A CHARACTERIZATION OF k-HYPONORMALITY VIA WEAK SUBNORMALITY

RAÚL E. CURTO, IL BONG JUNG, AND SANG SOO PARK

Abstract. An operator T acting on a Hilbert space $\mathcal{H}$ is said to be weakly subnormal if there exists an extension $\widehat{T}$ acting on $\mathcal{K} \supseteq \mathcal{H}$ such that $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ for all $f \in \mathcal{H}$. When such partially normal extensions exist, we denote by m.p.n.e.(T) the minimal one. On the other hand, for $k \geq 1$, T is said to be k-hyponormal if the operator matrix $([T^{*j}, T^i])_{i,j=1}^k$ is positive. We prove that a 2-hyponormal operator T always satisfies the inequality $T^*[T^*,T]T \leq \|T\|^2\,[T^*,T]$, and as a result T is automatically weakly subnormal. Thus, a hyponormal operator T is 2-hyponormal if and only if there exists B such that $BA^* = A^*T$ and $\begin{pmatrix} T & A \\ \mathbf{0} & B \end{pmatrix}$ is hyponormal, where $A := [T^*,T]^{1/2}$. More generally, we prove that T is $(k+1)$-hyponormal if and and only if T is weakly subnormal and m.p.n.e.(T) is k-hyponormal. As an application, we obtain a matricial representation of the minimal normal extension of a subnormal operator as a block staircase matrix.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert spaces and let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. For $S, T \in \mathcal{L}(\mathcal{H})$, we write $[S,T] := ST - TS$ for the commutator of S and T. An operator $T \in \mathcal{L}(\mathcal{H})$ is *normal* if $[T^*,T] = 0$, *hyponormal* if $[T^*,T] \geq 0$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Thus, T is subnormal if and only if there exist A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*,T] = AA^*, \\ A^*T = BA^*, \\ [B^*,B] + A^*A = 0. \end{cases} \tag{1.1}$$

In [1], the notion of weak subnormality was introduced, as follows. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exists A and B such that the first two conditions in (1.1) hold, that is, $[T^*,T] = AA^*$ and $A^*T = BA^*$; the operator $\widehat{T}$ is said to be a *partially normal extension* of T. Clearly, subnormal $\Rightarrow$ weakly subnormal $\Rightarrow$ hyponormal, with the converse implications not true in general [1, Examples 5.1 and 5.5]. As proved in [1], T is weakly subnormal if

---

1991 *Mathematics Subject Classification.* Primary 47B20, 47A20; Secondary 47B37, 47A13, 47A62.

Key words and phrases. weak subnormality, k-hyponormality, minimal normal extensions of subnormal operators.

The research of the first named author was partially supported by NSF grants DMS-9800931 and DMS-0099357.

The research of the second and third named authors was partially supported by a Korea Research Foundation Grant (KRF-2002-070-C00006).

and only if $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ for all $f \in \mathcal{H}$, if and only if $\mathcal{H} \subseteq \ker[\widehat{T}^*, \widehat{T}]$. Weak subnormality is invariant under unitary equivalence, translations, and restrictions to invariant subspaces.

Let T be a weakly subnormal operator on $\mathcal{H}$ and let $\widehat{T}$ be a partially normal extension of T acting on $\mathcal{K}$. $\widehat{T}$ is said to be a *minimal* partially normal extension of T (in symbols, $\widehat{T} = \text{m.p.n.e.}(T)$) if $\mathcal{K}$ has no proper subspace $\mathcal{L}$ containing $\mathcal{H}$ such that $\widehat{T}|_{\mathcal{L}}$ is also a partially normal extension of T. As shown in [1, Lemma 2.5], $\widehat{T} := \text{m.p.n.e.}(T)$ if and only if

$$\mathcal{K} = \vee\{\widehat{T}^{*n}h : h \in \mathcal{H},\ n = 0, 1\}. \tag{1.2}$$

Recall that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n copies). For a natural number $k \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, T is *k-hyponormal* if $(T, T^2, \cdots, T^k)$ is hyponormal. The k-tuple $\mathbf{T} = (T_1, \cdots, T_k)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_k T_k$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, k$, where $\mathbb{C}$ is the set of complex numbers. An operator T is *weakly k-hyponormal* if $(T, T^2, \cdots, T^k)$ is weakly hyponormal. (For information and basic results about these notions, the reader is referred to [2].)

It was shown in [1] that every 2-hyponormal weighted shift is weakly subnormal, but whether the same implication holds for arbitrary operators was left open. In this article we settle this, by establishing that every 2-hyponormal operator is indeed weakly subnormal (Theorem 2.7). This is special case of a more general result, in which we show that an operator T is $(k+1)$-hyponormal if and only if T is weakly subnormal and m.p.n.e.(T) is k-hyponormal (Theorem 3.2). Along the way we show that a 2-hyponormal operator always satisfies the inequality $T^*[T^*, T]T \leq \|T\|^2 [T^*, T]$ (Corollary 2.5). As an application of our results, we find a concrete matricial construction of the minimal normal extension of an arbitrary subnormal operator (cf. (4.3)); our model is especially simple when the operator is quasinormal (Corollary 4.4), and it is partially motivated by J. Stampfli's construction of the minimal normal extension of a subnormal weighted shift [3, pp. 368-373].

2. The Relation of Weak Subnormality to 2-hyponormality

Recall from [1, Cor. 2.3] that if T is weakly subnormal, then $T(\ker[T^*, T]) \subseteq \ker[T^*, T]$. For $T \in \mathcal{L}(\mathcal{H})$ we let

$$\Delta_2 := \begin{pmatrix} I & T^* \\ T & T^*T \end{pmatrix} \text{ and } \Delta_3 := \begin{pmatrix} I & T^* & T^{*2} \\ T & T^*T & T^{*2}T \\ T^2 & T^*T^2 & T^{*2}T^2 \end{pmatrix}.$$

Lemma 2.1. *Let T be a hyponormal operator on $\mathcal{H}$ with $\|T\| < 1$. Assume that*
(i) $T(\ker[T^*, T]) \subseteq \ker[T^*, T]$;
(ii) for $y \notin \ker[T^*, T]$ and $x \in \mathcal{H}$,

$$\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right) \leq \left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right). \tag{2.1}$$

Then

$$(T^*[T^*, T]Ty, y) \leq ([T^*, T]y, y) \quad (\text{all } y \in \mathcal{H}). \tag{2.2}$$

Proof. For $y \in \ker[T^*,T]$, condition (i) readily implies that $Ty \in \ker[T^*,T]$, so (2.2) clearly holds. Assume now that $y \notin \ker[T^*,T]$, and recall the identities

$$\left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \|x+T^*y\|^2 + ([T^*,T]y,y) \tag{2.3}$$

and

$$\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right) = \|Tx+T^*Ty\|^2 + (T^*[T^*,T]Ty,y). \tag{2.4}$$

Substituting x by $-T^*y$ in (2.3) and (2.4), the right-hand sides become $([T^*,T]y,y)$ and $\|[T^*,T]y\|^2 + (T^*[T^*,T]Ty,y)$, respectively. From (2.1) it follows at once that

$$(T^*[T^*,T]Ty,y) \le ([T^*,T]y,y).$$

To proof is now complete. □

Lemma 2.2. *Let T be a hyponormal operator on $\mathcal{H}$. Assume that*
(i) $T(\ker[T^*,T]) \subset \ker[T^*,T]$;
(ii) For $y \notin \ker[T^*,T]$ and $x \in \mathcal{H}$, and all $\delta > 0$,

$$\frac{1}{(\|T\|+\delta)^2}\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right) \le \left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Then,

$$(T^*[T^*,T]Ty,y) \le \|T\|^2 ([T^*,T]y,y) \quad \textit{for all } y \in \mathcal{H}.$$

Proof. Set $T_\delta := \frac{1}{\|T\|+\delta}T$. Since T_δ satisfies properties (i) and (ii) in Lemma 2.1, we have

$$(T_\delta^*[T_\delta^*,T_\delta]T_\delta y,y) \le ([T_\delta^*,T_\delta]y,y),$$

which implies

$$(T^*[T^*,T]Ty,y) \le (\|T\|+\delta)^2([T^*,T]y,y).$$

Since $\delta > 0$ is arbitrary, we obtain

$$(T^*[T^*,T]Ty,y) \le \|T\|^2 ([T^*,T]y,y)$$

for all $y \in \mathcal{H}$, as desired. □

It follows from [4, Lemma 2.6] that if $\alpha, \beta \ge 0$ and $0 \ne \gamma \in \mathbb{C}$, then

$$|z|^2\alpha + \beta + 2\mathrm{Re}(z\gamma) \ge 0 \ \ (\text{all } z \in \mathbb{C}) \Leftrightarrow |\gamma|^2 \le \alpha\beta. \tag{2.5}$$

Lemma 2.3. *Let T be a 2-hyponormal operator on $\mathcal{H}$, and let x_1, x_2, y_1, $y_2 \in \mathcal{H}$. Then*

(2.6)

$$\left|\left(\Delta_2 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} Tx_1 \\ Tx_2 \end{bmatrix}\right)\right|^2 \le \left(\Delta_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\left(\Delta_2 \begin{bmatrix} Ty_1 \\ Ty_2 \end{bmatrix}, \begin{bmatrix} Ty_1 \\ Ty_2 \end{bmatrix}\right).$$

Proof. Since T is 2-hyponormal, we have $\Delta_3 \ge 0$. For $z \in \mathbb{C}$, we must then have

$$0 \le \left(\Delta_3 \begin{bmatrix} x_1 \\ x_2 + zy_1 \\ zy_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 + zy_1 \\ zy_2 \end{bmatrix}\right)$$

$$= \left(\Delta_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + z\left(\Delta_2 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} Tx_1 \\ Tx_2 \end{bmatrix}\right)$$
$$+ \bar{z}\left(\Delta_2 \begin{bmatrix} Tx_1 \\ Tx_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) + |z|^2\left(\Delta_2 \begin{bmatrix} Ty_1 \\ Ty_2 \end{bmatrix}, \begin{bmatrix} Ty_1 \\ Ty_2 \end{bmatrix}\right).$$

The result now follows from (2.5). □

Proposition 2.4. *Let T be a 2-hyponormal operator on $\mathcal{H}$. Then*
(i) $T(\ker[T^*,T]) \subseteq \ker[T^*,T]$;
(ii) For $y \notin \ker[T^*,T]$ and $x \in \mathcal{H}$, and all $\delta > 0$,

$$\frac{1}{(\|T\|+\delta)^2}\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right) \le \left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Proof. (i) is [1, Lemma 2.2], so it is sufficient to establish (ii). Suppose there exist $y \notin \ker[T^*,T]$, $x \in \mathcal{H}$ and $\delta > 0$ such that

$$\left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) < \frac{1}{(\|T\|+\delta)^2}\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right). \tag{2.7}$$

By Lemma 2.3,

$$\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right)^2$$
$$\le \left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right)\left(\Delta_2 \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}, \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}\right)$$
$$< \frac{1}{(\|T\|+\delta)^2}\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right)\left(\Delta_2 \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}, \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}\right).$$

Thus,

$$\left(\Delta_2 \begin{bmatrix} Tx \\ Ty \end{bmatrix}, \begin{bmatrix} Tx \\ Ty \end{bmatrix}\right) < \frac{1}{(\|T\|+\delta)^2}\left(\Delta_2 \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}, \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}\right),$$

which implies that

$$\left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) < \frac{1}{(\|T\|+\delta)^4}\left(\Delta_2 \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}, \begin{bmatrix} T^2x \\ T^2y \end{bmatrix}\right). \tag{2.8}$$

Repeating this argument inductively, using (2.8) instead of (2.7), we obtain

$$\left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) < \frac{1}{(\|T\|+\delta)^{2n}}\left(\Delta_2 \begin{bmatrix} T^nx \\ T^ny \end{bmatrix}, \begin{bmatrix} T^nx \\ T^ny \end{bmatrix}\right)$$
$$\le \frac{\|\Delta_2\|}{(\|T\|+\delta)^{2n}}\left(\|T^nx\|^2 + \|T^ny\|^2\right)$$
$$\le \left(\frac{\|T\|}{\|T\|+\delta}\right)^{2n}\|\Delta_2\|\left(\|x\|^2+\|y\|^2\right),$$

for all $n \ge 1$. Since the right-hand side converges to 0 as $n \to \infty$, and since $\Delta_2 \ge 0$, we see that

$$\|x + T^*y\|^2 + ([T^*,T]y,y) = \left(\Delta_2 \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = 0.$$

It follows that $y \in \ker[T^*,T]$, a contradiction. This completes the proof. □

By combining Lemma 2.2 and Proposition 2.4, we obtain at once

Corollary 2.5. *Let T be 2-hyponormal. Then* $T^*[T^*,T]T \le \|T\|^2\,[T^*,T]$.

The theorem below is the main result of this section. Before we give its proof, we need one more ingredient. Assume $T \in \mathcal{L}(\mathcal{H})$ is hyponormal, that is, $[T^*, T] \geq 0$. Then

$$\ker[T^*, T] = \ker[T^*, T]^{\frac{1}{2}} = \ker[T^*, T]^2 \tag{2.9}$$

and

$$(\mathrm{ran}[T^*, T])^- = (\mathrm{ran}[T^*, T]^{\frac{1}{2}})^- = (\mathrm{ran}[T^*, T]^2)^-. \tag{2.10}$$

We write $\mathcal{R} \equiv \mathcal{R}_T = (\mathrm{ran}[T^*, T])^-$ and define $A \equiv A_T : \mathcal{R} \to \mathcal{H}$ by $Ax := [T^*, T]^{\frac{1}{2}}x$ $(x \in \mathcal{R})$. The proof of following lemma is straightforward.

Lemma 2.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be hyponormal, and let $A \in \mathcal{L}(\mathcal{R}, \mathcal{H})$ as above. Then*
(i) $A^*x = [T^*, T]^{\frac{1}{2}}x$ *(all $x \in \mathcal{H}$);*
(ii) $AA^*x = [T^*, T]x$ *(all $x \in \mathcal{H}$);*
(iii) $A^*Ax = [T^*, T]x$ *(all $x \in \mathcal{R}$);*
(iv) $\mathcal{R} = (\mathrm{ran}A^*A)^- = (\mathrm{ran}A)^- = (\mathrm{ran}A^*)^-$.

Theorem 2.7. *Let T be 2-hyponormal. Then T is weakly subnormal.*

Proof. In the notation of Lemma 2.6, recall that $Ax = [T^*, T]^{\frac{1}{2}}x$ for all $x \in \mathcal{R}$. Let $g \in \mathrm{ran}A^*$, say $g = A^*f$ for some $f \in \mathcal{H}$. Define a linear map $B : \mathrm{ran}A^* \to \mathrm{ran}A^*$ by $Bg = A^*Tf$ $(g \in \mathrm{ran}A^*)$. To see that B is well-defined, suppose $A^*f = 0$, i.e., $[T^*, T]^{\frac{1}{2}}f = 0$, by Lemma 2.6. Then $f \in \ker[T^*, T]$ (by (2.9), so Proposition 2.4(i) implies that $Tf \in \ker[T^*, T]$, so $[T^*, T]^{\frac{1}{2}}Tf = 0$, again by (2.9). Then $A^*Tf = [T^*, T]^{\frac{1}{2}}Tf = 0$, showing that B is well-defined. Next we claim that $\|Bg\| \leq \|T\| \, \|g\|$ for all $g \in \mathrm{ran}A^*A \subseteq \mathrm{ran}A^*$. Let $g = A^*Af$, with $f \in \mathcal{R}$. Then

$$\begin{aligned}
\|Bg\|^2 &= \|BA^*(Af)\|^2 = \|A^*TAf\|^2 = (A^*TAf, A^*TAf) \\
&= ([T^*, T]^{\frac{1}{2}}T[T^*, T]^{\frac{1}{2}}f, [T^*, T]^{\frac{1}{2}}T[T^*, T]^{\frac{1}{2}}f) \\
&= ([T^*, T]^{\frac{1}{2}}T^*[T^*, T]T[T^*, T]^{\frac{1}{2}}f, f) \\
&\leq \|T\|^2 ([T^*, T]f, [T^*, T]f) \qquad \text{(by Corollary 2.5)} \\
&= \|T\|^2 (A^*Af, A^*Af) \\
&= \|T\|^2 (g, g).
\end{aligned}$$

Since $(\mathrm{ran}A^*)^- = \mathcal{R} = (\mathrm{ran}A^*A)^-$, B can be extended to all of $\mathcal{R}$. Hence B is a bounded operator on $\mathcal{R}$ with $\|B\| \leq \|T\|$ such that $BA^* = A^*T$. If we let $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ relative to the decomposition $\mathcal{K} := \mathcal{H} \oplus \mathcal{R}$, then $\widehat{T}$ is a partially normal extension of T. Since $\mathcal{H} \oplus \mathcal{R} = \mathcal{H} \oplus (\mathrm{ran}[T^*, T])^-$, an application of Lemma 2.6 and (1.2) shows that $\widehat{T} = \text{m.p.n.e.}(T)$. □

Corollary 2.8. *Let T be 2-hyponormal, and let $\widehat{T} = \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ be the minimal partially normal extension of T. Then $\|B\| \leq \|T\|$.*

Proof. Straightforward from the proof of Theorem 2.7. □

3. k-HYPONORMALITY AND WEAK SUBNORMALITY

Let T be weakly subnormal on $\mathcal{H}$, and let $\widehat{T} = \text{m.p.n.e.}(T)$. Given $x \in \mathcal{H}$, we know that $\widehat{T}^*\widehat{T}x = \widehat{T}\widehat{T}^*x$. More generally, since $\widehat{T}^k x \in \mathcal{H}$ for all $k \in \mathbb{N}$, $k \geq 1$, we have

$$\widehat{T}^*\widehat{T}^k x = \widehat{T}^*\widehat{T}\widehat{T}^{k-1}x = \widehat{T}\widehat{T}^*\widehat{T}^{k-1}x = \ldots = \widehat{T}^k\widehat{T}^* x.$$

A straightforward computation then leads to

$$(\widehat{T}^{*j+1}\widehat{T}^{i+1}x, y) = (\widehat{T}\widehat{T}^{*j}\widehat{T}^i\widehat{T}^* x, y) \ \ (\text{all } x, y \in \mathcal{H} \text{ and all } i, j \in \mathbb{N}). \tag{3.1}$$

The following result is of independent interest.

Lemma 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be weakly subnormal, let $\{x_j\}_{j=0}^{k+1} \subseteq \mathcal{H}$, and define $z_0 := x_0 + \widehat{T}^* x_1$ and $z_j := \widehat{T}^* x_{j+1}$, $1 \leq j \leq k$. Then*

$$\sum_{i,j=0}^{k+1} (T^{*j}T^i x_j, x_i) = \sum_{i,j=0}^{k} (\widehat{T}^{*j}\widehat{T}^i z_j, z_i).$$

Proof. We first observe that

$$\begin{aligned}\sum_{i,j=0}^{k+1} (T^{*j}T^i x_j, x_i) &= \sum_{i,j=0}^{1} (T^{*j}T^i x_j, x_i) + \sum_{j=2}^{k+1}\{(T^{*j}x_j, x_0) \\ &\quad + (T^{*j}Tx_j, x_1\} + \sum_{i=2}^{k+1}\{(T^i x_0, x_i) + (T^*T^i x_1, x_i)\} \\ &\quad + \sum_{i,j=2}^{k+1} (T^{*j}T^i x_j, x_i) \\ &=: \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.\end{aligned}$$

We now express these four sums in terms of the variables z_j, as follows:

$$\Gamma_1 = \|x_0 + \widehat{T}^* x_1\|^2 = \|z_0\|^2,$$

$$\begin{aligned}\Gamma_2 &= \sum_{j=2}^{k+1}\{(\widehat{T}^{*j}x_j, x_0) + (\widehat{T}^{*j}\widehat{T}x_j, x_1)\} \\ &= \sum_{j=1}^{k}\{(\widehat{T}^{*j}\widehat{T}^* x_{j+1}, x_0) + (\widehat{T}^{*j}\widehat{T}^* x_{j+1}, \widehat{T}^* x_1)\} = \sum_{j=1}^{k}(\widehat{T}^{*j}z_j, z_0),\end{aligned}$$

$$\begin{aligned}\Gamma_3 &= \sum_{i=2}^{k+1}\{(\widehat{T}^i x_0, x_i) + (\widehat{T}^*\widehat{T}^i x_1, x_i)\} \\ &= \sum_{i=1}^{k}\{(\widehat{T}^i x_0, \widehat{T}^* x_{i+1}) + (\widehat{T}^i\widehat{T}^* x_1, \widehat{T}^* x_{i+1})\} = \sum_{i=1}^{k}(\widehat{T}^i z_0, z_i),\end{aligned}$$

$$\Gamma_4 = \sum_{i,j=1}^{k} (\widehat{T}\widehat{T}^{*j}\widehat{T}^i\widehat{T}^* x_{j+1}, x_{i+1}) = \sum_{i,j=1}^{k} (\widehat{T}^{*j}\widehat{T}^i z_j, z_i) \ \ (\text{using (3.1)}).$$

If we now combine the new expressions for the Γ_i's, we obtain

$$\begin{aligned}\sum_{i,j=0}^{k+1}(T^{*j}T^i x_j, x_i) &= \|z_0\|^2 + \sum_{j=1}^{k}(\widehat{T}^{*j} z_j, z_0) \\ &\quad + \sum_{i=1}^{k}(\widehat{T}^i z_0, z_i) + \sum_{i,j=1}^{k}(\widehat{T}^{*j}\widehat{T}^i z_j, z_i) \\ &= \sum_{i,j=0}^{k}(\widehat{T}^{*j}\widehat{T}^i z_j, z_i),\end{aligned}$$

as desired. □

Recall now that $T \in \mathcal{L}(\mathcal{H})$ is k-hyponormal if and only if $\sum_{i,j=0}^{k}(T^i x_j, T^j x_i) \geq 0$ for all $x_i, x_j \in \mathcal{H}$, $0 \leq i,j \leq k$ [2]; this is the analog, for k-hyponormality, of the Bram-Halmos condition for subnormality [5].

Theorem 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $k \geq 1$. The following statements are equivalent.*
(i) T is $(k+1)$-hyponormal;
(ii) T is weakly subnormal and $\widehat{T} :=$ m.p.n.e.(T) is k-hyponormal.

Proof. (ii) $\Rightarrow$ (i): Straightforward from Lemma 3.1.
(i) $\Rightarrow$ (ii): Since T is 2-hyponormal, we know by Theorem 2.7 that T is weakly subnormal. Let $\widehat{T} :=$ m.p.n.e.(T), acting on

$$\mathcal{K} = \vee\{\widehat{T}^{*n} h : h \in \mathcal{H},\ n = 0,1\},$$

by (1.2). Let $z_j := x_j + \widehat{T}^* y_j \in \mathcal{K}$, where $x_j, y_j \in \mathcal{H}$, $0 \leq i,j \leq k$. We shall prove that $\sum_{i,j=0}^{k}(\widehat{T}^{*j}\widehat{T}^i z_j, z_i) \geq 0$, and this will establish the k-hyponormality of $\widehat{T}$. Set $w_j := y_{j-1} + x_j$, $0 \leq j \leq k+1$ with $y_{-1} = x_{k+1} = 0$. Then we have

$$\begin{aligned}\sum_{i,j=0}^{k}(\widehat{T}^{*j}\widehat{T}^i z_j, z_i) &= \sum_{i,j=0}^{k}(\widehat{T}^{*j}\widehat{T}^i(x_j + \widehat{T}^* y_j), (x_i + \widehat{T}^* y_i)) \\ &= \sum_{i,j=0}^{k}\{(\widehat{T}^{*j}\widehat{T}^i x_j, x_i) + (\widehat{T}^{*j}\widehat{T}^i\widehat{T}^* y_j, x_i) \\ &\quad + (\widehat{T}^{*j}\widehat{T}^i x_j, \widehat{T}^* y_i) + (\widehat{T}^{*j}\widehat{T}^i\widehat{T}^* y_j, \widehat{T}^* y_i)\} \\ &= \sum_{i,j=0}^{k}\{(\widehat{T}^{*j}\widehat{T}^i x_j, x_i) + (\widehat{T}^{*j+1}\widehat{T}^i y_j, x_i) \\ &\quad + (\widehat{T}^{*j}\widehat{T}^{i+1} x_j, y_i) + (\widehat{T}^{*j+1}\widehat{T}^{i+1} y_j, y_i)\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j=0}^{k} (T^{*j}T^{i}x_j, x_i) + \sum_{i,j=0}^{k} (T^{*j+1}T^{i}y_j, x_i) \\
&+ \sum_{i,j=0}^{k} (T^{*j}T^{i+1}x_j, y_i) + \sum_{i,j=0}^{k} (T^{*j+1}T^{i+1}y_j, y_i) \\
&= \sum_{i,j=0}^{k+1} (T^{*j}T^{i}x_j, x_i) + \sum_{i,j=0}^{k+1} (T^{*j}T^{i}y_{j-1}, x_i) \\
&+ \sum_{i,j=0}^{k+1} (T^{*j}T^{i}x_j, y_{i-1}) + \sum_{i,j=0}^{k+1} (T^{*j}T^{i}y_{j-1}, y_{i-1})
\end{aligned}
$$

(the final equality using the convention $y_{-1} = x_{k+1} = 0$). Hence

$$
\begin{aligned}
\sum_{i,j=0}^{k} (\widehat{T}^{*j}\widehat{T}^{i}z_j, z_i) &= \sum_{i,j=0}^{k+1} \left(T^{*j}T^{i}\left(y_{j-1} + x_j\right), y_{i-1} + x_i\right) \\
&= \sum_{i,j=0}^{k+1} (T^{*j}T^{i}w_j, w_i) \geq 0
\end{aligned}
$$

(because T is $(k+1)$-hyponormal), as desired. $\square$

We conclude this section with an extension of [1, Theorem 1.2(iii)]; for related results, see [6, Section 2]. For $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$ for the spectrum of T and $r(T)$ for the spectral radius of T.

Corollary 3.3. ***Let*** *$T \in \mathcal{L}(\mathcal{H})$ be 2-hyponormal and let $\widehat{T} :=$ m.p.n.e.(T). Then* $\|\widehat{T}\| = \|T\|$.

Proof. By Theorem 3.2, $\widehat{T} \equiv \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is hyponormal, so $r(\widehat{T}) = \|\widehat{T}\|$. Since $\sigma(\widehat{T}) \subseteq \sigma(T) \cup \sigma(B)$ and $r(B) \leq \|B\| \leq \|T\|$ (by Corollary 2.8), we see that $\|\widehat{T}\| = r(\widehat{T}) \leq \max\{r(T), r(B)\} \leq \|T\|$. $\square$

4. Construction of the minimal normal extension of a subnormal operator

Let $k \geq 2$ and assume that $T \in \mathcal{L}(\mathcal{H})$ is $(k+1)$-hyponormal. By Theorem 2.7, T is weakly subnormal. Let

$$
\widehat{T} = \begin{pmatrix} T & A_1 \\ 0 & B_1 \end{pmatrix}
$$

be the m.p.n.e. of T on $\mathcal{H}_1 := \mathcal{H} \oplus \mathcal{R}_1$, where $\mathcal{R}_1 := (\text{ran}[T^*, T])^{-}$, $A_1 := [T^*, T]^{\frac{1}{2}}|_{\mathcal{R}_1}$ and $B_1 : \mathcal{R}_1 \to \mathcal{R}_1$ is given by $B_1(A_1^* f) := A_1^* T f$. Since $\widehat{T}$ is 2-hyponormal, it is also weakly subnormal, and we can repeat the process to obtain

$$
\widehat{\widehat{T}} = \begin{pmatrix} \widehat{T} & A_2 \\ 0 & B_2 \end{pmatrix}
$$

on $\mathcal{H}_2 := (\mathcal{H} \oplus \mathcal{R}_1) \oplus \mathcal{R}_2$, where $A_2 := [\widehat{T}^*, \widehat{T}]^{\frac{1}{2}}|_{\mathcal{R}_2}$, $\mathcal{R}_2 := (\text{ran}[\widehat{T}^*, \widehat{T}])^-$ and $B_2 : \mathcal{R}_2 \to \mathcal{R}_2$ satisfies $B_2(A_2^* f) = A_2^* \widehat{T} f$. For $n \geq 2$ we then write $\widehat{T}^{(n)} := \widehat{\widehat{T}^{(n-1)}}$. We thus have

$$\widehat{T}^{(n)} = \begin{pmatrix} \widehat{T}^{(n-1)} & A_n \\ 0 & B_n \end{pmatrix} \quad (1 \leq n \leq k)$$

relative to the decomposition $\mathcal{H}_n := (\mathcal{H} \oplus \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_{n-1}) \oplus \mathcal{R}_n$, where $A_n := [\widehat{T}^{(n-1)*}, \widehat{T}^{(n-1)}]^{\frac{1}{2}}|_{\mathcal{R}_n}$, $\mathcal{R}_n := (\text{ran}[\widehat{T}^{(n-1)*}, \widehat{T}^{(n-1)}])^-$ and $B_n : \mathcal{R}_n \to \mathcal{R}_n$ satisfies $B_n(A_n^* f) = A_n^* \widehat{T}^{(n-1)} f$. For notational convenience we set $\mathcal{H}_0 := \mathcal{H}$.

Lemma 4.1. *Let $k \geq 2$ and assume that $T \in \mathcal{L}(\mathcal{H})$ is $(k+1)$-hyponormal. Then $\widehat{T}^{(k)*}\mathcal{H}_{k-2} \subseteq \mathcal{H}_{k-1}$.*

Proof. Since $P_{\mathcal{H}_{k-1}} \widehat{T}^{(k)}|_{\mathcal{H}_k \ominus \mathcal{H}_{k-1}} = A_k$ and $P_{\mathcal{H}_{k-2}}[\widehat{T}^{(k-1)*}, \widehat{T}^{(k-1)}]^{\frac{1}{2}} = 0$, we have

$$P_{\mathcal{H}_{k-2}} \widehat{T}^{(k)}|_{\mathcal{H}_k \ominus \mathcal{H}_{k-1}} = P_{\mathcal{H}_{k-2}} A_k = P_{\mathcal{H}_{k-2}}[\widehat{T}^{(k-1)*}, \widehat{T}^{(k-1)}]^{\frac{1}{2}} = 0.$$

Let $f \in \mathcal{H}_{k-2}$ and $g \in \mathcal{H}_k \ominus \mathcal{H}_{k-1}$. Then

$$(\widehat{T}^{(k)*} f, g) = (f, \widehat{T}^{(k)} g) = (P_{\mathcal{H}_{k-2}} f, \widehat{T}^{(k)} g) = (f, P_{\mathcal{H}_{k-2}} \widehat{T}^{(k)} g) = 0,$$

which shows that $\widehat{T}^{(k)*} f \in \mathcal{H}_{k-1}$, as desired. □ □

Recall that if T is a subnormal operator on $\mathcal{H}$ and N is a normal extension of T on $\mathcal{K}$, then N is a *minimal* normal extension of T if and only if

$$\mathcal{K} = \vee \{N^{*k} h : h \in \mathcal{H},\ k \geq 0\} \quad ([7, \text{Proposition } 2.4]). \tag{4.1}$$

For subnormal weighted shifts, J. Stampfli gave in [3] an explicit construction of the minimal normal extension. Motivated in part by that construction, we will now describe a block staircase operator matrix model for the minimal normal extension of an arbitrary subnormal operator. Let T be subnormal on $\mathcal{H}$, and recall that T is $(k+1)$-hyponormal for all $k \geq 0$, so successive partially normal extensions exist. We recursively define $\widehat{T}^{(m)}$ as $\widehat{T}^{(0)} := T$ and $\widehat{T}^{(m+1)} := m.p.n.e.(\widehat{T}^{(m)})$. Let $\mathcal{H}_\infty := \mathcal{H} \oplus \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_n \oplus \cdots$ with the ℓ^2 inner product, that is, if $\widetilde{x} := x_0 \oplus x_1 \oplus \cdots \oplus x_m \oplus \cdots$ and $\widetilde{y} := y_0 \oplus y_1 \oplus \cdots \oplus y_m \oplus \cdots$, then $\langle \widetilde{x}, \widetilde{y} \rangle_{\mathcal{H}_\infty} := \sum_i (x_i, y_i)$. For $\widetilde{x} \equiv x_0 \oplus \cdots \oplus x_m \oplus 0 \oplus 0 \oplus \cdots \equiv \widetilde{x}_m \oplus 0 \oplus 0 \oplus \cdots$, we define $N\widetilde{x} := \widehat{T}^{(m)} \widetilde{x}_m$. For notational convenience, we will identify $\mathcal{H}_m$ with the subspace of $\mathcal{H}_\infty$ of all vectors $\widetilde{x} \equiv x_0 \oplus \cdots \oplus x_m \oplus 0 \oplus 0 \oplus \cdots$, and we shall write $\widetilde{x}_m \equiv P_{\mathcal{H}_m} \widetilde{x}$. Also, we shall regard $\mathcal{H}_m$ as a subspace of $\mathcal{H}_{m+1}$ Clearly N is linear on a dense linear submanifold of $\mathcal{H}_\infty$. Moreover, by Corollary 3.3,

$$\|N\widetilde{x}\| = \|\widehat{T}^{(m)} \widetilde{x}_m\| \leq \|\widehat{T}^{(m-1)}\| \|\widetilde{x}_m\| = \cdots = \|T\| \, \|\widetilde{x}\| \tag{4.2}$$

whenever $\widetilde{x} \equiv \widetilde{x}_m \oplus 0 \oplus 0 \oplus \cdots$. Therefore, N can be extended to a bounded linear operator on $\mathcal{H}_\infty$, which we will also denote by N. By (4.2), $\|N\| = \|T\|$.

Lemma 4.2. $N^* \widetilde{x} = \widehat{T}^{(m+1)*} \widetilde{x}_m$, *for all* $\widetilde{x} \equiv \widetilde{x}_m \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{H}_\infty$.

Proof. Since $\widehat{T}^{(\ell+k)}|_{\mathcal{H}_\ell} = \widehat{T}^{(\ell)}$ for all $k \geq 1$, $\widehat{T}^{(\ell)*} = P_{\mathcal{H}_\ell} \widehat{T}^{(\ell+k)*}|_{\mathcal{H}_\ell}$. By Lemma 4.1, $\widehat{T}^{(m+2)*} \widetilde{x}_m \in \mathcal{H}_{m+1}$, so

$$\begin{aligned}(\widehat{T}^{(m+1)*} \widetilde{x}_m, \widetilde{y}) &= (P_{\mathcal{H}_{m+1}} \widehat{T}^{(m+2)*} \widetilde{x}_m, \widetilde{y}) \\ &= (\widehat{T}^{(m+2)*} \widetilde{x}_m, P_{\mathcal{H}_{m+1}} \widetilde{y}) \\ &= (\widehat{T}^{(m+2)*} \widetilde{x}_m, \widetilde{y})\end{aligned}$$

for any $\widetilde{y} \in \mathcal{H}_\infty$, which implies that $\widehat{T}^{(m+1)*}\widetilde{x}_m = \widehat{T}^{(m+2)*}\widetilde{x}_m$ for all $\widetilde{x}_m \in \mathcal{H}_m$. Also, since $\mathcal{H}_m \subseteq \mathcal{H}_{m+k}$ for any $k \geq 1$, we can continue this process to obtain $\widehat{T}^{(m+1)*}\widetilde{x}_m = \widehat{T}^{(m+1+k)*}\widetilde{x}_m$ for all $\widetilde{x}_m \in \mathcal{H}_m$. Given $\widetilde{x}_m \in \mathcal{H}_m$ and $\widetilde{y}_\ell \in \mathcal{H}_\ell$, and if we let $n := \max\{m+1, \ell\}$, we obtain

$$\begin{aligned}(N^*\widetilde{x}_m, \widetilde{y}_\ell) &= (\widetilde{x}_m, N\widetilde{y}_\ell) = (\widetilde{x}_m, \widehat{T}^{(n)}\widetilde{y}_\ell) \\ &= (\widehat{T}^{(n)*}\widetilde{x}_m, \widetilde{y}_\ell) = (\widehat{T}^{(m+1)*}\widetilde{x}_m, \widetilde{y}_\ell),\end{aligned}$$

which readily implies that $N^*\widetilde{x} \equiv N^*\widetilde{x}_m = \widehat{T}^{(m+1)*}\widetilde{x}_m$, as desired. □

We are now ready to present the staircase model for the minimal normal extension of a subnormal operator.

Theorem 4.3. *Let T be a subnormal operator on $\mathcal{H}$ and let N be the operator on $\mathcal{H}_\infty := \mathcal{H} \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots$ defined as $N\widetilde{x} := \widehat{T}^{(m)}\widetilde{x}_m$, for $\widetilde{x} \equiv \widetilde{x}_m \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{H}_\infty$. Then N is a minimal normal extension of T with $\|N\| = \|T\|$. In particular, N has the following block staircase operator matrix form:*

$$N = \begin{pmatrix} T & A_1 & 0 & & & \\ 0 & B_1 & C_2 & 0 & & \\ & 0 & B_2 & C_3 & 0 & \\ & & 0 & B_3 & C_4 & \ddots \\ & & & 0 & B_4 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}, \tag{4.3}$$

where

$$\widehat{T} = \begin{pmatrix} T & A_1 \\ 0 & B_1 \end{pmatrix}, \quad \widehat{T}^{(k+1)} = \begin{pmatrix} \widehat{T}^{(k)} & A_k \\ 0 & B_k \end{pmatrix}, \ k = 1, 2, ...,$$

$\mathcal{R}_k := (\text{ran}[\widehat{T}^{(k-1)*}, \widehat{T}^{(k-1)}])^-$, $A_k := [\widehat{T}^{(k-1)*}, \widehat{T}^{(k-1)}]^{\frac{1}{2}}|_{\mathcal{R}_k}$ regarded as an operator from $\mathcal{R}_k$ to $\mathcal{H}_{k-1}$, and $C_k := (A_k^*|_{\mathcal{R}_{k-1}})^* : \mathcal{R}_k \to \mathcal{R}_{k-1}$.

Proof. We first establish that N is normal. Let $\widetilde{x} \equiv \widetilde{x}_m \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{H}_\infty$. Then

$$\begin{aligned}\|N\widetilde{x}\|^2 &= \|\widehat{T}^{(m)}\widetilde{x}_m\|^2 = (\widehat{T}^{(m)*}\widehat{T}^{(m)}\widetilde{x}_m, \widetilde{x}_m) \\ &= (\widehat{T}^{(m+1)*}\widehat{T}^{(m+1)}\widetilde{x}_m, \widetilde{x}_m) = (\widehat{T}^{(m+1)}\widehat{T}^{(m+1)*}\widetilde{x}_m, \widetilde{x}_m) \\ &= (NN^*\widetilde{x}, \widetilde{x}) = \|N^*\widetilde{x}\|^2.\end{aligned}$$

Hence N is a normal operator on $\mathcal{H}_\infty$. Now we turn to the proof of the minimality of N. By (1.2) and Lemma 4.2,

$$(\{x_0 + N^*x_1 : x_i \in \mathcal{H},\ i = 0, 1\})^- = (\{x_0 + \widehat{T}^*x_1 : x_i \in \mathcal{H},\ i = 0, 1\})^- = \mathcal{H}_1.$$

To use mathematical induction, we assume that

$$(\{\sum_{i=0}^{n} N^{*i}x_i : x_i \in \mathcal{H},\ i = 0, ..., n\})^- = \mathcal{H}_n.$$

Then

$$\begin{aligned}\mathcal{H}_{n+1} &= (\{\sum_{i=0}^{n} N^{*i}p_i + \widehat{T}^{(n+1)*}(\sum_{j=0}^{n} N^{*j}q_j) : p_i, q_j \in \mathcal{H},\ 0 \le i,j \le n\})^- \\ &= (\{\sum_{i=0}^{n} N^{*i}p_i + N^{*}(\sum_{j=0}^{n} N^{*j}q_j) : p_i, q_j \in \mathcal{H},\ 0 \le i,j \le n\})^- \quad \text{(by Lemma 4.2)} \\ &= (\{\sum_{i=0}^{n} N^{*i}z_i + N^{*n+1}z_{n+1} : z_i \in \mathcal{H},\ 0 \le i \le n\})^- \\ &= (\{\sum_{i=0}^{n+1} N^{*i}x_i : x_i \in \mathcal{H},\ i = 0, ..., n+1\})^- .\end{aligned}$$

Thus $\vee\{N^{*k}h : h \in \mathcal{H},\ k \ge 0\} = \mathcal{H}_\infty$. By (4.1), N is a minimal normal extension of T. Finally, the block staircase matrix form in (4.3) follows easily from the construction of $\widehat{T}^{(n)}$ and [1, Lemma 2.8]. Thus, the proof is complete. □

We conclude this section with a special but important case of Theorem 4.3. Recall that an operator T is said to be *quasinormal* if T commutes with T^*T or, equivalently, if $[T^*, T]T = 0$.

Corollary 4.4. *Let T be a weakly subnormal operator on $\mathcal{H}$, and let $\widehat{T} := m.p.n.e.(T)$. Then*
(i) T is quasinormal if and only if $\widehat{T} = \begin{pmatrix} T & A_1 \\ 0 & 0 \end{pmatrix}$ (A_1 as in Theorem 4.3);
(ii) if T is quasinormal, then $\widehat{T}$ is quasinormal;
(iii) if T is quasinormal, then the minimal normal extension N of T is unitarily equivalent to the following operator matrix

$$\begin{pmatrix} T & A_1 & 0 & & & \\ 0 & 0 & C_2 & 0 & & \\ & 0 & 0 & C_3 & 0 & \\ & & 0 & 0 & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}.$$

Proof. (i) ($\Rightarrow$) Since $A_1^*T = B_1A_1^*$, we see that

$$0 = [T^*, T]T = A_1A_1^*T = A_1B_1A_1^*.$$

Also, since $(\text{ran}A_1^*)^- = \mathcal{R}_1$, $A_1B_1 = 0$, so $B_1^*A_1^* = 0$, and a fortiori $B_1^* = 0$, or $B_1 = 0$.
($\Leftarrow$) Since $A_1^*T = 0$, we have $A_1A_1^*T = [T^*, T]T = 0$, which shows that T is quasinormal.
(ii) By a simple computation, $[\widehat{T}^*, \widehat{T}]\widehat{T} = 0$. Hence $\widehat{T}$ is quasinormal.
(iii) This follows easily from Theorem 4.3. □

References

[1] R.E. Curto and W.Y. Lee, *Towards a model theory for 2-hyponormal operators*, Integral Equations Operator Theory, to appear.
[2] R. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, Proc. Symposia Pure Math. **51** (1990), 69-91.

[3] J. Stampfli, *Which weighted shifts are subnormal*, Pacific J. Math. **17** (1966), 367-379.
[4] J.B. Conway and W. Szymanski, *Linear combination of hyponormal operators,* Rocky Mountain J. **18** (1988), 695-705.
[5] J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75-94.
[6] R. Curto, I.S. Hwang and W.Y. Lee, *Weak subnormality of operators*, Archiv der Math., to appear.
[7] J.B. Conway, *The Theory of Subnormal Operators,* Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

Department of Mathematics, The University of Iowa, College of Liberal Arts and Sciences, Iowa City, Iowa 52242, USA
E-mail address: curto@math.uiowa.edu

Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu 702-701, Korea
E-mail address: ibjung@knu.ac.kr

Department of Mathematics, College of Natural Sciences, Kyungpook National University, Taegu 702-701, Korea
E-mail address: pss4855@hanmail.net