Recursively determined representing measures for bivariate truncated moment sequences

(joint work with Lawrence Fialkow; to appear in JOT)

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The Classical (Full) Moment Problem

Let $\beta \equiv \beta^{(\infty)} = \{\beta_i\}_{i \in \mathbb{Z}_+^m}$ denote a m-dimensional real multisequence, and let K (closed) $\subseteq \mathbb{R}^m$. The (full) K-moment problem asks for necessary and sufficient conditions on β to guarantee the existence of a positive Borel measure μ supported in K such that

$$\beta_i = \int x^i d\mu \qquad (i \in \mathbb{Z}_+^m);$$

μ is called a **rep. meas.** for β.

Associated with β is a moment matrix $M \equiv M(\infty)$, defined by

$$M_{ij} := \beta_{i+j} \qquad (i, j \in \mathbb{Z}_+^m).$$

The Truncated Moment Problem (TMP)

Let $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}_+^m,\ |i|+|j| \leq 2n}$ denote a m-dimensional real multisequence, and let K (closed) $\subseteq \mathbb{R}^m$. The (truncated) K-moment problem asks for necessary and sufficient conditions on β to guarantee the existence of a positive Borel measure μ supported in K such that

$$\beta_i = \int x^i d\mu \qquad (i \in \mathbb{Z}_+^m,\ |i| + |j| \leq 2n);$$

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Basic Positivity Condition

$\mathcal{P}_n$: polynomials p over $\mathbb{R}$ with $\deg p \leq n$

- Given $p \in \mathcal{P}_n$, $p(x) \equiv \sum_{0 \leq i+j \leq n} a_i x^i$,

$$0 \leq \int p(x)^2 d\mu(x)$$

$$= \sum_{ij} a_i a_j \int x^{i+j} d\mu(x) = \sum_{ij} a_i a_j \beta_{i+j}.$$

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- Now recall that we're working in d real variables. To understand this **"matricial" positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, X_1, \ldots, X_m, X_1^2, X_2X_1, \ldots, X_m^2, \ldots$$

The entries are given by

$$M(n)_{i,j} := \beta_{i+j}.$$

Then

$$(matricial\ positivity)\ \sum_{ij} a_i a_j \beta_{i+j} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\beta) \geq 0.$$

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Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, several years ago we were able to build a family $\beta_{00}, \beta_{01,} \beta_{10}, ..., \beta_{06}, ..., \beta_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. Later on, we also did this for $n = 2$ (in this case $M(2)$ is not invertible).

For example, for moment problems in $\mathbb{R}^2$,

$$M(1) = \begin{pmatrix} \beta_{00} & \beta_{01} & \beta_{10} \\ \beta_{01} & \beta_{02} & \beta_{11} \\ \beta_{10} & \beta_{11} & \beta_{20} \end{pmatrix},$$

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$$M(2) = \begin{pmatrix} \beta_{00} & \beta_{01} & \beta_{10} & \beta_{02} & \beta_{11} & \beta_{20} \\ \beta_{01} & \beta_{02} & \beta_{11} & \beta_{03} & \beta_{12} & \beta_{21} \\ \beta_{10} & \beta_{11} & \beta_{20} & \beta_{12} & \beta_{21} & \beta_{30} \\ \beta_{02} & \beta_{03} & \beta_{12} & \beta_{04} & \beta_{13} & \beta_{22} \\ \beta_{11} & \beta_{12} & \beta_{21} & \beta_{13} & \beta_{22} & \beta_{31} \\ \beta_{20} & \beta_{21} & \beta_{30} & \beta_{22} & \beta_{31} & \beta_{40} \end{pmatrix}.$$

- In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

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- The link between TMP and FMP is provided by a result of Stochel (2001):

Theorem (Stochel's Theorem)

$\beta^{(\infty)}$ has a rep. meas. supported in a closed set $K \subseteq \mathbb{R}^d$ if and only if, for each n, $\beta^{(2n)}$ has a rep. meas. supported in K.

For moment problems in $\mathbb{C}$,

$$M(3) = \begin{pmatrix} 1 & Z & \bar{Z} & Z^2 & \bar{Z}Z & \bar{Z}^2 & \vdots & Z^3 & \bar{Z}Z^2 & \bar{Z}^2Z & \bar{Z}^3 \\ \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} & \vdots & \gamma_{03} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} & \vdots & \gamma_{13} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} & \vdots & \gamma_{04} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} & \vdots & \gamma_{23} & \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} & \vdots & \gamma_{14} & \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} & \vdots & \gamma_{05} & \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{30} & \gamma_{31} & \gamma_{40} & \gamma_{32} & \gamma_{41} & \gamma_{50} & \vdots & \gamma_{33} & \gamma_{42} & \gamma_{51} & \gamma_{60} \\ \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} & \gamma_{41} & \vdots & \gamma_{24} & \gamma_{33} & \gamma_{42} & \gamma_{51} \\ \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} & \gamma_{32} & \vdots & \gamma_{15} & \gamma_{24} & \gamma_{33} & \gamma_{42} \\ \gamma_{03} & \gamma_{04} & \gamma_{13} & \gamma_{05} & \gamma_{14} & \gamma_{23} & \vdots & \gamma_{06} & \gamma_{15} & \gamma_{24} & \gamma_{33} \end{pmatrix}.$$

For moment problems in $\mathbb{R}^2$,

$$M(3) = \begin{pmatrix}
1 & X & Y & X^2 & XY & Y^2 & \vdots & X^3 & X^2Y & XY^2 & Y^3 \\
\beta_{00} & \beta_{01} & \beta_{10} & \beta_{02} & \beta_{11} & \beta_{20} & \vdots & \beta_{03} & \beta_{12} & \beta_{21} & \beta_{30} \\
\beta_{01} & \beta_{02} & \beta_{11} & \beta_{03} & \beta_{12} & \beta_{21} & \vdots & \beta_{04} & \beta_{13} & \beta_{22} & \beta_{31} \\
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- TMP is more general than FMP:

  fewer moments $\Longrightarrow$ less data

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- Helpful tool: Smul'jan's Theorem on positivity of 2×2 matrices

Positivity of Block Matrices

Theorem

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases} .$$

Moreover, rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ =rank $A \Leftrightarrow C = W^*AW$.

Corollary

Assume rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank } A$. *Then*

$$A \geq 0 \Leftrightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Abstract of New Results

A theorem of Bayer and Teichmann [BT] implies that if a finite real multisequence $\beta \equiv \beta^{(2d)}$ has a representing measure, then the associated moment matrix M_d admits positive, recursively generated moment matrix extensions $M_{d+1},\ M_{d+2}, \ldots$.

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Typical Problem: Given a 3-D body, let X-rays act on the body at different angles, collecting the information on a screen. One then seeks to obtain a constructive, optimal way to approximate the body, or in some cases to reconstruct the body.

Functional Calculus

For $p \in \mathcal{P}_n$, $p(x,y) \equiv \sum_{0 \leq i+j \leq n} a_{ij} x^i y^j$ define

$$p(X,Y) := \sum a_{ij} X^i Y^j \equiv M(n)\hat{p},$$

where $\hat{p} := (a_{00} \cdots a_{0n} \cdots a_{n0})^T$.

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If there exists a rep. meas. μ, then

$$p(X,Y) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p := \{(x,y) \in \mathbb{R}^2 : \ p(x,y) = 0\}.$$

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The following is our analogue of recursiveness for the TCMP

$$\text{(RG)} \quad \text{If} p, q, pq \in \mathcal{P}_n, \textbf{ and } p(X,Y) = 0,$$

$$\textbf{then } (pq)(X,Y) = 0.$$

Truncated Moment Problems in Two Real Variables

TMP	Complex	Real
moments	$\gamma_{ij} \in \mathbb{C}, \quad \gamma_{ji} = \bar{\gamma_{ij}}$	$\beta_{ij} \in \mathbb{R}$
moment matrix	$M(n)$	$M(n)$
functional calculus	$p(Z, \bar{Z})$	$p(X, Y)$
algebraic variety	$\mathcal{V}(\gamma) := \bigcap_{p(Z,\bar{Z})=0} \mathcal{Z}_p$	$\mathcal{V}(\beta) := \bigcap_{p(X,Y)=0} \mathcal{Z}_p$

Singular TMP; Real Case

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where $\mathcal{Z}_p$ is the zero set of p.

- Always true:

$$r := \text{rank}\, \mathcal{M}(n) \leq \text{card supp } \mu \leq v := \text{card}\, \mathcal{V}(\beta),$$

so if the variety is finite there's a natural candidate for supp μ, i.e., supp $\mu = \mathcal{V}(\beta)$

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  $\text{rank } M(n) \leq \text{rank } M(n+1) \leq \text{card } \mathcal{V}(M(n+1)) \leq \text{card } \mathcal{V}(M(n))$
- General case.

Localizing Matrices

Consider the **full, complex** MP

$$\int \bar{z}^i z^j \, d\mu = \gamma_{ij} \ \ (i,j \geq 0),$$

where supp $\mu \subseteq K$, for K a closed subset of $\mathbb{C}$.

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- Riesz-Haviland:

  There exists μ with supp $\mu \subseteq K \Leftrightarrow \Lambda_\gamma(p) \geq 0$ for all p such that $p|_K \geq 0$.

If q is a polynomial in z and $\bar{z}$, and

$$K \equiv K_q := \{z \in \mathbb{C} : q(z, \bar{z}) \geq 0\},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for μ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p}\, d\mu \geq 0 \ \ \text{(all } p\text{)}.$$

- K. Schmüdgen (1991): If K_q is compact, $\Lambda_\gamma(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all p, then there exists μ with supp $\mu \subseteq K_q$.

Quick Overview of Known Results for TCMP

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0, \infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a, b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

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- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.

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- J. Lasserre, M. Laurent and others to convert polynomial optimization into an instance of semidefinite programming.

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- More recently, we have begun to use our methods to solve FULL moment problems, by first solving truncated MP's, and then applying J. Stochel's limiting argument.
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- In the specific case of $K :=$ supp μ, a semi-algebraic set determined by a finite collection of complex polynomials $\mathcal{P} = \{p_i(z, \bar{z})\}_{i=1}^m$, i.e.,

$$K = K_P := \{z \in \mathbb{C} : p_i(z, \bar{z}) \geq 0, 1 \leq i \leq m\},$$

we obtain an existence criterion expressed in terms of positivity and extension properties of the moment matrix $M(n)(\gamma)$ associated to γ and of the localizing matrix M_{p_i} corresponding to each p_i.

Case of Flat Data

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

β is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

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To find μ concretely, let $r :=$rank $M(n)$ and look for the relation

$$Z^r = c_0 1 + c_1 Z + \ldots + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + \ldots + c_{r-1} z^{r-1})$$

and solve the Vandermonde equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

Localization of Support: Main Theorem

Theorem

(RC-LF, 2000) *Let* $M(n) \geq 0$ *and suppose* $\deg(q) = 2k$ *or* $2k - 1$ *for some* $k \leq n$. *Then* $\exists\ \mu$ *with* $\operatorname{rank} M(n)$ *atoms and* $\operatorname{supp}\ \mu \subseteq K_q$ *if and only if* $\exists$ *a flat extension* $M(n+1)$ *for which* $M_q(n+k) \geq 0$. *In this case,* $\exists\ \mu$ *with exactly* $\operatorname{rank} M(n) - \operatorname{rank}\ M_q(n+k)$ *atoms in* $\mathcal{Z}(q)$.

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Remark

M. Laurent (2005) has found an alternative proof, using ideas from real algebraic geometry.

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A\,1$ (Dirac measure)

The Quartic Moment Problem: Complex Case

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- $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$\begin{aligned} D &= 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \text{ and } C = \bar{B} \\ &\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2 \\ &\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}. \end{aligned}$$

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Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

(a)	$\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$	parabola; $y = x^2$
(b)	$\bar{W}^2 = -4i1 + W^2$	hyperbola; $yx = 1$
(c)	$\bar{W}^2 = W^2$	pair of intersect. lines; $yx = 0$
(d)	$\bar{W}W = 1$	unit circle; $x^2 + y^2 = 1$.

Theorem (The Quartic MP: Complex Case)

(RC-L. Fialkow, 2005) Let $\gamma^{(4)}$ *be given, and assume* $M(2) \geq 0$ *and* $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ *is a basis for* $\mathcal{C}_{M(2)}$. *Then* $\gamma^{(4)}$ *admits a rep. meas.* μ. *Moreover, it is possible to find* μ *with* card supp $\mu = \text{rank } M(2)$, *except in some cases when* $\mathcal{V}(\gamma^{(4)})$ *is a pair of intersecting lines; in such cases, there exist* μ *with* card supp $\mu \leq 6$.

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Corollary

Assume that $M(2) \geq 0$ *and that* rank $M(2) \leq$ card $\mathcal{V}(\gamma^{(4)})$. *Then* $M(2)$ *admits a representing measure.*

Extremal MP; $r = v$

Recall that the *algebraic variety* of β is

$$\mathcal{V} \equiv \mathcal{V}(\beta) := \bigcap_{p(X,Y)=0} \mathcal{Z}_p,$$

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If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$.

Here Λ is the Riesz functional, given by $\Lambda(x^i y^j) := \beta_{ij}$

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Consistency implies

(Recursiveness) $p, q, pq \in \mathcal{P}_n,\ M(n)\hat{p} = 0 \Longrightarrow M(n)(pq)\hat{} = 0.$

In general, *Positivity*, *Consistency* and the *Variety Condition* are **not** sufficient. However,

Theorem (The Extremal Case)

(RC, L. Fialkow and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ **extremal**, *i.e.*, $r = v$, *the following are equivalent:*

(i) β *has a representing measure;*

(ii) β *has a unique representing measure, which is* rank $M(n)$*-atomic (minimal);*

(iii) $M(n) \geq 0$ *and* β *is consistent.*

A Basic Algorithm

The extension of this result to general representing measures follows from a theorem of C. Bayer and J. Teichmann: If β has a representing measure, then it has a finitely atomic representing measure.

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For the class of bivariate *recursively determinate* moment matrices, we now present a detailed analysis of an algorithm that can be used in numerical examples to determine the existence or nonexistence of flat extensions (and representing measures).

This algorithm determines the existence or nonexistence of positive, recursively generated extensions $M_{d+1}, \ldots, M_{2d-1}$, at least one of which must be a flat extension in the case when there is a measure.

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One of our main results shows that there are sequences $\beta^{(2d)}$ for which the first flat extension occurs at M_{2d-1}, so all of the above extensions must be computed in order to recognize that there is a measure.

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This result stands in sharp contrast to traditional truncated moment theorems (concerning representing measures supported in $\mathbb{R}$, $[a, b]$, $[0, +\infty)$, or in a planar curve of degree 2), which express the existence of a measure in terms of tests closely related to the original moment data.

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We will show that under mild additional hypotheses on M_d, the implementation of each extension step, from M_{d+j} to M_{d+j+1}, leading to a flat extension M_{d+k+1}, consists of simply verifying a matrix positivity condition.

Recursively Determinate Matrices

A bivariate moment matrix M_d is *recursively determinate* if there are column dependence relations of the form

$$X^n = p(X, Y) \quad (p \in \mathcal{P}_{n-1},\ n \le d)$$

and

$$Y^m = q(X, Y) \quad (q \in \mathcal{P}_m,\ q \text{ has no } y^m \text{ term},\ m \le d),$$

or with similar relations with the roles of p and q reversed.

We next present an example which illustrates the algorithm in a case leading to a measure.

Example. Let $d = 3$ and consider

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & \mathbf{c} \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & \mathbf{c} & \delta \end{pmatrix}.$$

We have $M_3 \succeq 0$, $M_2 \succ 0$, and

$$\text{rank } M_3 = 8 \Longleftrightarrow \delta = 2026881 - 2844c + c^2.$$

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Let

$$r_1(x, y) = y - x^3$$

and

$$r_2(x, y) = y^3 - q(x, y).$$

With these two column relations in hand, our results guarantees the existence of a unique RG extension M_4. To test the positivity of M_4, we calculate the determinant of the 9×9 matrix consisting of the rows and columns of M_4 indexed by the monomials $1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2$. A straightforward calculation using *Mathematica* shows that three cases arise:

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(i) $c < 1429$: here $M_4 \not\succeq 0$, so M_3 admits no representing measure;

(ii) $c = 1429$: here M_4 is a flat extension of M_3, so M_3 admits an 8-atomic representing measure;

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To illustrate this case, we take the special value $c = 1430$, so that $q(x, y) \equiv -5x + 20y - 21x^2y + 8xy^2$. Let $\alpha := \frac{1}{2}\sqrt{5 - 2\sqrt{5}}$ and $\gamma := \sqrt{5}\alpha$.

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M_3 satisfies the hypothesis of our Theorem with $n = m = 3$, so we proceed to generate the RG extension M_4.

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(first in $\begin{pmatrix} M_3 & B(4) \end{pmatrix}$,

This extension is uniquely determined by imposing the column relations

$$\begin{aligned} X^4 &= XY \ \ (\text{from } X^3 = Y) \\ X^3Y &= Y^2 \ \ (\text{from } X^3 = Y) \\ XY^3 &= (xq)(X,Y) \ \ (\text{from } Y^3 = q(X,Y)) \\ Y^4 &= (yq)(X,Y) \ \ (\text{from } Y^3 = q(X,Y)) \end{aligned}$$

(first in $\begin{pmatrix} M_3 & B(4) \end{pmatrix}$, then in $\begin{pmatrix} B(4)^T & C(4) \end{pmatrix}$).

A calculation shows that, as expected, these relations unambiguously define a positive moment matrix M_4 with

$$\text{rank } M_4 = 9 \ (> 8 = \text{rank } M_3).$$

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It follows that M_3 admits no flat extension M_4, so we proceed to construct the *RG* extension M_5, uniquely determined by imposing the relations

$$
\begin{aligned}
X^5 &= X^2Y, \\
X^4Y &= XY^2, \\
X^3Y^2 &= Y^3, \\
X^2Y^3 &= (x^2q)(X,Y), \\
XY^4 &= (xyq)(X,Y), \\
Y^5 &= (y^2q)(X,Y).
\end{aligned}
$$

A calculation of these columns (first in $\begin{pmatrix} M_4 & B(5) \end{pmatrix}$, then in $\begin{pmatrix} B(5)^T & C(5) \end{pmatrix}$), shows that, as again expected, they do fit together to unambiguously define a moment matrix M_5.

A calculation of these columns (first in $\begin{pmatrix} M_4 & B(5) \end{pmatrix}$, then in $\begin{pmatrix} B(5)^T & C(5) \end{pmatrix}$), shows that, as again expected, they do fit together to unambiguously define a moment matrix M_5. From the form of $q(x, y)$, we see that M_5 is actually a flat extension of M_4, in keeping with the above discussion. Corresponding to this flat extension is the unique, 9-atomic, representing measure $\mu \equiv \mu_{M_5}$.

Clearly, supp $\mu = \mathcal{V}$, so μ is of the form $\mu = \sum_{i=1}^{9} \rho_i \delta_{(x_i, x_i^3)}$. To compute the densities, we use the Vandermonde method and find $\rho_1 = \frac{1}{5} = 0.2$, $\rho_2 = \rho_5 = \frac{-1+\sqrt{5}}{8\sqrt{5}} \approx 0.069$, $\rho_3 = \rho_4 = \frac{1+\sqrt{5}}{8\sqrt{5}} \approx 0.181$, $\rho_6 = \rho_9 = \frac{5+3\sqrt{5}}{40\sqrt{5}} \approx 0.131$, and $\rho_7 = \rho_8 = \frac{-5+3\sqrt{5}}{40\sqrt{5}} \approx 0.019$.

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$n + m - d - 1 = 3 + 3 - 2 - 1 = 2$ or as

$1 + \text{card } \mathcal{V} - \text{rank } M_3 = 1 + 9 - 8 = 2$, which is consistent with our earlier discussion. □

Theorem

Suppose the bivariate moment matrix $M_d(\beta)$ *is positive and recursively generated, with column dependence relations generated entirely the monomials* X^n *and* Y^m *via recursiveness and linearity. Then there exists a unique moment matrix block* $B(d+1)$ *such that* $\begin{pmatrix} M_d & B(d+1) \end{pmatrix}$ *is recursively generated and* $Ran\ B(d+1) \subseteq Ran\ M_d$.

Corollary

If M_d satisfies the hypotheses of the previous Theorem, then there exists a unique moment matrix block $C \equiv C(n+1)$ consistent with the structure of an RG extension M_{d+1}.

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By combining the previous Theorem and Corollary, we immediately obtain the first of our main results, which follows.

Theorem

If M_d is positive, with column relations generated entirely by X^n and Y^m via recursiveness and linearity, then M_d admits a unique RG extension M_{d+1}, i.e., Ran $B(n+1) \subseteq$ Ran M_d, and M_{d+1} is recursively generated.

Corollary

If M_d satisfies the above mentioned hypotheses and $d = n + m - 2$, then M_d admits a flat moment matrix extension M_{d+1} (and β admits a rank M_d-atomic representing measure).

We continue with an example which shows that our Theorem is no longer valid if we permit column dependence relations in M_d in addition to those generated by X^n and Y^m.

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Example. We define M_3 by setting

$$
\begin{aligned}
\beta_{00} &= \beta_{20} = \beta_{02} = 1 \\
\beta_{11} &= \beta_{30} = \beta_{21} = \beta_{03} = 0 \\
\beta_{12} &= \beta_{40} = 2 \\
\beta_{31} &= \beta_{13} = 0 \\
\beta_{22} &= 5 \\
\beta_{04} &= 22 \\
\beta_{50} &= -1
\end{aligned}
$$

$$
\begin{aligned}
\beta_{41} &= -2 \\
\beta_{32} &= 13 \\
\beta_{23} &= 3 \\
\beta_{14} &= \frac{894}{13} \\
\beta_{05} &= \frac{336}{13} \\
\beta_{60} &= 178 \\
\beta_{51} &= 139 \\
\beta_{42} &= 159 \\
\beta_{33} &= \frac{1657}{13} \\
\beta_{24} &= \frac{4298}{13} \\
\beta_{15} &= r
\end{aligned}
$$

$$\beta_{06} \quad = \quad \gamma \equiv \frac{443272376768 - 2742712830r - 4826809r^2}{41327767}.$$

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Thus, we have

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 5 & 0 & 22 \\ 1 & 0 & 0 & 2 & 0 & 5 & -1 & -2 & 13 & 3 \\ 0 & 0 & 2 & 0 & 5 & 0 & -2 & 13 & 3 & \frac{894}{13} \\ 1 & 2 & 0 & 5 & 0 & 22 & 13 & 3 & \frac{894}{13} & \frac{336}{13} \\ 0 & 2 & 0 & -1 & -2 & 13 & 178 & 139 & 159 & \frac{1657}{13} \\ 0 & 0 & 5 & -2 & 13 & 3 & 139 & 159 & \frac{1657}{13} & \frac{4298}{13} \\ 2 & 5 & 0 & 13 & 3 & \frac{894}{13} & 159 & \frac{1657}{13} & \frac{4298}{13} & r \\ 0 & 0 & 22 & 3 & \frac{894}{13} & \frac{336}{13} & \frac{1657}{13} & \frac{4298}{13} & r & \gamma \end{pmatrix}.$$

It is straightforward to check that M_3 is positive, recursively generated, and recursively determinate, with $M_2 \succ 0$, It is straightforward to check that M_3 is positive, recursively generated, and recursively determinate, with $M_2 \succ 0$, rank $M_3 = 7$ and column dependence relations

$$X^3 = p(X, Y) := 40 \cdot 1 - 24X + 4Y - 53X^2 - 2XY + 13Y^2,$$

$$X^2 Y = t(X, Y) := 35 \cdot 1 - 22X - Y - 46X^2 + 3XY + 11Y^2,$$

It is straightforward to check that M_3 is positive, recursively generated, and recursively determinate, with $M_2 \succ 0$, rank $M_3 = 7$ and column dependence relations

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$$X^2Y = t(X,Y) := 35 \cdot 1 - 22X - Y - 46X^2 + 3XY + 11Y^2,$$

and

$$Y^3 = q(X,Y) := d_1 \cdot 1 + d_2 X + d_3 Y + d_4 X^2 + d_5 XY + d_6 Y^2 + d_7 XY^2,$$

where $d_1 = \frac{3(487658-1651r)}{1447}$, $d_2 = \frac{3(-342075+1157r)}{1447}$, $d_3 = \frac{2(-2131598+6591r)}{18811}$, $d_4 = \frac{-2000094+6773r}{1447}$, $d_5 = \frac{2338519-6591r}{18811}$, $d_6 = \frac{2(-316575+1079r)}{1447}$, $d_7 = \frac{-48015+169r}{1447}$.

A calculation shows that

$$\langle (yp)(X,Y) - (xt)(X,Y), XY^2 \rangle = \frac{-49462 + 169r}{13},$$

so for $r \neq \frac{49462}{169}$, X^3Y is not well-defined.

A calculation shows that

$$\langle (yp)(X,Y) - (xt)(X,Y), XY^2 \rangle = \frac{-49462 + 169r}{13},$$

so for $r \neq \frac{49462}{169}$, $X^3 Y$ is not well-defined. Thus, the conclusions of our Theorem do not hold for M_3 (and thus there is no representing measure). □

Remark

Therefore, we have been able to build a concrete example which shows that our Theorem is no longer valid if we permit column dependence relations in M_d in addition to those generated by X^n and Y^m.

By contrast, we prove that if $M_d \in RD$, with *all* column dependence relations of strictly lower degree, then M_d does admit an RG extension.

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Theorem

Suppose M_d is positive and recursively generated, and satisfies

$$X^n = p(X,Y) \quad (p \in \mathcal{P}_{n-1},\ n \leq d)$$

and

$$Y^m = q(X,Y) \quad (q \in \mathcal{P}_{m-1},\ q \text{ has no } y^m \text{ term},\ m \leq d).$$

By contrast, we prove that if $M_d \in RD$, with *all* column dependence relations of strictly lower degree, then M_d does admit an RG extension.

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If each column relation in M_d can be expressed as $X^iY^j = r(X,Y)$ with $\deg\ r < i+j$, then M_d admits a unique RG extension.

Failure at the Second Stage

We now construct a positive, recursively generated, recursively determinate $M_4(\beta^{(8)})$ which admits a positive, recursively generated extension M_5, but such that M_5 fails to admit a positive, recursively generated extension M_6. By the Bayer-Teichmann Theorem, $\beta^{(8)}$ has no representing measure.

We define M_4 by defining its component blocks in the decomposition

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We set

$$\beta_{00} = \beta_{20} = \beta_{02} = \beta_{22} = 1,$$

$$\beta_{40} = \beta_{04} = \beta_{42} = \beta_{24} = 2,$$

$$\beta_{60} = \beta_{06} = 5,$$

and all other moments up to degree 6 set to 0, so that

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 5 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 5 \end{pmatrix}.$$

We next set

$$B(4) = \begin{pmatrix} 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 5 \\ a & b & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & g & h \end{pmatrix},$$

where $\beta_{70} = a$, $\beta_{61} = b$, $\beta_{16} = g$, $\beta_{07} = h$, and all other degree 7 moments equal 0.

We complete the definition of a recursively determinate M_4 by extending the basic relations to the columns of $\begin{pmatrix} B(4)^T & C(4) \end{pmatrix}$, leading to

$$C(4) = \begin{pmatrix} 13 + a^2 + b^2 & ab & 5 & 0 & 4 \\ ab & 5 & 0 & 4 & 0 \\ 5 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & 5 & gh \\ 4 & 0 & 5 & gh & 13 + g^2 + h^2 \end{pmatrix}.$$

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Since $M_3 \succ 0$ (positive and invertible), we see that $M_4 \succeq 0$ with rank 13 if and only if

$$\Delta(4) \equiv C(4) - B(4)^T M_3^{-1} B(4) \succ 0.$$

This positivity is equivalent to the positivity of the compression of $\Delta(4)$ to rows and columns indexed by X^3Y, X^2Y^2, XY^3, i.e.,

$$[\Delta(4)]_{\{X^3Y, X^2Y^2, XY^4\}} \equiv \begin{pmatrix} 1-b^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-g^2 \end{pmatrix} \succ 0.$$

Thus, if b and g satisfy $1 - b^2 > 0$ and $1 - g^2 > 0$, then M_4 is positive, recursively generated, and recursively determinate, with rank $M_4 = 13$, so M_4 satisfies the hypotheses of our main Theorem.

Thus, if b and g satisfy $1 - b^2 > 0$ and $1 - g^2 > 0$, then M_4 is positive, recursively generated, and recursively determinate, with rank $M_4 = 13$, so M_4 satisfies the hypotheses of our main Theorem. We next seek to extend M_4 to a positive and recursively generated M_5. In view of the basic hypotheses, this can only be accomplished by defining

$$X^5 := (xp)(X, Y)$$

and

$$Y^5 := (yq)(X, Y).$$

This leads to a unique $B(5)$ and $C(5)$, and the resulting $M(5)$ is positive semi-definite. One can then define, uniquely, the recursively generated extension $M(6)$, which fails to be positive semi-definite for some choices of b and g.

Maximum Number of Extension Steps

Theorem

For $d \geq 1$, there exists a moment matrix M_d, satisfying the conditions of our main Theorem, for which the extension algorithm determines successive positive, recursively generated extensions $M_{d+1}, \ldots, M_{2d-1}$, and for which the first flat extension occurs at M_{2d-1}. Moreover, each extension M_{d+i} satisfies the conditions of the Theorem, so to continue the sequence it is only necessary to verify that the RG extension M_{d+i+1} is positive semidefinite.

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Proof uses the Division Algorithm of Algebraic Geometry in a nontrivial way.

Lemma

(The Division Algorithm in $\mathbb{R}[x_1, \cdots, x_n]$) Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$ and let $F = (f_1, \cdots, f_s)$ be an ordered s-tuple of polynomials in $\mathbb{R}[x_1, \cdots, x_n]$. Then every $f \in \mathbb{R}[x_1, \cdots, x_n]$ can be written as

$$f = a_1 f_1 + \cdots + a_s f_s + r,$$

where $a_i, \in \mathbb{R}[x_1, \cdots, x_n]$, and either $r = 0$ or r is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in $f_1, \cdots, f_s$.
Furthermore, if $a_i f_i \neq 0$, then we have multideg $(f) \geq$ multideg $(a_i f_i)$.

Lemma

(T. Sauer, 1997) For $N \geq 1$ let $v_1, \cdots, v_N$ be distinct points in $\mathbb{R}^2$, and consider the multivariable Vandermonde matrix $V_N := (v_i^\alpha)_{1 \leq i \leq N, \alpha \in \mathbb{Z}_+^2, |\alpha| \leq N-1}$, of size $N \times \frac{N(N+1)}{2}$. Then the rank of V_N equals N.

Corollary

Let $\mathbf{x} \equiv \{x_1, \ldots, x_m\}$ *and* $\mathbf{y} \equiv \{y_1, \ldots, y_n\}$ *be sets of distinct real numbers, and consider the grid* $\mathbf{x} \times \mathbf{y} := \{(x_i, y_j)\}_{1 \le i \le m, 1 \le j \le n}$ *consisting of* $N := mn$ *distinct points in* $\mathbb{R}^2$. *Then the generalized Vandermonde matrix* $V_{\mathbf{x} \times \mathbf{y}}$, *obtained from* V_N *by removing all columns indexed by monomials divisible by* x^m *or* y^n, *is invertible.*

The following result is a special case of Alon's Combinatorial Nullstellensatz.

Corollary

Let $G \equiv \mathbf{x} \times \mathbf{y}$ *be a grid as in Corollary 20, let* $N := mn$*, and let* $p \in \mathbb{R}[x, y]$ *be such that* $\deg_x p < m$ *and* $\deg_y p < n$*. Assume also that* $p|_G \equiv 0$*. Then* $p \equiv 0$*.*

Proposition

Let $P(x,y) := (x - x_1)\cdots(x - x_d)$ *and let* $Q(x,y) := (y - y_1)\cdots(y - y_d)$. *If* $\rho :=$ *multideg* $(f) \geq d$ *and* $f|\mathcal{V}((P,Q)) \equiv 0$, *then there exists* $u, v \in \mathcal{P}_{\rho-d}$ *such that* $f = uP + vQ$.

Proof.

Let $\mathcal{V} := \mathcal{V}((P,Q))$. By Lemma 18, we can write $f = uP + vQ + r$, where multideg $(uP) \leq \rho$ and multideg $(vQ) \leq \rho$. It follows that $u, v \in \mathcal{P}_{\rho-d}$ and that $r|\mathcal{V} \equiv 0$. Moreover, r is a linear combination, with coefficients in $\mathbb{R}$, of monomials, none of which is divisible by any of the leading terms in P and Q, that is, they are not divisible by x^d and y^d. Therefore, r satisfies the hypotheses of Corollary 21 with $m = n = d$. By Corollary 21, $r \equiv 0$. Thus, $f = uP + vQ$, as desired. □

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- But our new results stop short of covering some important cases, even for a class of moment matrices $M(3)$.
- The Basic Algorithm, though, does provide a tool for examining all TMP's.