for all $n \neq 0$. For $\mathbf{k} = (0,0)$, we have

$$H_{(0,0)} = \begin{pmatrix} (x_0x_1)^2 - (x_0^2)^2 & a^2y^2 - x_0^2y^2 \\ a^2y^2 - x_0^2y^2 & y^2 - y^4 \end{pmatrix}.$$

Since $(x_0x_1)^2 - (x_0^2)^2 > 0$ and

$$\det H_{(0,0)} = y^4(\frac{2}{3} - \frac{9}{16}y^2)(1-y^2) - (a^2 - \frac{3}{4}y^2)^2 = \frac{y^4}{48}\{(72a^2-59)y^2 + 32 - 48a^4\},$$

we obtain the desired result. $\square$

In the following theorem, we summarize the results in Propositions 4.4 and 4.5, and provide a new family of examples to settle Conjecture 4.1 in the negative.

Theorem 4.6. *If* $\sqrt{\frac{1}{2-a^2}} < y \leq \frac{\sqrt{32-48a^4}}{\sqrt{59-72a^2}}$, *then the 2-variable weighted shift* $\mathbf{T}$ *given by Figure 1 is hyponormal but not subnormal.*

For $k \geq 2$ we let

$$H_k(y) := \frac{1}{2} \begin{pmatrix} \frac{2}{y^2} & \frac{3}{2} & 2 & 2a^2 & \frac{4}{3} & \frac{5}{4} & \cdots & \frac{k+2}{k+1} \\ \frac{3}{2} & \frac{4}{3} & 2a^2 & 2a^2 & \frac{5}{4} & \frac{6}{5} & \cdots & \frac{k+3}{k+2} \\ 2 & 2a^2 & 2 & 2a^2 & 2a^2 & 2a^2 & \cdots & 2a^2 \\ 2a^2 & 2a^2 & 2a^2 & 2a^2 & 2a^2 & 2a^2 & \cdots & 2a^2 \\ \frac{4}{3} & \frac{5}{4} & 2a^2 & 2a^2 & \frac{6}{5} & \frac{7}{6} & \cdots & \frac{k+4}{k+3} \\ \frac{5}{4} & \frac{6}{5} & 2a^2 & 2a^2 & \frac{7}{6} & \frac{8}{7} & \cdots & \frac{k+5}{k+4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k+2}{k+1} & \frac{k+3}{k+2} & 2a^2 & 2a^2 & \frac{k+4}{k+3} & \frac{k+5}{k+4} & \cdots & \frac{2k+2}{2k+1} \end{pmatrix}_{(k+3)\times(k+3)}.$$

We then have:

Theorem 4.7. *The 2-variable weighted shift* $\mathbf{T}$ *given by Figure 1 is* k*-hyponormal* ($k \geq 2$) *if and only if* $0 < y \leq D(k) := \sqrt{\frac{\frac{(k+1)^2}{2k(k+2)} - a^2}{a^4 - \frac{5}{2}a^2 + \frac{(k+1)^2}{2k(k+2)} + \frac{2k^2+4k+3}{4(k+1)^2}}}$.

Remark 4.8. Since $D(k+1) < D(k)$ for every $k \geq 2$, it follows that for $D(k+1) < y \leq D(k)$, the associated 2-variable weighted shift $\mathbf{T}$ is k-hyponormal but not $(k+1)$-hyponormal.

Proof. By Theorem 2.4(e), $\mathbf{T}$ is k-hyponormal if and only if

$$M_{\mathbf{k}}(k) = (\gamma_{\mathbf{k}+(m,n)+(p,q)})_{\substack{0\leq n+m\leq k \\ 0\leq p+q\leq k}} \geq 0,$$

for all $\mathbf{k} \in \mathbb{Z}_+^2$. By [CuYo1, Theorem 5.2 and Remark 5.3] we need to verify that $M_{\mathbf{k}}(k) \geq 0$ for $\mathbf{k} = (0,0)$. A direct computation shows that this is equivalent to $H_k(y) \geq 0$ and, in turn, equivalent to $\det H_k(y) \geq 0$ and the fact that $\mathbf{T}$ is $(k-1)$-hyponormal. Now let

$$A_k := \frac{1}{2} \begin{pmatrix} \frac{4}{3} & 2a^2 & 2a^2 & \frac{5}{4} & \frac{6}{5} & \cdots & \frac{k+3}{k+2} \\ 2a^2 & 2 & 2a^2 & 2a^2 & 2a^2 & \cdots & 2a^2 \\ 2a^2 & 2a^2 & 2a^2 & 2a^2 & 2a^2 & \cdots & 2a^2 \\ \frac{5}{4} & 2a^2 & 2a^2 & \frac{6}{5} & \frac{7}{6} & \cdots & \frac{k+4}{k+3} \\ \frac{6}{5} & 2a^2 & 2a^2 & \frac{7}{6} & \frac{8}{7} & \cdots & \frac{k+5}{k+4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k+3}{k+2} & 2a^2 & 2a^2 & \frac{k+4}{k+3} & \frac{k+5}{k+4} & \cdots & \frac{2k+2}{2k+1} \end{pmatrix}_{(k+2)\times(k+2)}.$$

Note that we can easily calculate $\det A_k$ and $\det H_k(1)$. Indeed, observe that

$$\begin{cases} \det A_k = a_k \cdot a^2(1-a^2)(\frac{(k+1)^2}{2k(k+2)} - a^2) \\ \det H_k(1) = a_k \cdot a^2(a^2-1)\{(1-a^2)(\frac{1}{2} - a^2) + \frac{1}{4(k+1)^2}\} \\ a_k := \frac{(1!2!\cdots(k-1)!)^2 2!3!\cdots(k+1)!}{2^{k-1}(k+2)!(k+3)!\cdots(2k+1)!}k(k+2). \end{cases} \tag{4.1}$$

On the other hand, by the cofactor expansion along the first row or the first column, we have

$$\det H_k(y) = \frac{1}{y^2}\det A_k + \det H_k(1) - \det A_k = (\frac{1}{y^2} - 1)\det A_k + \det H_k(1).$$

Since $\det A_k \geq 0$,

$$\det H_k(y) \geq 0 \Leftrightarrow y^2 \leq \frac{\det A_k}{\det A_k - \det H_k(1)}.$$

Thus,

$$\det H_k(y) \geq 0 \Leftrightarrow y^2 \leq \frac{\frac{(k+1)^2}{2k(k+2)} - a^2}{a^4 - \frac{5}{2}a^2 + \frac{(k+1)^2}{2k(k+2)} + \frac{2k^2+4k+3}{4(k+1)^2}}.$$

Therefore, we see that $\mathbf{T}$ is k-hyponormal ($k \geq 2$) if and only if $0 < y \leq \sqrt{\frac{\frac{(k+1)^2}{2k(k+2)} - a^2}{a^4 - \frac{5}{2}a^2 + \frac{(k+1)^2}{2k(k+2)} + \frac{2k^2+4k+3}{4(k+1)^2}}}$, as desired. □

Corollary 4.9. *Let* $\mathbf{T}$ *be the* 2*-variable weighted shift given by Figure 1, and assume that* $\mathbf{T}$ *is* k*-hyponormal for every* $k \geq 2$*. Then* $0 < y \leq \sqrt{\frac{1}{2-a^2}}$.

Proof. We know that $0 < y \leq D(k)$ for every $k \geq 2$, so

$$y \leq \lim_{k\to\infty} D(k) = \sqrt{\frac{\frac{1}{2} - a^2}{a^4 - \frac{5}{2}a^2 + 1}} = \sqrt{\frac{\frac{1}{2} - a^2}{(a^2 - \frac{1}{2})(a^2 - 2)}} = \sqrt{\frac{1}{2 - a^2}},$$

as desired. □

Remark 4.10. The results in Proposition 4.4 and Corollary 4.9 illustrate Theorem 2.3 (the multivariable Bram-Halmos criterion); that is, the pair $\mathbf{T}$ is subnormal if and only if it is k-hyponormal for every $k \geq 1$.

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