Recursively determined
representing measures for
bivariate truncated moment sequences
(joint work with Lawrence Fialkow; to appear in JOT)

Raúl Curto

IWOTA, UNSW, Sydney, July 17, 2012
The Classical (Full) Moment Problem

Let \( \beta \equiv \beta(\infty) = \{\beta_i\}_{i \in \mathbb{Z}_+^m} \) denote a \( m \)-dimensional real multisequence, and let \( K \) (closed) \( \subseteq \mathbb{R}^m \). The (full) \( K \)-moment problem asks for necessary and sufficient conditions on \( \beta \) to guarantee the existence of a positive Borel measure \( \mu \) supported in \( K \) such that

\[
\beta_i = \int x^i d\mu \quad (i \in \mathbb{Z}_+^m);
\]

\( \mu \) is called a rep. meas. for \( \beta \).

Associated with \( \beta \) is a moment matrix \( M \equiv M(\infty) \), defined by

\[
M_{ij} := \beta_{i+j} \quad (i, j \in \mathbb{Z}_+^m).
\]
The Truncated Moment Problem (TMP)

Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}^m_+, |i|+|j|\leq 2n}$ denote a $m$-dimensional real multisequence, and let $K$ (closed) $\subseteq \mathbb{R}^m$. The (truncated) $K$-moment problem asks for necessary and sufficient conditions on $\beta$ to guarantee the existence of a positive Borel measure $\mu$ supported in $K$ such that

$$\beta_i = \int x^i \, d\mu \quad (i \in \mathbb{Z}^m_+, |i| + |j| \leq 2n);$$

$\mu$ is called a rep. meas. for $\beta$.

Associated with $\beta$ is a moment matrix $M \equiv M(n)$, defined by

$$M_{ij} := \beta_{i+j} \quad (i, j \in \mathbb{Z}^m_+, |i| + |j| \leq 2n).$$
Basic Positivity Condition

$\mathcal{P}_n$: polynomials $p$ over $\mathbb{R}$ with $\deg p \leq n$

- Given $p \in \mathcal{P}_n$, $p(x) \equiv \sum_{0 \leq i+j \leq n} a_{ij}x^i$,

\[
0 \leq \int p(x)^2 d\mu(x)
= \sum_{ij} a_{ij} \int x^{i+j} d\mu(x)
= \sum_{ij} a_{ij} \beta_{i+j}.
\]
Basic Positivity Condition

\( \mathcal{P}_n : \) polynomials \( p \) over \( \mathbb{R} \) with \( \deg p \leq n \)

- Given \( p \in \mathcal{P}_n, \ p(x) \equiv \sum_{0 \leq i+j \leq n} a_{ij}x^i, \)

\[
0 \leq \int p(x)^2 \, d\mu(x) = \sum_{ij} a_{ij}a_{ij} \int x^{i+j} \, d\mu(x) = \sum_{ij} a_{ij}a_{ij} \beta_{i+j}.
\]

- Now recall that we’re working in \( d \) real variables. To understand this “matricial” positivity, we introduce the following lexicographic order on the rows and columns of \( M(n) \):

\[
1, X_1, \ldots, X_m, X_1^2, X_2X_1, \ldots, X_m^2, \ldots
\]
The entries are given by

\[ M(n)_{i,j} := \beta_{i+j}. \]

Then

(\textit{matricial positivity})  \[ \sum_{ij} a_i a_j \beta_{i+j} \geq 0 \]

\[ \iff M(n) \equiv M(n)(\beta) \geq 0. \]
The entries are given by

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Then

(matricial positivity) \[ \sum_{ij} a_i a_j \beta_{i+j} \geq 0 \]

[ \iff M(n) \equiv M(n)(\beta) \geq 0. \]

**Positivity Condition is not sufficient:**

By modifying an example of K. Schm"udgen, several years ago we were able to build a family \( \beta_{00}, \beta_{01}, \beta_{10}, \ldots, \beta_{06}, \ldots, \beta_{60} \) with positive invertible moment matrix \( M(3) \) but no rep. meas. Later on, we also did this for \( n = 2 \) (in this case \( M(2) \) is not invertible).
For example, for moment problems in $\mathbb{R}^2$,

$$M(1) = \begin{pmatrix}
\beta_{00} & \beta_{01} & \beta_{10} \\
\beta_{01} & \beta_{02} & \beta_{11} \\
\beta_{10} & \beta_{11} & \beta_{20}
\end{pmatrix},$$

$$M(2) = \begin{pmatrix}
\beta_{00} & \beta_{01} & \beta_{02} & \beta_{10} \\
\beta_{01} & \beta_{02} & \beta_{03} & \beta_{11} & \beta_{20} \\
\beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} & \beta_{21} \\
\beta_{02} & \beta_{03} & \beta_{12} & \beta_{13} & \beta_{22} & \beta_{30} \\
\beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} & \beta_{22} & \beta_{31} \\
\beta_{20} & \beta_{21} & \beta_{22} & \beta_{23} & \beta_{31} & \beta_{40}
\end{pmatrix}. $$
For example, for moment problems in $\mathbb{R}^2$, 

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\beta_{10} & \beta_{11} & \beta_{20} & \beta_{12} & \beta_{21} & \beta_{30} \\
\beta_{02} & \beta_{03} & \beta_{12} & \beta_{04} & \beta_{13} & \beta_{22} \\
\beta_{11} & \beta_{12} & \beta_{21} & \beta_{13} & \beta_{22} & \beta_{31} \\
\beta_{20} & \beta_{21} & \beta_{30} & \beta_{22} & \beta_{31} & \beta_{40}
\end{pmatrix}.$$
In general,

\[ M(n + 1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \]

Similarly, one can build \( M(\infty) \equiv M(\infty)(\beta) \equiv M(\beta) \).
In general,
\[ M(n + 1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix} \]
Similarly, one can build \( M(\infty) \equiv M(\infty)(\beta) \equiv M(\beta) \).

The link between TMP and FMP is provided by a result of Stochel (2001):

**Theorem (Stochel’s Theorem)**

\( \beta(\infty) \) has a rep. meas. supported in a closed set \( K \subseteq \mathbb{R}^d \) if and only if, for each \( n \), \( \beta^{(2n)} \) has a rep. meas. supported in \( K \).
For moment problems in $\mathbb{C}$,

$$M(3) = \begin{pmatrix}
1 & Z & \bar{Z} & Z^2 & \bar{Z}Z & \bar{Z}^2 & \vdots & Z^3 & \bar{Z}Z^2 & \bar{Z}^2Z & \bar{Z}^3 \\
\gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \vdots & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\
\gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \vdots & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\
\gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \vdots & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\
\gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \vdots & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\
\gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \vdots & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \vdots & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\
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\gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \vdots & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33}
\end{pmatrix}.$$
For moment problems in $\mathbb{R}^2$, 

$$M(3) = \begin{pmatrix} 
1 & X & Y & X^2 & XY & Y^2 & \cdots & X^3 & X^2Y & XY^2 & Y^3 \\
\beta_{00} & \beta_{01} & \beta_{10} & \beta_{02} & \beta_{11} & \beta_{20} & \cdots & \beta_{03} & \beta_{12} & \beta_{21} & \beta_{30} \\
\beta_{01} & \beta_{02} & \beta_{11} & \beta_{03} & \beta_{12} & \beta_{21} & \cdots & \beta_{04} & \beta_{13} & \beta_{22} & \beta_{31} \\
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\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
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\beta_{30} & \beta_{31} & \beta_{40} & \beta_{32} & \beta_{41} & \beta_{50} & \cdots & \beta_{33} & \beta_{42} & \beta_{51} & \beta_{60} 
\end{pmatrix}$$

Raúl Curto (IWOTA, 7/17/12)
General Idea to Study TMP

- TMP is more general than FMP:
  - fewer moments $\implies$ less data

Stochel: link between TMP and FMP

Existing approaches are directed at enlarging the data by acquiring new moments, and eventually making the problem into one of flat data type (i.e., with intrinsic recursiveness).

This naturally leads to a full MP.

If such a flat extension of the initial data cannot be accomplished, then TMP has no representing measure.

Helpful tool: Smul'jan's Theorem on positivity of $2 \times 2$ matrices
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- If such a flat extension of the initial data cannot be accomplished, then TMP has no representing measure.
- Helpful tool: Smul’jan’s Theorem on positivity of $2 \times 2$ matrices
Theorem (Smul’jan, 1959)

\[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix} \succeq 0 \iff \begin{cases}
A \succeq 0 \\
B = AW \\
C \succeq W^*AW
\end{cases}.
\]

Moreover, rank \[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix}
\] = rank \( A \) \(\iff\) \( C = W^*AW \).
**Corollary**

Assume \( \text{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank} \ A \). Then

\[
A \geq 0 \iff \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.
\]
A theorem of Bayer and Teichmann [BT] implies that if a finite real multisequence $\beta \equiv \beta^{(2d)}$ has a representing measure, then the associated moment matrix $M_d$ admits positive, recursively generated moment matrix extensions $M_{d+1}, M_{d+2}, \ldots$. For a bivariate recursively determinate $M_d$, we show that the existence of positive, recursively generated extensions $M_{d+1}, \ldots, M_{2d-1}$ is sufficient for a measure. Examples illustrate that all of these extensions may be required to show that $\beta$ has a measure. We describe in detail a constructive procedure for determining whether such extensions exist. Under mild additional hypotheses, we show that $M_d$ admits an extension $M_{d+1}$ which has many of the properties of a positive, recursively generated extension.
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Abstract of New Results

A theorem of Bayer and Teichmann [BT] implies that if a finite real multisequence $\beta \equiv \beta^{(2d)}$ has a representing measure, then the associated moment matrix $M_d$ admits positive, recursively generated moment matrix extensions $M_{d+1}, M_{d+2}, \ldots$. For a bivariate recursively determinate $M_d$, we show that the existence of positive, recursively generated extensions $M_{d+1}, \ldots, M_{2d-1}$ is sufficient for a measure. Examples illustrate that all of these extensions may be required to show that $\beta$ has a measure.
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Some Applications of TMP

- Subnormal Operator Theory (unilateral weighted shifts)
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**Typical Problem:** Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.
For $p \in \mathcal{P}_n$, $p(x, y) \equiv \sum_{0 \leq i + j \leq n} a_{ij} x^i y^j$ define
\[
p(X, Y) := \sum a_{ij} X^i Y^j \equiv M(n) \hat{p},
\]
where $\hat{p} := (a_{00} \cdots a_{0n} \cdots a_{n0})^T$. 

For $p \in \mathcal{P}_n$, $p(x, y) \equiv \sum_{0 \leq i + j \leq n} a_{ij} x^i y^j$ define

$$p(X, Y) := \sum a_{ij} X^i Y^j \equiv M(n)p,$$

where $\hat{p} := (a_{00} \cdots a_{0n} \cdots a_{n0})^T$.

If there exists a rep. meas. $\mu$, then

$$p(X, Y) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p := \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}.$$
For $p \in \mathcal{P}_n$, $p(x, y) \equiv \sum_{0 \leq i + j \leq n} a_{ij}x^iy^j$ define

$$p(X, Y) := \sum a_{ij}X^i Y^j \equiv M(n)\hat{p},$$

where $\hat{p} := (a_{00} \cdots a_{0n} \cdots a_{n0})^T$.

If there exists a rep. meas. $\mu$, then

$$p(X, Y) = 0 \iff \text{supp } \mu \subseteq \mathcal{Z}_p := \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}.$$

The following is our analogue of recursiveness for the TCMP

\[(RG) \quad \text{If } p, q, pq \in \mathcal{P}_n, \text{ and } p(X, Y) = 0, \]

\[\text{then } (pq)(X, Y) = 0.\]
## Truncated Moment Problems in Two Real Variables

<table>
<thead>
<tr>
<th>TMP</th>
<th>Complex</th>
<th>Real</th>
</tr>
</thead>
<tbody>
<tr>
<td>moments</td>
<td>$\gamma_{ij} \in \mathbb{C}, \gamma_{ji} = \overline{\gamma_{ij}}$</td>
<td>$\beta_{ij} \in \mathbb{R}$</td>
</tr>
<tr>
<td>moment matrix</td>
<td>$M(n)$</td>
<td>$M(n)$</td>
</tr>
<tr>
<td>functional calculus</td>
<td>$p(Z, \overline{Z})$</td>
<td>$p(X, Y)$</td>
</tr>
<tr>
<td>algebraic variety</td>
<td>$\mathcal{V}(\gamma) := \bigcap_{p(Z, \overline{Z})=0} \mathbb{Z}_p$</td>
<td>$\mathcal{V}(\beta) := \bigcap_{p(X, Y)=0} \mathbb{Z}_p$</td>
</tr>
</tbody>
</table>
Singular TMP; Real Case

- Given a finite family of moments, build moment matrix
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Identify all column relations
Singular TMP; Real Case

- Given a finite family of moments, build moment matrix
- Identify all column relations
- Build algebraic variety

\[ V \equiv V(\beta) \equiv V(M(n)) := \bigcap_{p(X,Y)=0} \mathcal{Z}_p, \]

where \( \mathcal{Z}_p \) is the zero set of \( p \).
Singular TMP; Real Case

• Given a finite family of moments, build moment matrix
• Identify all column relations
• Build algebraic variety

\[ \mathcal{V} \equiv \mathcal{V}(\beta) \equiv \mathcal{V}(M(n)) := \bigcap_{p(X,Y) = 0} \mathcal{Z}_p, \]

where \( \mathcal{Z}_p \) is the zero set of \( p \).

• Always true:

\[ r := \text{rank } M(n) \leq \text{card } \text{supp } \mu \leq \nu := \text{card } \mathcal{V}(\beta), \]

so if the variety is finite there’s a natural candidate for \( \text{supp } \mu \), i.e.,

\[ \text{supp } \mu = \mathcal{V}(\beta) \]
Finite rank case
Singular TMP

- Finite rank case
- Flat case
Finite rank case

Flat case

Extremal case
Singular TMP

- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
Finite rank case

Flat case

Extremal case

Recursively generated relations

Strategy: Build positive extension, repeat, and eventually extremal

\[ \text{rank } M(n) \leq \text{rank } M(n + 1) \leq \text{card } \mathcal{V}(M(n + 1)) \leq \text{card } \mathcal{V}(M(n)) \]
Singular TMP

- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
  - Strategy: Build positive extension, repeat, and eventually extremal
  - \[ \text{rank } M(n) \leq \text{rank } M(n+1) \leq \text{card } \mathcal{V}(M(n+1)) \leq \text{card } \mathcal{V}(M(n)) \]
- General case.
Consider the **full, complex** MP

\[ \int \bar{z}^i z^j \, d\mu = \gamma_{ij} \quad (i, j \geq 0), \]

where \( \text{supp} \ \mu \subseteq K \), for \( K \) a closed subset of \( \mathbb{C} \).

- **The Riesz functional** is given by

  \[ \Lambda_{\gamma}(\bar{z}^i z^j) := \gamma_{ij} \quad (i, j \geq 0). \]
Localizing Matrices

Consider the **full, complex** MP

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- The **Riesz functional** is given by

  \[ \Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij} \quad (i, j \geq 0). \]

- **Riesz-Haviland:**

  There exists \( \mu \) with supp \( \mu \subseteq K \iff \Lambda_\gamma(p) \geq 0 \) for all \( p \) such that \( p|_K \geq 0 \).
If $q$ is a polynomial in $z$ and $\bar{z}$, and

$$K \equiv K_q := \{ z \in \mathbb{C} : q(z, \bar{z}) \geq 0 \},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for $\mu$ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p} \, d\mu \geq 0 \text{ (all } p).$$

- K. Schmüdgen (1991): If $K_q$ is compact, $\Lambda_{\gamma}(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all $p$, then there exists $\mu$ with $\text{supp } \mu \subseteq K_q$. 
We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

\[
\text{supp } \mu \subseteq \mathbb{R} \quad \text{(Hamburger TMP)} \\
\text{supp } \mu \subseteq [0, \infty) \quad \text{(Stieltjes TMP)} \\
\text{supp } \mu \subseteq [a, b] \quad \text{(Hausdorff TMP)} \\
\text{supp } \mu \subseteq \mathbb{T} \quad \text{(Toeplitz TMP)}
\]
Quick Overview of Known Results for TCMP

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

\[
\begin{align*}
\text{supp } \mu &\subseteq \mathbb{R} \quad \text{(Hamburger TMP)} \\
\text{supp } \mu &\subseteq [0, \infty) \quad \text{(Stieltjes TMP)} \\
\text{supp } \mu &\subseteq [a, b] \quad \text{(Hausdorff TMP)} \\
\text{supp } \mu &\subseteq \mathbb{T} \quad \text{(Toeplitz TMP)}
\end{align*}
\]

- Along the way we have developed new machinery for analyzing TMP’s in **one or several real or complex variables**.
Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
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Our results have been applied by S. McCullough to obtain a dilation-type structure theorem in Fejér-Riesz factorization theory; J. Lasserre to obtain concrete necessary and sufficient conditions on the coefficients of a polynomial so that all of its zeros lie in a prescribed semi-algebraic subset of the plane; and J. Lasserre, M. Laurent and others to convert polynomial optimization into an instance of semidefinite programming.
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In the specific case of $K := \text{supp } \mu$, a semi-algebraic set determined by a finite collection of complex polynomials $\mathcal{P} = \{p_i(z, \bar{z})\}_{i=1}^m$, i.e.,

$$K = K_\mathcal{P} := \{z \in \mathbb{C} : p_i(z, \bar{z}) \geq 0, 1 \leq i \leq m\},$$

we obtain an existence criterion expressed in terms of positivity and extension properties of the moment matrix $M(n)(\gamma)$ associated to $\gamma$ and of the localizing matrix $M_{p_i}$ corresponding to each $p_i$. 
**Case of Flat Data**

Recall: If \( \mu \) is a rep. meas. for \( M(n) \), then \( \text{rank } M(n) \leq \text{card supp } \mu \).

\( \beta \) is flat if \( M(n) = \begin{pmatrix}
M(n-1) & M(n-1)W \\
W^*M(n-1) & W^*M(n-1)W
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Theorem (RC-L. Fialkow, 1996) If $\beta$ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.
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To find $\mu$ concretely, let $r := \text{rank } M(n)$ and look for the relation...
\[ Z^r = c_0 1 + c_1 Z + \ldots + c_{r-1} Z^{r-1}. \]

We then define
\[ p(z) := z^r - (c_0 + \ldots + c_{r-1} Z^{r-1}) \]

and solve the Vandermonde equation

\[
\begin{pmatrix}
1 & \ldots & 1 \\
 z_0 & \ldots & z_{r-1} \\
 \vdots & \ddots & \vdots \\
 z_0^{r-1} & \ldots & z_{r-1}^{r-1}
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\
\rho_1 \\
\vdots \\
\rho_{r-1}
\end{pmatrix}
= 
\begin{pmatrix}
\gamma_{00} \\
\gamma_{01} \\
\vdots \\
\gamma_{0r-1}
\end{pmatrix}.
\]

Then
\[
\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.
\]
**Theorem**

(RC-LF, 2000) Let $M(n) \geq 0$ and suppose $\deg(q) = 2k$ or $2k - 1$ for some $k \leq n$. Then $\exists \mu$ with rank $M(n)$ atoms and $\text{supp } \mu \subseteq K_q$ if and only if $\exists$ a flat extension $M(n + 1)$ for which $M_q(n + k) \geq 0$. In this case, $\exists \mu$ with exactly rank $M(n) - \text{rank } M_q(n + k)$ atoms in $\mathcal{Z}(q)$. 

**Remark**

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**Theorem**

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M. Laurent (2005) has found an alternative proof, using ideas from real algebraic geometry.
Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

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- $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$D = 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \quad \text{and} \quad C = \bar{B}$$

$$\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2$$

$$\Rightarrow \bar{W}W = 1 \quad \text{(circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}.$$
The functional calculus we have constructed is such that $p(Z, \bar{Z}) = 0$ implies $\text{supp } \mu \subseteq \mathcal{Z}(p)$. 

When \{1, Z, \bar{Z}, Z^2, \bar{Z} Z\} is a basis for $C^M(2)$, the associated algebraic variety is the zero set of a real quadratic equation in $x := \text{Re}[z]$ and $y := \text{Im}[z]$. Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

(a) $\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$; parabola;

(b) $\bar{W}^2 = -4iW^2$; hyperbola;

(c) $\bar{W}^2 = W^2$; pair of intersecting lines;

(d) $\bar{W}W = 1$; unit circle; $x^2 + y^2 = 1$. 

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Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

\[(a) \quad \bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W \quad \text{parabola; } y = x^2\]

\[(b) \quad \bar{W}^2 = -4i1 + W^2 \quad \text{hyperbola; } yx = 1\]

\[(c) \quad \bar{W}^2 = W^2 \quad \text{pair of intersect. lines; } yx = 0\]

\[(d) \quad \bar{W}W = 1 \quad \text{unit circle; } x^2 + y^2 = 1.\]
**Theorem (The Quartic MP: Complex Case)**

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ be given, and assume $M(2) \geq 0$ and \{1, $Z$, $\bar{Z}$, $Z^2$, $\bar{Z}Z$\} is a basis for $C_{M(2)}$. Then $\gamma^{(4)}$ admits a rep. meas. $\mu$.

Moreover, it is possible to find $\mu$ with card supp $\mu = \text{rank } M(2)$, except in some cases when $\mathcal{V}(\gamma^{(4)})$ is a pair of intersecting lines; in such cases, there exist $\mu$ with card supp $\mu \leq 6$. 

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**Corollary**

Assume that $M(2) \geq 0$ and that $\text{rank } M(2) \leq \text{card } \mathcal{V}(\gamma^{(4)})$. Then $M(2)$ admits a representing measure.
Recall that the *algebraic variety* of $\beta$ is

$$
\mathcal{V} \equiv \mathcal{V}(\beta) := \bigcap_{p(X,Y)=0} \mathcal{Z}_p,
$$

where $\mathcal{Z}_p$ is the zero set of $p$. 

If $p \in \mathcal{P}_n$ satisfies

$$
M(n)^{\hat{p}} = 0 \iff \text{supp} \mu \subseteq \mathcal{Z}_p,
$$

thus $\text{supp} \mu \subseteq \mathcal{V}$, so $r := \text{rank } M(n)$ and $v := \text{card } \mathcal{V}$ satisfy $r \leq \text{card } \text{supp } \mu \leq v$.

If $p \in \mathcal{P}_2^n$ and $p|\mathcal{V} \equiv 0$, then $\Lambda(p) = \int p \ d\mu = 0$. Here $\Lambda$ is the Riesz functional, given by $\Lambda(x_i y_j) := \beta_{ij}$. 

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- If \( \beta \) admits a representing measure \( \mu \), then

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p \in \mathcal{P}_n \text{ satisfies } M(n)\hat{p} = 0 \iff \text{supp } \mu \subseteq \mathcal{Z}_p
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Thus $\text{supp } \mu \subseteq \mathcal{V}$, so $r := \text{rank } M(n)$ and $v := \text{card } \mathcal{V}$ satisfy

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  Thus $\text{supp } \mu \subseteq V$, so $r := \text{rank } M(n)$ and $v := \text{card } V$ satisfy

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If $p \in \mathcal{P}_{2n}$ and $p|_V \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$.

Here $\Lambda$ is the *Riesz functional*, given by $\Lambda(x^i y^j) := \beta_{ij}$.
Basic necessary conditions for the existence of a representing measure

(Positivity) $M(n) \geq 0$

Consistency implies (Recursiveness)

Consistency: $p \in \mathcal{P}_n$, $V \equiv 0 \Rightarrow \Lambda(p) = 0$.

Variety Condition: $r \leq v$, i.e., rank $M(n) \leq \text{card } V$. 

Recursively Determinate TMP
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Basic necessary conditions for the existence of a representing measure

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(Consistency) \( p \in \mathcal{P}_{2n}, \ p|_\mathcal{V} \equiv 0 \implies \Lambda(p) = 0 \)

(Variety Condition) \( r \leq \nu, \) i.e., rank \( M(n) \leq \text{card } \mathcal{V}. \)

Consistency implies

(Recursiveness) \( p, q, pq \in \mathcal{P}_n, \ M(n)\hat{p} = 0 \implies M(n)(pq)\hat{r} = 0. \)
In general, *Positivity*, *Consistency* and the *Variety Condition* are not sufficient. However,

**Theorem (The Extremal Case)**

(RC, L. Fialkow and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ extremal, i.e., $r = v$, the following are equivalent:

(i) $\beta$ has a representing measure;

(ii) $\beta$ has a unique representing measure, which is rank $M(n)$-atomic (minimal);

(iii) $M(n) \geq 0$ and $\beta$ is consistent.
The extension of this result to general representing measures follows from a theorem of C. Bayer and J. Teichmann: If $\beta$ has a representing measure, then it has a finitely atomic representing measure.
A Basic Algorithm

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Also, $\text{rank } M(n) \leq \text{rank } M(n+1) \leq \text{card } \mathcal{V}(n+1) \leq \text{card } \mathcal{V}(n)$.

Thus, eventually, a soluble TMP must be flat or extremal.
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Also, $\text{rank } M(n) \leq \text{rank } M(n + 1) \leq \text{card } \mathcal{V}(n + 1) \leq \text{card } \mathcal{V}(n)$.

Thus, eventually, a soluble TMP must be flat or extremal.

At present, for a general moment matrix, there is no known concrete test for the existence of a flat extension $M_{d+k+1}$.
The extension of this result to general representing measures follows from a theorem of C. Bayer and J. Teichmann: If $\beta$ has a representing measure, then it has a finitely atomic representing measure. Also, $\text{rank } M(n) \leq \text{rank } M(n + 1) \leq \text{card } \mathcal{V}(n + 1) \leq \text{card } \mathcal{V}(n)$. Thus, eventually, a soluble TMP must be flat or extremal. At present, for a general moment matrix, there is no known concrete test for the existence of a flat extension $M_{d+k+1}$. For the class of bivariate recursively determinate moment matrices, we now present a detailed analysis of an algorithm that can be used in numerical examples to determine the existence or nonexistence of flat extensions (and representing measures).
This algorithm determines the existence or nonexistence of positive, recursively generated extensions $M_{d+1}, \ldots, M_{2d-1}$, at least one of which must be a flat extension in the case when there is a measure.
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One of our main results shows that there are sequences $\beta^{(2d)}$ for which the first flat extension occurs at $M_{2d-1}$, so all of the above extensions must be computed in order to recognize that there is a measure.
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One of our main results shows that there are sequences $\beta^{(2d)}$ for which the first flat extension occurs at $M_{2d-1}$, so all of the above extensions must be computed in order to recognize that there is a measure.

This result stands in sharp contrast to traditional truncated moment theorems (concerning representing measures supported in $\mathbb{R}$, $[a, b]$, $[0, +\infty)$, or in a planar curve of degree 2), which express the existence of a measure in terms of tests closely related to the original moment data.
Here we see that, at least within the framework of moment matrix extensions, we may need to go far from the original data to resolve the existence of a measure.
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We will show that under mild additional hypotheses on $M_d$, the implementation of each extension step, from $M_{d+j}$ to $M_{d+j+1}$, leading to a flat extension $M_{d+k+1}$, consists of simply verifying a matrix positivity condition.
A bivariate moment matrix $M_d$ is *recursively determinate* if there are column dependence relations of the form

$$X^n = p(X, Y) \quad (p \in \mathcal{P}_{n-1}, \; n \leq d)$$

and

$$Y^m = q(X, Y) \quad (q \in \mathcal{P}_m, \; q \text{ has no } y^m \text{ term}, \; m \leq d),$$

or with similar relations with the roles of $p$ and $q$ reversed.
We next present an example which illustrates the algorithm in a case leading to a measure.

**Example.** Let $d = 3$ and consider

\[ M_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\
0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\
1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\
0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\
0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\
0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & c \\
0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & c & \delta
\end{pmatrix}. \]
We have $M_3 \succeq 0$, $M_2 \succ 0$, and

$$\text{rank } M_3 = 8 \iff \delta = 2026881 - 2844c + c^2.$$

When rank $M_3 = 8$, then the two column relations are $X_3 = Y$ and $Y_3 = q(X, Y)$, where $q(x, y) := (5715 - 4c)x + 10(-1428 + c)y - 3(-2853 + 2c)x^2y + (-1422 + c)xy^2$. Let $r_1(x, y) = y - x^3$ and $r_2(x, y) = y^3 - q(x, y)$.
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$$Y^3 = q(X, Y),$$

where $q(x, y) :=$

$$(5715 - 4c)x + 10(-1428 + c)y - 3(-2853 + 2c)x^2y + (-1422 + c)xy^2.$$ 

Let

$$r_1(x, y) = y - x^3$$

and

$$r_2(x, y) = y^3 - q(x, y).$$
With these two column relations in hand, our results guarantees the existence of a unique $RG$ extension $M_4$. To test the positivity of $M_4$, we calculate the determinant of the $9 \times 9$ matrix consisting of the rows and columns of $M_4$ indexed by the monomials $1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2$. A straightforward calculation using Mathematica shows that three cases arise:
With these two column relations in hand, our results guarantees the existence of a unique \( RG \) extension \( M_4 \). To test the positivity of \( M_4 \), we calculate the determinant of the \( 9 \times 9 \) matrix consisting of the rows and columns of \( M_4 \) indexed by the monomials \( 1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2 \). A straightforward calculation using Mathematica shows that three cases arise:

(i) \( c < 1429 \): here \( M_4 \not\succcurlyeq 0 \), so \( M_3 \) admits no representing measure;
With these two column relations in hand, our results guarantees the existence of a unique $RG$ extension $M_4$. To test the positivity of $M_4$, we calculate the determinant of the $9 \times 9$ matrix consisting of the rows and columns of $M_4$ indexed by the monomials $1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2$. A straightforward calculation using *Mathematica* shows that three cases arise:

(i) $c < 1429$: here $M_4 \not\succeq 0$, so $M_3$ admits no representing measure;

(ii) $c = 1429$: here $M_4$ is a flat extension of $M_3$, so $M_3$ admits an 8-atomic representing measure;
(iii) $c > 1429$: here $M_4$ is a positive RG extension of $M_3$ with rank 9.
(iii) $c > 1429$: here $M_4$ is a positive RG extension of $M_3$ with rank 9. Although $M_4$ is not a flat extension of $M_3$, it nevertheless satisfies the hypotheses of our Theorem, so $M_4$ admits a flat extension $M_5$, and therefore $M_3$ has a 9-atomic representing measure. Moreover, since the original algebraic variety $\mathcal{V} \equiv \mathcal{V}(M_3)$ associated with $M_3$, $\mathcal{Z}_r_1 \cap \mathcal{Z}_r_2$, can have at most 9 points (by Bézout’s Theorem), it follows that $\mathcal{V} = \mathcal{V}(M_5)$. 
(iii) $c > 1429$: here $M_4$ is a positive RG extension of $M_3$ with rank 9. Although $M_4$ is not a flat extension of $M_3$, it nevertheless satisfies the hypotheses of our Theorem, so $M_4$ admits a flat extension $M_5$, and therefore $M_3$ has a 9-atomic representing measure. Moreover, since the original algebraic variety $\mathcal{V} \equiv \mathcal{V}(M_3)$ associated with $M_3$, $\mathcal{Z}_{r_1} \cap \mathcal{Z}_{r_2}$, can have at most 9 points (by Bézout’s Theorem), it follows that $\mathcal{V} = \mathcal{V}(M_5)$. This algebraic variety must have exactly 9 points, and thus constitutes the support of the unique representing measure for $M_3$. 
To illustrate this case, we take the special value $c = 1430$, so that $q(x, y) \equiv -5x + 20y - 21x^2y + 8xy^2$. Let $\alpha := \frac{1}{2} \sqrt{5} - 2\sqrt{5}$ and $\gamma := \sqrt{5}\alpha$. 
To illustrate this case, we take the special value $c = 1430$, so that
$q(x, y) \equiv -5x + 20y - 21x^2y + 8xy^2$. Let $\alpha := \frac{1}{2}\sqrt{5 - 2\sqrt{5}}$ and $\gamma := \sqrt{5}\alpha$. A calculation shows that $V = \{(x_i, x_i^3)\}_{i=1}^9$, where $x_1 = 0,$
$x_2 = \frac{1}{2}(-1 - \sqrt{5}) \approx -1.618,$
$x_3 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.618,$
$x_4 = -x_3 \approx 0.618,$
$x_5 = -x_2 \approx 1.618,$
$x_6 = -\alpha - \gamma \approx -1.176,$
$x_7 = -\alpha + \gamma \approx 0.449,$
$x_8 = -x_7 \approx -0.449$ and $x_9 = -x_6 \approx 1.176.$
To illustrate this case, we take the special value $c = 1430$, so that $q(x, y) \equiv -5x + 20y - 21x^2y + 8xy^2$. Let $\alpha := \frac{1}{2}\sqrt{5 - 2\sqrt{5}}$ and $\gamma := \sqrt{5}\alpha$. A calculation shows that $V = \{(x_i, x_i^3)\}_{i=1}^9$, where $x_1 = 0$, $x_2 = \frac{1}{2}(-1 - \sqrt{5}) \approx -1.618$, $x_3 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.618$, $x_4 = -x_3 \approx 0.618$, $x_5 = -x_2 \approx 1.618$, $x_6 = -\alpha - \gamma \approx -1.176$, $x_7 = -\alpha + \gamma \approx 0.449$, $x_8 = -x_7 \approx -0.449$ and $x_9 = -x_6 \approx 1.176$.

$M_3$ satisfies the hypothesis of our Theorem with $n = m = 3$, so we proceed to generate the $RG$ extension $M_4$. 

Raúl Curto (IWOTA, 7/17/12)  
Recursively Determinate TMP
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Y^4 &= (yq)(X, Y) \quad \text{(from } Y^3 = q(X, Y)) 
\end{align*}
\]

(first in \( \begin{pmatrix} M_3 & B(4) \end{pmatrix} \)).
This extension is uniquely determined by imposing the column relations

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Y^4 = (yq)(X, Y) \quad \text{(from } Y^3 = q(X, Y))
\]

(first in \( \left( \begin{array}{cc} M_3 & B(4) \end{array} \right) \), then in \( \left( \begin{array}{cc} B(4)^T & C(4) \end{array} \right) \)).
A calculation shows that, as expected, these relations unambiguously define a positive moment matrix $M_4$ with

$$\text{rank } M_4 = 9 \ (> 8 = \text{rank } M_3).$$
A calculation shows that, as expected, these relations unambiguously define a positive moment matrix $M_4$ with

$$\text{rank } M_4 = 9 \ (> 8 = \text{rank } M_3).$$

It follows that $M_3$ admits no flat extension $M_4$, so we proceed to construct the $RG$ extension $M_5$, uniquely determined by imposing the relations
\[
X^5 = X^2 Y, \\
X^4 Y = XY^2, \\
X^3 Y^2 = Y^3, \\
X^2 Y^3 = (x^2 q)(X, Y), \\
XY^4 = (xyq)(X, Y), \\
Y^5 = (y^2 q)(X, Y).
\]
A calculation of these columns (first in \( \begin{pmatrix} M_4 & B(5) \end{pmatrix} \), then in \( \begin{pmatrix} B(5)^T & C(5) \end{pmatrix} \)), shows that, as again expected, they do fit together to unambiguously define a moment matrix \( M_5 \).
A calculation of these columns (first in $\begin{pmatrix} M_4 & B(5) \end{pmatrix}$, then in $\begin{pmatrix} B(5)^T & C(5) \end{pmatrix}$), shows that, as again expected, they do fit together to unambiguously define a moment matrix $M_5$. From the form of $q(x, y)$, we see that $M_5$ is actually a flat extension of $M_4$, in keeping with the above discussion. Corresponding to this flat extension is the unique, 9-atomic, representing measure $\mu \equiv \mu_{M_5}$.
Clearly, supp $\mu = \mathcal{V}$, so $\mu$ is of the form $\mu = \sum_{i=1}^{9} \rho_i \delta_{(x_i,x_3^i)}$. To compute the densities, we use the Vandermonde method and find $\rho_1 = \frac{1}{5} = 0.2$, $\rho_2 = \rho_5 = \frac{-1+\sqrt{5}}{8\sqrt{5}} \approx 0.069$, $\rho_3 = \rho_4 = \frac{1+\sqrt{5}}{8\sqrt{5}} \approx 0.181$, $\rho_6 = \rho_9 = \frac{5+3\sqrt{5}}{40\sqrt{5}} \approx 0.131$, and $\rho_7 = \rho_8 = \frac{-5+3\sqrt{5}}{40\sqrt{5}} \approx 0.019$. 

Thus, the existence of a representing measure for $\beta(\mathcal{V})$ is established on the basis of the extensions $M_4$ and $M_5$. Note that in this case, the actual number of extensions leading to a flat extension can be computed as either $n + m - d - 1 = 3 + 3 - 2 - 1 = 2$ or as $1 + \text{card} V - \text{rank} M_3 = 1 + 9 - 8 = 2$, which is consistent with our earlier discussion.
Clearly, supp $\mu = \mathcal{V}$, so $\mu$ is of the form $\mu = \sum_{i=1}^{9} \rho_i \delta_{(x_i, x_3)}$. To compute the densities, we use the Vandermonde method and find $\rho_1 = \frac{1}{5} = 0.2$, $\rho_2 = \rho_5 = \frac{-1+\sqrt{5}}{8\sqrt{5}} \approx 0.069$, $\rho_3 = \rho_4 = \frac{1+\sqrt{5}}{8\sqrt{5}} \approx 0.181$, $\rho_6 = \rho_9 = \frac{5+3\sqrt{5}}{40\sqrt{5}} \approx 0.131$, and $\rho_7 = \rho_8 = \frac{-5+3\sqrt{5}}{40\sqrt{5}} \approx 0.019$. Thus, the existence of a representing measure for $\beta^{(6)}$ is established on the basis of the extensions $M_4$ and $M_5$. Note that in this case, the actual number of extensions leading to a flat extension can be computed as either $n + m - d - 1 = 3 + 3 - 2 - 1 = 2$ or as $1 + \text{card } \mathcal{V} - \text{rank } M_3 = 1 + 9 - 8 = 2$, which is consistent with our earlier discussion. \qed
Theorem

Suppose the bivariate moment matrix $M_d(\beta)$ is positive and recursively generated, with column dependence relations generated entirely the monomials $X^n$ and $Y^m$ via recursiveness and linearity. Then there exists a unique moment matrix block $B(d + 1)$ such that $\begin{pmatrix} M_d & B(d + 1) \end{pmatrix}$ is recursively generated and $\text{Ran } B(d + 1) \subseteq \text{Ran } M_d$. 
Corollary

If $M_d$ satisfies the hypotheses of the previous Theorem, then there exists a unique moment matrix block $C \equiv C(n + 1)$ consistent with the structure of an RG extension $M_{d+1}$.
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If $M_d$ satisfies the hypotheses of the previous Theorem, then there exists a unique moment matrix block $C \equiv C(n + 1)$ consistent with the structure of an RG extension $M_{d+1}$.

By combining the previous Theorem and Corollary, we immediately obtain the first of our main results, which follows.

**Theorem**

If $M_d$ is positive, with column relations generated entirely by $X^n$ and $Y^m$ via recursiveness and linearity, then $M_d$ admits a unique RG extension $M_{d+1}$, i.e., $\text{Ran } B(n + 1) \subseteq \text{Ran } M_d$, and $M_{d+1}$ is recursively generated.
Corollary

If $M_d$ satisfies the above mentioned hypotheses and $d = n + m - 2$, then $M_d$ admits a flat moment matrix extension $M_{d+1}$ (and $\beta$ admits a rank $M_d$-atomic representing measure).
We continue with an example which shows that our Theorem is no longer valid if we permit column dependence relations in $M_d$ in addition to those generated by $X^n$ and $Y^m$.
We continue with an example which shows that our Theorem is no longer valid if we permit column dependence relations in $M_d$ in addition to those generated by $X^n$ and $Y^m$.

**Example.** We define $M_3$ by setting

\[
\begin{align*}
\beta_{00} &= \beta_{20} = \beta_{02} = 1 \\
\beta_{11} &= \beta_{30} = \beta_{21} = \beta_{03} = 0 \\
\beta_{12} &= \beta_{40} = 2 \\
\beta_{31} &= \beta_{13} = 0 \\
\beta_{22} &= 5 \\
\beta_{04} &= 22 \\
\beta_{50} &= -1
\end{align*}
\]
\[
\begin{align*}
\beta_{41} &= -2 \\
\beta_{32} &= 13 \\
\beta_{23} &= 3 \\
\beta_{14} &= \frac{894}{13} \\
\beta_{05} &= \frac{336}{13} \\
\beta_{60} &= 178 \\
\beta_{51} &= 139 \\
\beta_{42} &= 159 \\
\beta_{33} &= \frac{1657}{13} \\
\beta_{24} &= \frac{4298}{13} \\
\beta_{15} &= r
\end{align*}
\]
\[ \beta_{06} = \gamma \equiv \frac{443272376768 - 2742712830r - 4826809r^2}{41327767}. \]
\[
\beta_{06} = \gamma \equiv \frac{443272376768 - 2742712830r - 4826809r^2}{41327767}.
\]

Thus, we have

\[
M_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 5 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 5 & 0 & 22 \\
1 & 0 & 0 & 2 & 0 & 5 & -1 & -2 & 13 & 3 \\
0 & 0 & 2 & 0 & 5 & 0 & -2 & 13 & 3 & \frac{894}{13} \\
1 & 2 & 0 & 5 & 0 & 22 & 13 & 3 & \frac{894}{13} & \frac{336}{13} \\
0 & 2 & 0 & -1 & -2 & 13 & 178 & 139 & 159 & \frac{1657}{13} \\
0 & 0 & 5 & -2 & 13 & 3 & 139 & 159 & \frac{1657}{13} & \frac{4298}{13} \\
2 & 5 & 0 & 13 & 3 & \frac{894}{13} & 159 & \frac{1657}{13} & \frac{4298}{13} & r \\
0 & 0 & 22 & 3 & \frac{894}{13} & \frac{336}{13} & \frac{1657}{13} & \frac{4298}{13} & r & \gamma
\end{pmatrix}.
\]
It is straightforward to check that $M_3$ is positive, recursively generated, and recursively determinate, with $M_2 \succ 0$, 
\[
X_3 = p(X, Y) := 40 \cdot 1 - 24 X + 4 Y - 53 X^2 - 2 XY + 13 Y^2,
\]
\[
X_2 Y = t(X, Y) := 35 \cdot 1 - 22 X - Y - 46 X^2 + 3 XY + 11 Y^2,
\]
and
\[
Y_3 = q(X, Y) := d_1 \cdot 1 + d_2 X + d_3 Y + d_4 X^2 + d_5 XY + d_6 Y^2 + d_7 XY^2,
\]
where
\[
d_1 = 3(487658 - 1651 r) 1447,
\]
\[
d_2 = 3(-342075+1157 r) 1447,
\]
\[
d_3 = 2(-2131598+6591 r) 18811,
\]
\[
d_4 = -2000094+6773 r 1447,
\]
\[
d_5 = 2338519 - 6591 r 18811,
\]
\[
d_6 = 2(-316575+1079 r) 1447,
\]
\[
d_7 = -48015+169 r 1447.
\]
It is straightforward to check that $M_3$ is positive, recursively generated, and recursively determinate, with $M_2 \succ 0$, rank $M_3 = 7$ and column dependence relations

$$X^3 = p(X, Y) := 40 \cdot 1 - 24X + 4Y - 53X^2 - 2XY + 13Y^2,$$

$$X^2Y = t(X, Y) := 35 \cdot 1 - 22X - Y - 46X^2 + 3XY + 11Y^2,$$
It is straightforward to check that $M_3$ is positive, recursively generated, and recursively determinate, with $M_2 \succ 0$, rank $M_3 = 7$ and column dependence relations

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\]

and

\[
Y^3 = q(X, Y) := d_1 \cdot 1 + d_2X + d_3Y + d_4X^2 + d_5XY + d_6Y^2 + d_7XY^2,
\]

where

\[
\begin{align*}
d_1 &= \frac{3(487658-1651r)}{1447}, \\
d_2 &= \frac{3(-342075+1157r)}{1447}, \\
d_3 &= \frac{2(-2131598+6591r)}{18811}, \\
d_4 &= \frac{-2000094+6773r}{1447}, \\
d_5 &= \frac{2338519-6591r}{18811}, \\
d_6 &= \frac{2(-316575+1079r)}{1447}, \\
d_7 &= \frac{-48015+169r}{1447}.
\end{align*}
\]
A calculation shows that

$$\langle (yp)(X, Y) - (xt)(X, Y), XY^2 \rangle = \frac{-49462 + 169r}{13},$$

so for $r \neq \frac{49462}{169}$, $X^3Y$ is not well-defined.
A calculation shows that

\[ \langle (yp)(X, Y) - (xt)(X, Y), XY^2 \rangle = \frac{-49462 + 169r}{13}, \]

so for \( r \neq \frac{49462}{169} \), \( X^3 Y \) is not well-defined. Thus, the conclusions of our Theorem do not hold for \( M_3 \) (and thus there is no representing measure).
Remark
Therefore, we have been able to build a concrete example which shows that our Theorem is no longer valid if we permit column dependence relations in $M_d$ in addition to those generated by $X^n$ and $Y^m$. 
By contrast, we prove that if $M_d \in RD$, with all column dependence relations of strictly lower degree, then $M_d$ does admit an $RG$ extension.
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**Theorem**

Suppose $M_d$ is positive and recursively generated, and satisfies

$$X^n = p(X, Y) \quad (p \in \mathcal{P}_{n-1}, \ n \leq d)$$

and

$$Y^m = q(X, Y) \quad (q \in \mathcal{P}_{m-1}, \ q \text{ has no } y^m \text{ term}, \ m \leq d).$$
By contrast, we prove that if $M_d \in RD$, with all column dependence relations of strictly lower degree, then $M_d$ does admit an $RG$ extension.

**Theorem**

Suppose $M_d$ is positive and recursively generated, and satisfies

$$X^n = p(X, Y) \quad (p \in \mathcal{P}_{n-1}, \ n \leq d)$$

and

$$Y^m = q(X, Y) \quad (q \in \mathcal{P}_{m-1}, \ q \ has \ no \ y^m \ term, \ m \leq d).$$

If each column relation in $M_d$ can be expressed as $X^i Y^j = r(X, Y)$ with $\deg r < i + j$, then $M_d$ admits a unique $RG$ extension.
We now construct a positive, recursively generated, recursively determinate $M_4(\beta^{(8)})$ which admits a positive, recursively generated extension $M_5$, but such that $M_5$ fails to admit a positive, recursively generated extension $M_6$. By the Bayer-Teichmann Theorem, $\beta^{(8)}$ has no representing measure.
We define $M_4$ by defining its component blocks in the decomposition

\[ M_4 = \begin{pmatrix} M_3 & B(4) \\ B(4)^T & C(4) \end{pmatrix} \].
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$$M_4 = \begin{pmatrix} M_3 & B(4) \\ B(4)^T & C(4) \end{pmatrix}.$$ 

We set

$$\beta_{00} = \beta_{20} = \beta_{02} = \beta_{22} = 1,$$

$$\beta_{40} = \beta_{04} = \beta_{42} = \beta_{24} = 2,$$

$$\beta_{60} = \beta_{06} = 5,$$

and all other moments up to degree 6 set to 0, so that
\[ M_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 5 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 5 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 5 \\
\end{pmatrix} \]
We next set

\[
B(4) = \begin{pmatrix}
2 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
5 & 0 & 2 & 0 & 2 \\
0 & 2 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 & 5 \\
a & b & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g \\
0 & 0 & 0 & g & h
\end{pmatrix},
\]

where \(\beta_{70} = a\), \(\beta_{61} = b\), \(\beta_{16} = g\), \(\beta_{07} = h\), and all other degree 7 moments equal 0.
We complete the definition of a recursively determinate $M_4$ by extending the basic relations to the columns of $\begin{pmatrix} B(4)^T & C(4) \end{pmatrix}$, leading to

$$C(4) = \begin{pmatrix} 13 + a^2 + b^2 & ab & 5 & 0 & 4 \\ ab & 5 & 0 & 4 & 0 \\ 5 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & 5 & gh \\ 4 & 0 & 5 & gh & 13 + g^2 + h^2 \end{pmatrix}.$$
We complete the definition of a recursively determinate $M_4$ by extending the basic relations to the columns of $\begin{pmatrix} B(4)^T & C(4) \end{pmatrix}$, leading to

$$C(4) = \begin{pmatrix}
13 + a^2 + b^2 & ab & 5 & 0 & 4 \\
ab & 5 & 0 & 4 & 0 \\
5 & 0 & 4 & 0 & 5 \\
0 & 4 & 0 & 5 & gh \\
4 & 0 & 5 & gh & 13 + g^2 + h^2
\end{pmatrix}.$$ 

Since $M_3 \succ 0$ (positive and invertible), we see that $M_4 \succeq 0$ with rank 13 if and only if

$$\Delta(4) \equiv C(4) - B(4)^T M_3^{-1} B(4) \succeq 0.$$
This positivity is equivalent to the positivity of the compression of $\Delta(4)$ to rows and columns indexed by $X^3Y$, $X^2Y^2$, $XY^3$, i.e.,

$$[\Delta(4)]_{\{X^3Y, X^2Y^2, XY^3\}} \equiv \begin{pmatrix} 1 - b^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - g^2 \end{pmatrix} \succ 0.$$
Thus, if $b$ and $g$ satisfy $1 - b^2 > 0$ and $1 - g^2 > 0$, then $M_4$ is positive, recursively generated, and recursively determinate, with rank $M_4 = 13$, so $M_4$ satisfies the hypotheses of our main Theorem.
Thus, if $b$ and $g$ satisfy $1 - b^2 > 0$ and $1 - g^2 > 0$, then $M_4$ is positive, recursively generated, and recursively determinate, with rank $M_4 = 13$, so $M_4$ satisfies the hypotheses of our main Theorem. We next seek to extend $M_4$ to a positive and recursively generated $M_5$. In view of the basic hypotheses, this can only be accomplished by defining

$$X^5 := (xp)(X, Y)$$

and

$$Y^5 := (yq)(X, Y).$$
This leads to a unique $B(5)$ and $C(5)$, and the resulting $M(5)$ is positive semi-definite. One can then define, uniquely, the recursively generated extension $M(6)$, which fails to be positive semi-definite for some choices of $b$ and $g$. 
**Theorem**

For $d \geq 1$, there exists a moment matrix $M_d$, satisfying the conditions of our main Theorem, for which the extension algorithm determines successive positive, recursively generated extensions $M_{d+1}, \ldots, M_{2d-1}$, and for which the first flat extension occurs at $M_{2d-1}$. Moreover, each extension $M_{d+i}$ satisfies the conditions of the Theorem, so to continue the sequence it is only necessary to verify that the RG extension $M_{d+i+1}$ is positive semidefinite.
Theorem

For $d \geq 1$, there exists a moment matrix $M_d$, satisfying the conditions of our main Theorem, for which the extension algorithm determines successive positive, recursively generated extensions $M_{d+1}, \ldots, M_{2d-1}$, and for which the first flat extension occurs at $M_{2d-1}$. Moreover, each extension $M_{d+i}$ satisfies the conditions of the Theorem, so to continue the sequence it is only necessary to verify that the RG extension $M_{d+i+1}$ is positive semidefinite.

Proof uses the Division Algorithm of Algebraic Geometry in a nontrivial way.
(The Division Algorithm in $\mathbb{R}[x_1, \cdots, x_n]$) Fix a monomial order $>$ on $\mathbb{Z}_n^\geq_0$ and let $F = (f_1, \cdots, f_s)$ be an ordered $s$-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or $r$ is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $f_1, \cdots, f_s$.

Furthermore, if $a_i f_i \neq 0$, then we have $\text{multideg}(f) \geq \text{multideg}(a_i f_i)$. 

**Lemma**
Lemma

(T. Sauer, 1997) For $N \geq 1$ let $v_1, \cdots, v_N$ be distinct points in $\mathbb{R}^2$, and consider the multivariable Vandermonde matrix $V_N := (v_i^\alpha)_{1 \leq i \leq N, \alpha \in \mathbb{Z}_+^2, |\alpha| \leq N-1}$, of size $N \times \frac{N(N+1)}{2}$. Then the rank of $V_N$ equals $N$. 
Corollary

Let $\mathbf{x} \equiv \{x_1, \ldots, x_m\}$ and $\mathbf{y} \equiv \{y_1, \ldots, y_n\}$ be sets of distinct real numbers, and consider the grid $\mathbf{x} \times \mathbf{y} := \{(x_i, y_j)\}_{1 \leq i \leq m, 1 \leq j \leq n}$ consisting of $N := mn$ distinct points in $\mathbb{R}^2$. Then the generalized Vandermonde matrix $V_{\mathbf{x} \times \mathbf{y}}$, obtained from $V_N$ by removing all columns indexed by monomials divisible by $x^m$ or $y^n$, is invertible.
The following result is a special case of Alon’s Combinatorial Nullstellensatz.

**Corollary**

Let $G \equiv x \times y$ be a grid as in Corollary 20, let $N := mn$, and let $p \in \mathbb{R}[x, y]$ be such that $\deg_x p < m$ and $\deg_y p < n$. Assume also that $p|_G \equiv 0$. Then $p \equiv 0$. 

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**Proposition**

Let $P(x, y) := (x - x_1) \cdots (x - x_d)$ and let

$Q(x, y) := (y - y_1) \cdots (y - y_d)$. If $\rho := \text{multideg} \,(f) \geq d$ and $f|_{\mathcal{V}}((P, Q)) \equiv 0$, then there exists $u, v \in P_{\rho - d}$ such that $f = uP + vQ$.

**Proof.**

Let $\mathcal{V} := \mathcal{V}((P, Q))$. By Lemma 18, we can write $f = uP + vQ + r$, where $\text{multideg} \,(uP) \leq \rho$ and $\text{multideg} \,(vQ) \leq \rho$. It follows that $u, v \in P_{\rho - d}$ and that $r|_{\mathcal{V}} \equiv 0$. Moreover, $r$ is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $P$ and $Q$, that is, they are not divisible by $x^d$ and $y^d$. Therefore, $r$ satisfies the hypotheses of Corollary 21 with $m = n = d$. By Corollary 21, $r \equiv 0$. Thus, $f = uP + vQ$, as desired. \qed
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- The Basic Algorithm, though, does provide a tool for examining all TMP’s.