PROPAGATION PHENOMENA
FOR HYPONORMAL 2-VARIABLE WEIGHTED SHIFTS

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Abstract. We study the class of hyponormal 2-variable weighted shifts with two consecutive equal weights in the weight sequence of one of the coordinate operators. We show that under natural assumptions on the coordinate operators, the presence of consecutive equal weights leads to horizontal or vertical flatness, in a way that resembles the situation for 1-variable weighted shifts. In 1-variable, it is well known that flat weighted shifts are necessarily subnormal (with finitely atomic Berger measures). By contrast, we exhibit a large collection of flat (i.e., horizontally and vertically flat) 2-variable weighted shifts which are hyponormal but not subnormal. Moreover we completely characterize the hyponormality and subnormality of symmetrically flat contractive 2-variable weighted shifts.

1. Introduction

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions on a pair of commuting subnormal operators on Hilbert space to admit a joint normal extension. In previous work we have proved that the (joint) hyponormality of the pair, while being a necessary condition, is by no means sufficient ([CuYo1], [CuYo2]). We have also established that in a very special situation, hyponormality is indeed sufficient [CuYo1, Theorem 5.2 and Remark 5.3]. This involves 2-variable weighted shifts with weight sequences which are constant except for the 0-th row in the index set $\mathbb{Z}_+^2$. One is then tempted to claim that a similar result might be true for weight sequences which are constant in a slightly smaller domain of indices, e.g., those indices $\mathbf{k} \in \mathbb{Z}_+^2$ with $k_1, k_2 \geq 1$. However, in this paper we show that such is not the case, that is, hyponormality and subnormality are quite different even in those cases.

For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_k := \alpha_k e_{k+1}$ (all $k \geq 0$), where $\{e_k\}_{k=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. A quadratically hyponormal weighted shift W_α with $\alpha_{k+1} = \alpha_k$ for some $k \geq 1$ must necessarily be (i) *flat* (i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$), and (ii) subnormal. For 2-variable weighted shifts associated with weight sequences $\{\alpha_{\mathbf{k}}\}, \{\beta_{\mathbf{k}}\} \in \ell^\infty(\mathbb{Z}_+^2)$, we first establish the correct analogue of (i) (Theorem 3.3), and we then show that there is a rich family of sequences $\{\alpha_{\mathbf{k}}\}, \{\beta_{\mathbf{k}}\}$ giving rise to flat, non-subnormal, hyponormal 2-variable weighted shifts; this is in sharp contrast with the 1-variable situation. The optimality of Theorem 3.3 is established through an elaborate construction which uses Bergman-like weighted shifts (Theorem 3.14). Finally, in Section 5 we completely characterize the hyponormality and subnormality of symmetrically flat contractive 2-variable weighted shifts, which sheds new light on the relationship between flatness and subnormality.

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Recall that a bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$. For $k \geq 1$ and $T \in \mathcal{B}(\mathcal{H})$, T is *k-hyponormal* if $(I, T, \cdots, T^k)$ is (jointly) hyponormal. Additionally, T is *weakly k-hyponormal* if $p(T)$ is hyponormal for every polynomial p of degree at most k. Thus, if T is k-hyponormal then T is weakly k-hyponormal, and "hyponormality" "1-hyponormality" and "weak 1-hyponormality" are all identical notions ([Ath]). On the other hand, results in ([CMX]), ([Cu2]) and ([McCP]) show that if T is weakly 2-hyponormal (also called *quadratically hyponormal*), then T need not be 2-hyponormal. The Bram-Halmos characterization of subnormality ([Con, III.1.9]) can be paraphrased as follow: T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([CMX, Proposition 1.9]). In particular, each subnormal operator is *polynomially hyponormal* (i.e., weakly k-hyponormal for every $k \geq 1$). The converse implication, whether T polynomially hyponormal $\Rightarrow$ T subnormal, was settled in the negative in ([CuPu]); indeed, it was shown that there exists a polynomially hyponormal operator which is not 2-hyponormal. Previously, S. McCullough and V. Paulsen had established ([McCP]) that one can find a non-subnormal polynomially hyponormal operator if and only if one can find a unilateral weighted shift with the same property. Thus, although the existence proof in ([CuPu]) is abstract, by combining the results in ([CuPu]) and ([McCP]) we now know that there exists a polynomially hyponormal unilateral weighted shift which is not subnormal.

For $S, T \in \mathcal{B}(\mathcal{H})$ we let $[S, T] := ST - TS$. We say that a commuting n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [Ath], [CMX]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.

For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$ and W_α the associated unilateral weighted shift, the *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot ... \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{cases}.$$

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. If $\alpha_{k+1} = \alpha_k$ for all $k \geq 1$, W_α is called *flat*. On occasion, we will write $shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$ to denote the weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$. We also denote by $U_+ := shift(1, 1, 1, \cdots)$ the (unweighted) unilateral shift, and for $0 < a < 1$ we let $S_a := shift(a, 1, 1, \cdots)$; the shift S_a is the prototypical flat weighted shift, and it is subnormal.

Similarly, consider double-indexed positive bounded sequences $\{\alpha_{\mathbf{k}}\}, \{\beta_{\mathbf{k}}\} \in \ell^{\infty}(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Recall that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define the 2-*variable weighted shift* $\mathbf{T}$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \text{ (all } \mathbf{k}). \tag{1.1}$$

In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+^2$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, $T_1 \cong W_\alpha \bigotimes I$ and $T_2 \cong I \bigotimes W_\beta$, and $\mathbf{T}$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [Con, III.8.16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$ (called the *Berger measure* of W_α) such that $\gamma_k(\alpha) := \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 = \int t^k d\xi(t)$ $(k \geq 1)$. For instance, the Berger measures of U_+ and S_a are δ_1 and $(1-a^2)\delta_0 + a^2\delta_1$, respectively, where δ_x denotes the point-mass probability measure with support the singleton $\{x\}$.

If W_α is subnormal, and if for $h \geq 1$ we let $\mathcal{M}_h := \bigvee\{e_k : k \geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$. For $h = 2$, one can use this to prove the following result.

Lemma 1.1. *Let $T \equiv shift(\beta_0, \beta_1, \cdots)$ be a subnormal weighted shift, with Berger measure η, and let $T_{\mathcal{M}}$ be its restriction to $\mathcal{M} := \vee\{e_2, e_3, \cdots\}$. Then $\beta_1^2 = (\|\frac{1}{t}\|_{L^1(\eta_{\mathcal{M}})})^{-1}$.*

Proof. We have

$$\left\|\frac{1}{t}\right\|_{L^1(\eta_{\mathcal{M}})} = \int \frac{1}{t}(\frac{1}{\gamma_2} t^2 d\eta(t)) = \frac{1}{\gamma_2}\int t d\eta(t) = \frac{\gamma_1}{\gamma_2} = \frac{1}{\beta_1^2},$$

as desired. $\square$

Corollary 1.2. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting 2-variable weighted shift, assume that T_2 is subnormal, and assume that there exists $k_1 \geq 0$ such that $\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2)+\varepsilon_2}$ for all $k_2 \geq 2$. Then $\beta_{(k_1,1)} = \beta_{(k_1,1)+\varepsilon_1}$.*

Proof. Consider $S := shift(\beta_{(k_1,2)}, \beta_{(k_1,3)}, \cdots)$ and $S' := shift(\beta_{(k_{1+1},2)}, \beta_{(k_1+1,3)}, \cdots)$. Since T_2 is subnormal, we know that both S and S' are subnormal, with Berger measures η and η', respectively. Since $\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2)+\varepsilon_2}$, the commuting property (1.1) readily implies that $\beta_{(k_1,k_2)} = \beta_{(k_1,k_2)+\varepsilon_1}$ for all $k_2 \geq 2$, that is $S = S'$, that is, $\eta = \eta'$. By Lemma 1.1, we must have

$$\beta_{(k_1,1)}^2 = (\left\|\frac{1}{t}\right\|_{L^1(\eta)})^{-1} = (\left\|\frac{1}{t}\right\|_{L^1(\eta')})^{-1} = \beta_{(k_1,1)+\varepsilon_1}^2,$$

as desired. $\square$

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2. Propagation Phenomena for 1-Variable Weighted Shifts

In this section, we review some basic propagation phenomena for 1-variable weighted shifts, and we then develop the results for the 2-variable case in Sections 3 and 4. J. Stampfli showed in [Sta] that for a subnormal weighted shift W_α, a propagation phenomenon occurs which forces the flatness of W_α whenever two equal weights are present.

Proposition 2.1. *(Subnormality, One-variable Case) ([Sta]) Let W_α be a subnormal weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then W_α is flat.*

The first named author showed that in the presence of 2-hyponormality (resp. quadratic hyponormality) of weighted shifts, a propagation phenomenon also occurs which forces the flatness of W_α whenever two equal weights (resp. three equal weights) are present.

Proposition 2.2. *(2-hyponormality, One-variable Case) ([Cu2]) Let W_α be a 2-hyponormal weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then W_α is flat.*

Proposition 2.3. *(Quadratic Hyponormality, One-variable Case) ([Cu2]) Let W_α be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$, and assume that W_α is quadratically hyponormal. If $\alpha_k = \alpha_{k+1} = \alpha_{k+2}$ for some $k \geq 0$, then W_α is flat.*

Y. Choi later improved Proposition 2.3, as follows.

Proposition 2.4. *(Quadratic Hyponormality, One-variable Case, Improved Version) ([Choi]) Let W_α be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$, and assume that W_α is quadratically hyponormal. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 1$, then W_α is flat.*

Moreover, Y. Choi showed that, in the presence of polynomially hyponormality, two consecutive equal weights again force flatness.

Proposition 2.5. *(Polynomially hyponormality) ([Choi]) Let W_α be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^{\infty}$, and assume that W_α is polynomially hyponormal. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then W_α is flat.*

3. Propagation in the 2-variable Hyponormal Case

In this section, we show that if a commuting, (jointly) hyponormal pair $\mathbf{T} \equiv (T_1, T_2)$ with T_1 quadratically hyponormal satisfies $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1, T_2(I \otimes U_+^{k_2-1}))$ is horizontally flat (see Definition 3.1 below); this is the content of Theorem 3.3. We also prove that Theorem 3.3 is optimal in the following sense: the propagation does not extend either to the left (0-th column) or down (below k_2-th level).

We begin with:

Definition 3.1. *A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ is horizontally flat (resp. vertically flat) if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$). We say that $\mathbf{T}$ is flat if $\mathbf{T}$ is horizontally and vertically flat (cf. Figure 1), and we say that $\mathbf{T}$ is symmetrically flat if $\mathbf{T}$ is flat and $\alpha_{11} = \beta_{11}$.*

Lemma 3.2. *([Cu1])(Six-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences α and β. Then*

$$[\mathbf{T}^*, \mathbf{T}] \geq 0 \Leftrightarrow (([T_j^*, T_i]e_{\mathbf{k}+\varepsilon_j}, e_{\mathbf{k}+\varepsilon_i}))_{i,j=1}^2 \geq 0 \ (\textit{all } \mathbf{k} \in \mathbf{Z}_+^2)$$

$$\Leftrightarrow \begin{pmatrix} \alpha_{\mathbf{k}+\varepsilon_1}^2 - \alpha_{\mathbf{k}}^2 & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta_{\mathbf{k}+\varepsilon_2}^2 - \beta_{\mathbf{k}}^2 \end{pmatrix} \geq 0 \ (\textit{all } \mathbf{k} \in \mathbf{Z}_+^2).$$

Theorem 3.3. *Let $\mathbf{T} \equiv (T_1, T_2)$ be a commuting, hyponormal 2-variable weighted shift.*
(i) If T_1 is quadratically hyponormal and $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1, T_2(I \otimes U_+^{k_2-1}))$ is horizontally flat.
(ii) If, instead, T_2 is quadratically hyponormal and $\beta_{(k_1,k_2)+\varepsilon_2} = \beta_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1(U_+^{k_1-1} \otimes I), T_2)$ is vertically flat.

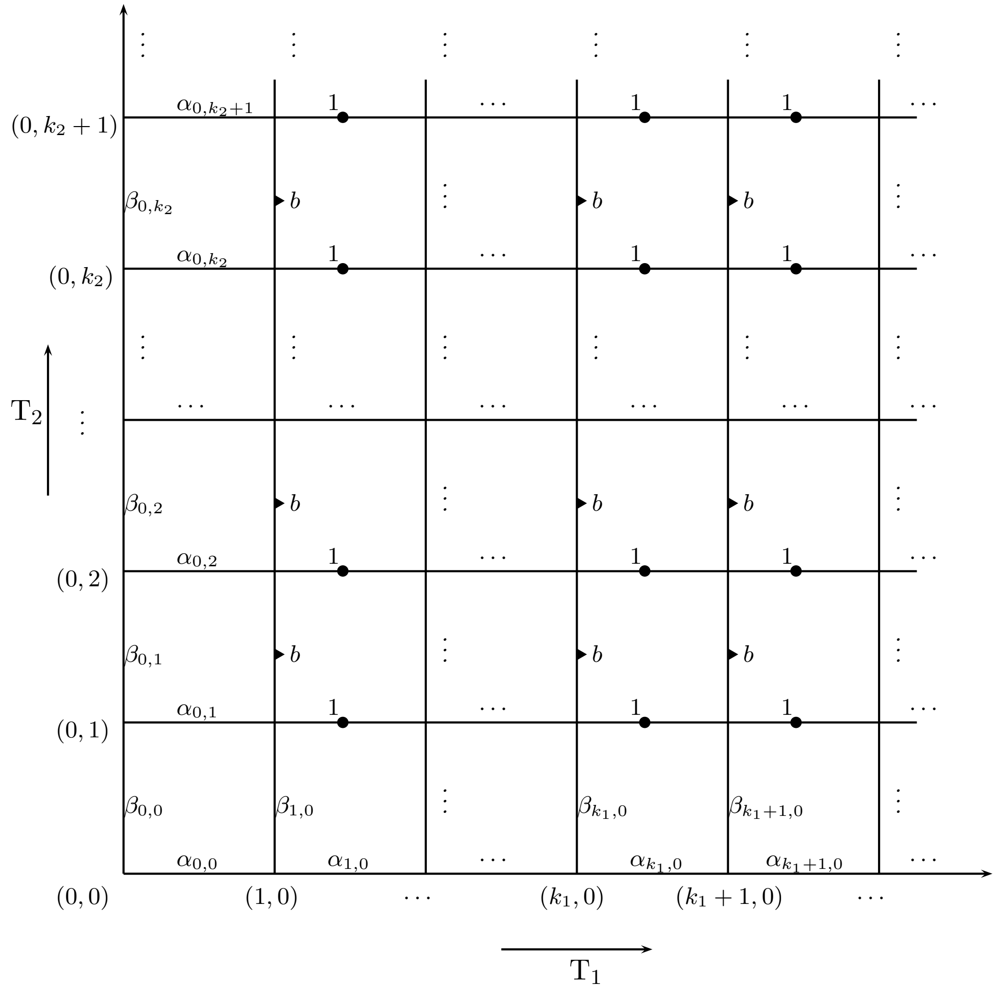

FIGURE 1. Weight diagram of a flat 2-variable weighted shift (with round dots for horizontal flatness, triangular dots for vertical flatness)

Proof. Without loss of generality, we only prove (i). Consider the restricted weight diagram based at (k_1, k_2) (see Figure 3).

Recall that, by joint hyponormality, we have

$$\begin{pmatrix} \alpha^2_{(k_1,k_2)+\varepsilon_1} - \alpha^2_{(k_1,k_2)} & \alpha_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)+\varepsilon_1} - \beta_{(k_1,k_2)}\alpha_{(k_1,k_2)} \\ \alpha_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)+\varepsilon_1} - \beta_{(k_1,k_2)}\alpha_{(k_1,k_2)} & \beta^2_{(k_1,k_2)+\varepsilon_2} - \beta^2_{(k_1,k_2)} \end{pmatrix} \geq 0.$$

Since $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$, it follows that

$$\alpha_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)+\varepsilon_1} = \beta_{(k_1,k_2)}\alpha_{(k_1,k_2)}. \tag{3.1}$$

By the commuting property (1.1),

$$\alpha_{(k_1,k_2)}\beta_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)}. \tag{3.2}$$

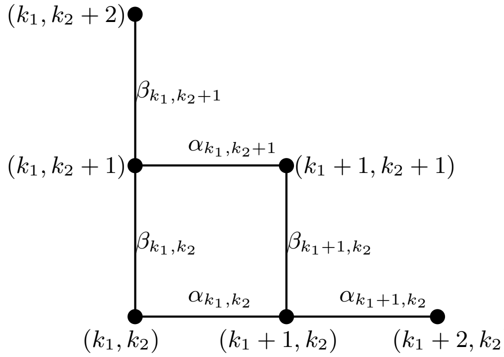

FIGURE 2. Weight diagram for the Six-point Test

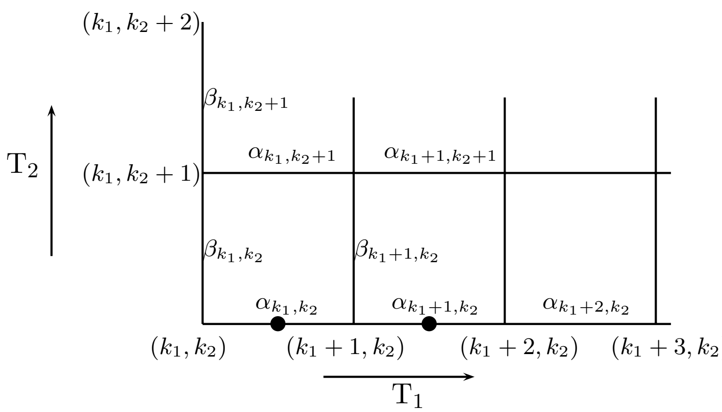

FIGURE 3. Weight diagram of the 2-variable weighted shift in Theorem 3.3 (the two solid black dots represent equal weights)

Therefore

$$\alpha^2_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)} = \alpha_{(k_1,k_2)+\varepsilon_2}(\alpha_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)}) = \alpha_{(k_1,k_2)+\varepsilon_2}(\alpha_{(k_1,k_2)}\beta_{(k_1,k_2)+\varepsilon_1}) \text{ (by (3.2))}$$

$$= \alpha_{(k_1,k_2)}(\alpha_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)+\varepsilon_1}) = \alpha_{(k_1,k_2)}(\beta_{(k_1,k_2)}\alpha_{(k_1,k_2)}) \text{ (by (3.1))}.$$

Thus, $\alpha^2_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)} = \alpha_{(k_1,k_2)}(\beta_{(k_1,k_2)}\alpha_{(k_1,k_2)})$, which implies that $\alpha_{(k_1,k_2)+\varepsilon_2} = \alpha_{(k_1,k_2)}$. We now recall Theorem 2.4, which says that flatness can be propagated to the right, that is, $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)+2\varepsilon_1}$. It follows that $\alpha_{(k_1,k_2)+\varepsilon_1+\varepsilon_2} = \alpha_{(k_1,k_2)+\varepsilon_2}$, and then two equal weights occurs at level k_2+1, which then implies $\alpha_{(k_1,k_2)+2\varepsilon_2} = \alpha_{(k_1,k_2)+\varepsilon_2} = \alpha_{(k_1,k_2)}$. It is now easy to see that for every level $\ell \geq k_2$ we must have $\alpha_{(k_1,\ell)} = \alpha_{(k_1,k_2)}$ (all $k_1 \geq 1$). Using Theorem 2.4 to propagate these equalities to the left, we eventually conclude that

$$\alpha_{(k_1,\ell)} = \alpha_{(1,k_2)} \ (k_1 \geq 1, \ell \geq k_2).$$

We thus obtain that $(T_1, T_2)|_{\bigvee\{e_{(k_1,\ell)}:\ k_1\geq 1,\ \ell\geq k_2\}}$ is unitarily equivalent to $(\alpha_{(1,k_2)}U_+ \otimes I, I \otimes W_\eta)$, where $\eta_k := \beta_{1,k+k_2} (k \geq 0)$. This can be rephrased as saying that $(T_1, T_2(I \otimes U_+^{k_2-1}))$ is horizontally flat, as desired. $\square$

Remark 3.4. The proof of Theorem 3.3 shows that for $\mathbf{T} \equiv (T_1, T_2)$ commuting and hyponormal, and for $k_1, k_2 \geq 0$,

$$\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} \Rightarrow \beta_{(k_1,k_2)} = \beta_{(k_1,k_2)+\varepsilon_1} \tag{3.3}$$

(by (3.1) and (3.2)). Moreover, if $k_2 \geq 1$,

$$\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} \text{ and } \alpha_{(k_1,k_2)+\varepsilon_1-\varepsilon_2} = \alpha_{(k_1,k_2)-\varepsilon_2} \Rightarrow \alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)+\varepsilon_1-\varepsilon_2}.$$

Remark 3.5. The proof of Theorem 3.3 also reveals that asking $\mathbf{T} \equiv (T_1, T_2)$ to be jointly hyponormal is significantly stronger than asking both T_1 and T_2 to be hyponormal. For, consider the 2-variable weighted shift whose weight diagram is given by Figure 4. In [CuYo1, Theorem 5.2], we

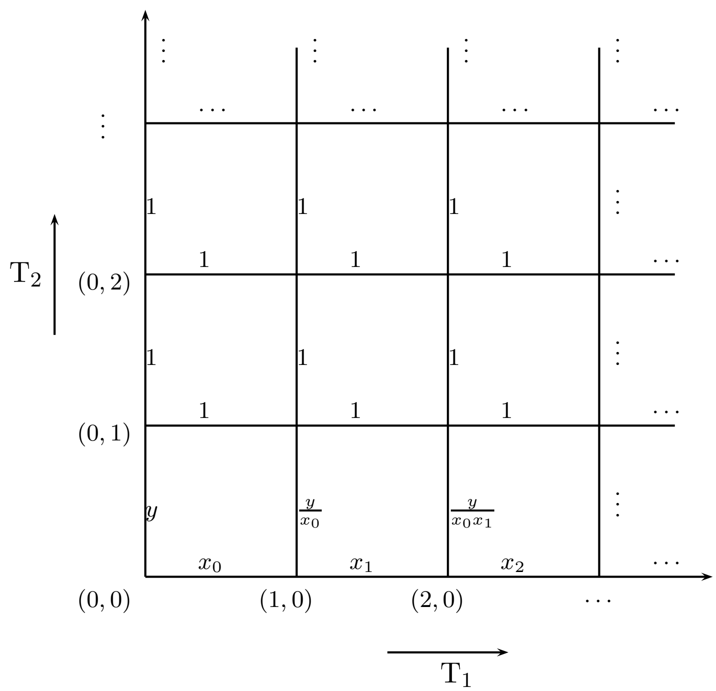

FIGURE 4. Weight diagram of the 2-variable weighted shift in Remark 3.5

established that in the case when $\|W_\alpha\| \leq 1$, $\mathbf{T}$ is subnormal if and only if $\mathbf{T}$ is hyponormal. Thus, a necessary condition for the hyponormality of $\mathbf{T}$ is the subnormality of $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$. For $0 < a < 1$, let $x_0 \equiv x_1 := a$ and let $x_k := 1$ $(k \geq 2)$. Clearly W_0 is hyponormal and not subnormal, and if we take $0 < y \leq a^2$ we can guarantee that each of T_1 and T_2 is hyponormal, yet $\mathbf{T}$ is not. An alternative way to see this is observe that if $\mathbf{T}$ were hyponormal then α_{01} would equal a, since $\alpha_{00} = \alpha_{10}$.

We will now show that Theorem 3.3 is optimal in the following sense: the propagation does not necessarily extend either to the left (0-th column) or down (below k_2-th level). To demonstrate this optimality, we first introduce the class of Bergman-like weighted shifts.

Definition 3.6. *For $\ell \geq 1$, the Bergman-like weighted shift on $\ell^2(\mathbb{Z}_+)$ is $B_+^{(\ell)} := shift(\{\sqrt{\ell - \frac{1}{k+2}} : k \geq 0\})$; that is,*

$$B_+^{(\ell)} e_k := \sqrt{\ell - \frac{1}{k+2}} e_{k+1} \ \ (k \geq 0).$$

In particular, $B_+^{(1)} \equiv B_+ := shift(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots)$ is the Bergman shift.

Remark 3.7. (i) B_+ is subnormal with Berger measure $d\xi(s) := ds$ on $[0,1]$.
(ii) ([CuYo3]) $B_+^{(2)}$ is subnormal with Berger measure $d\xi(s) := \frac{sds}{\pi\sqrt{2s-s^2}}$ on $[0,2]$.

Our next step is to show that $B_+^{(\ell)}$ ($\ell \geq 1$) is always 2-hyponormal. To this end, we need two preliminary results.

Lemma 3.8. *(Nested Determinants Test; Special Case) If $a > 0$ and $\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0$, then*

$$\begin{pmatrix} a & b & c \\ b & c & d \\ c & d & e \end{pmatrix} \geq 0 \Leftrightarrow \det \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & e \end{pmatrix} \geq 0.$$

Proof. Straightforward from Choleski's Algorithm ([Atk]). □

Lemma 3.9. *([Cu2])Let $W_\alpha e_k = \alpha_k e_{k+1}$ ($k \geq 0$) be a hyponormal weighted shift. The following statements are equivalent:*
(i) W_α is 2-hyponormal.
(ii) The matrix

$$(([W_\alpha^{*j}, W_\alpha^i] e_{k+j}, e_{k+i}))_{i,j=1}^2$$

is positive semi-definite for all $k \geq -1$.
(iii) The matrix

$$(\gamma_k \gamma_{k+i+j} - \gamma_{k+i}\gamma_{k+j})_{i,j=1}^2$$

is positive semi-definite for all $k \geq 0$, where as usual $\gamma_0 = 1$, $\gamma_n = \alpha_0^2 \cdot \cdots \cdot \alpha_{n-1}^2$ ($n \geq 1$).
(iv) The Hankel matrix

$$H(2;k) := (\gamma_{k+i+j-2})_{i,j=1}^3$$

is positive semi-definite for all $k \geq 0$.

We now use symbolic manipulation to prove the following result.

Theorem 3.10. *All Bergman-like weighted shifts $B_+^{(\ell)}$ (all $\ell \geq 1$) are 2-hyponormal.*

Proof. By Lemma 3.8 and Lemma 3.9, to prove that $B_+^{(\ell)}$ is 2-hyponormal it suffices to see that $\det H(2;k) > 0$ for all $k \geq 0$. Now,

$$\begin{aligned} \det H(2;k) &= \gamma_k^3 \det \begin{pmatrix} 1 & \alpha_k^2 & \alpha_k^2\alpha_{k+1}^2 \\ \alpha_k^2 & \alpha_k^2\alpha_{k+1}^2 & \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 \\ \alpha_k^2\alpha_{k+1}^2 & \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 & \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2\alpha_{k+3}^2 \end{pmatrix} \\ &= \gamma_k^3 \frac{2(\ell+1)((k+2)\ell-1)^2((k+3)\ell-1)}{(k+2)^3(k+3)^3(k+4)^2(k+5)} > 0, \end{aligned}$$

as desired. □

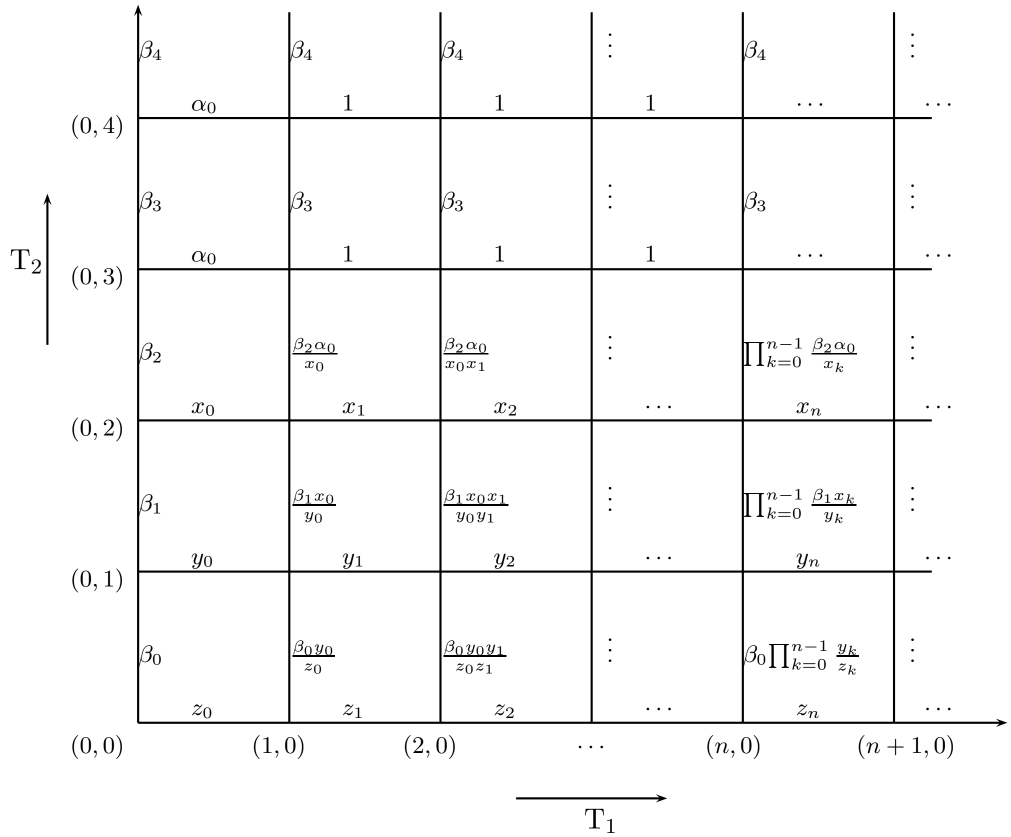

FIGURE 5. Weight diagram of the 2-variable weighted shift in Lemma 3.13

Corollary 3.11. *For every $\ell \geq 1$, the Bergman-like weighted shift $B_+^{(\ell)}$ is quadratically hyponormal.*

Remark 3.12. In ([CPY]), we prove a much stronger result: all Bergman-like weighted shifts $B_+^{(\ell)}$ (all $\ell \geq 1$) are subnormal.

Theorem 3.14 below says that the amount of propagation provided by Theorem 3.3 is maximum; briefly, we say that Theorem 3.3 is optimal. Observe that for the 2-variable weighted shift in Figure 6, we have $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ (all $k_1 \geq 1, k_2 \geq 2$), yet $\alpha_{(k_1,k_2)} < \alpha_{(k_1,k_2)+\varepsilon_1}$ for all $k_1 \geq 0$ and $k_2 = 0, 1$ and $\alpha_{(0,k_2)} < \alpha_{(1,k_2)}$ for all $k_2 \geq 0$. In other words, the trivial weight structure present in the subspace $\bigvee\{e_{(k_1,k_2)} : k_1 \geq 1, k_2 \geq 2\}$ cannot be expanded either to the left (0th column) or down (first row). First, we need an auxiliary result, of independent interest.

Lemma 3.13. *Consider the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ given by Figure 5, where $shift(x_0, x_1, x_2, \cdots)$ and $shift(y_0, y_1, y_2, \cdots)$ are Bergman-like weighted shifts. Assume that $(T_1, T_2)|_{\mathcal{M}}$ is jointly hyponormal, where $\mathcal{M}$ is the subspace associated to indices $\mathbf{k}$ with $k_2 \geq 1$. Then there exists a Bergman-like weighted shift $shift(z_0, z_1, z_2, \cdots)$ and a hyponormal weighted shift $W_\beta := shift(\beta_0, \beta_1, \beta_2, \cdots)$ ($\beta_n < \beta_{n+1}$ for all $n \geq 0$) such that $\mathbf{T}$ is jointly hyponormal.*

Proof. Let

$$\begin{aligned}
&shift(x_0, x_1, x_2, \cdots) \equiv shift(\{\sqrt{p - \tfrac{1}{n+2}} : n \geq 0\}),\\
&shift(y_0, y_1, y_2, \cdots) \equiv shift(\{\sqrt{q - \tfrac{1}{n+2}} : n \geq 0\}) \text{ and}\\
&shift(z_0, z_1, z_2, \cdots) \equiv shift(\{\sqrt{r - \tfrac{1}{n+2}} : n \geq 0\}), \text{ for some integers } p < q < r.
\end{aligned}$$

Since the restriction of (T_1, T_2) to $\vee\{e_{(k_1,k_2)} : k_2 \geq 1\}$ is jointly hyponormal, it suffices to apply the Six-point Test (Lemma 3.2) to $\mathbf{k} = (n, 0)$, with $n \geq 0$.

Case 1: $\mathbf{k} = (0, 0)$. Here

$$M(0,0) := \begin{pmatrix} z_1^2 - z_0^2 & \frac{y_0^2\beta_0}{z_0} - \beta_0 \cdot z_0 \\ \frac{y_0^2\beta_0}{z_0} - \beta_0 \cdot z_0 & \beta_1^2 - \beta_0^2 \end{pmatrix} \geq 0$$

$$\Leftrightarrow z_0^2(z_1^2 - z_0^2)(\beta_1^2 - \beta_0^2) \geq \beta_0^2(z_0^2 - y_0^2)^2$$

$$\Leftrightarrow (r - \tfrac{1}{2})(\beta_1^2 - \beta_0^2) \geq 6\beta_0^2(r - q)^2.$$

If we choose β_0 such that $\beta_0 \leq \beta_1$ and $\beta_1^2 \geq \frac{12(r-q)^2+2r-1}{2r-1}\beta_0^2$, we obtain $M(0,0) \geq 0$.

Case 2: $\mathbf{k} = (n, 0)$ $(n \geq 1)$. Here

$$M(n,0) := \begin{pmatrix} z_{n+1}^2 - z_n^2 & y_n\beta_0 \prod_{k=0}^{n} \frac{y_k}{z_k} - z_n\beta_0 \prod_{k=0}^{n-1} \frac{y_k}{z_k} \\ y_n\beta_0 \prod_{k=0}^{n} \frac{y_k}{z_k} - z_n\beta_0 \prod_{k=0}^{n-1} \frac{y_k}{z_k} & \beta_1^2 \prod_{k=0}^{n-1}(\frac{x_k}{y_k})^2 - \beta_0^2 \prod_{k=0}^{n-1}(\frac{y_k}{z_k})^2 \end{pmatrix} \geq 0$$

$$\Leftrightarrow (z_{n+1}^2 - z_n^2)(\beta_1^2 \prod_{k=0}^{n-1}(\tfrac{x_k}{y_k})^2 - \beta_0^2 \prod_{k=0}^{n-1}(\tfrac{y_k}{z_k})^2) \geq \beta_0^2(y_n \prod_{k=0}^{n} \tfrac{y_k}{z_k} - z_n \prod_{k=0}^{n-1} \tfrac{y_k}{z_k})^2$$

$$\Leftrightarrow z_n^2(z_{n+1}^2 - z_n^2)(\beta_1^2 \prod_{k=0}^{n-1}(\tfrac{x_k}{y_k})^2 - \beta_0^2 \prod_{k=0}^{n-1}(\tfrac{y_k}{z_k})^2) \geq \beta_0^2 \prod_{k=0}^{n-1}(\tfrac{y_k}{z_k})^2(y_n^2 - z_n^2)^2$$

$$\Leftrightarrow z_n^2(z_{n+1}^2 - z_n^2)(\beta_1^2 \prod_{k=0}^{n-1}(\tfrac{x_k z_k}{y_k^2})^2 - \beta_0^2) \geq \beta_0^2(y_n^2 - z_n^2)^2$$

If we choose p, q and r such that $\frac{x_k z_k}{y_k^2} \geq 3$, then $\frac{r(n+2)-1}{(n+2)^2(n+3)}((\frac{12(r-q)^2+2r-1}{2r-1})\prod_{k=0}^{n-1}(\frac{x_k z_k}{y_k^2})^2 - 1) \geq (r-q)^2$ (all $n \geq 1$), which implies $M(n,0) \geq 0$ (all $n \geq 1$).

By Cases 1 and 2, it follows that (T_1, T_2) is jointly hyponormal. □

Theorem 3.14. *For every $k_2 \geq 1$ and $0 < \alpha_0 < 1$ there exist*
(i) a family $\{B_+^{(\ell_i)}\}_{i=0}^{k_2-1}$ of Bergman-like weighted shifts, and
(ii) a subnormal weighted shift $W_\beta := shift(\beta_0, \beta_1, \beta_2, \cdots)$ (with $\beta_n < \beta_{n+1}$ for all $n \geq 0$),
such that the commuting 2-variable weighted shift $\mathbf{T}$ with a weight diagram whose first k_2 rows are $B_+^{(\ell_0)}, \cdots, B_+^{(\ell_{k_2-1})}$, whose remaining rows are S_{α_0}, and whose 0-th column is given by W_β, is (jointly) hyponormal (see Figure 6 for the case $k_2 = 2$).

Proof. We divide the proof into three cases, according to the value of k_2.

Case 1: $k_2 = 1$. For $p \geq 1$ let $\alpha_{m,0} \equiv x_m := \sqrt{p - \frac{1}{m+2}}$ $(m \geq 0)$. Since the restriction of (T_1, T_2) to $\bigvee\{e_{(k_1,k_2)} : k_2 \geq 1\}$ must be unitarily equivalent to $(S_{\alpha_0} \otimes I, I \otimes shift(\beta_1, \beta_2, \cdots))$, to guarantee the hyponormality of (T_1, T_2) it suffices to apply the Six-point Test (Lemma 3.2) to $\mathbf{k} = (m, 0)$, with $m \geq 0$. □

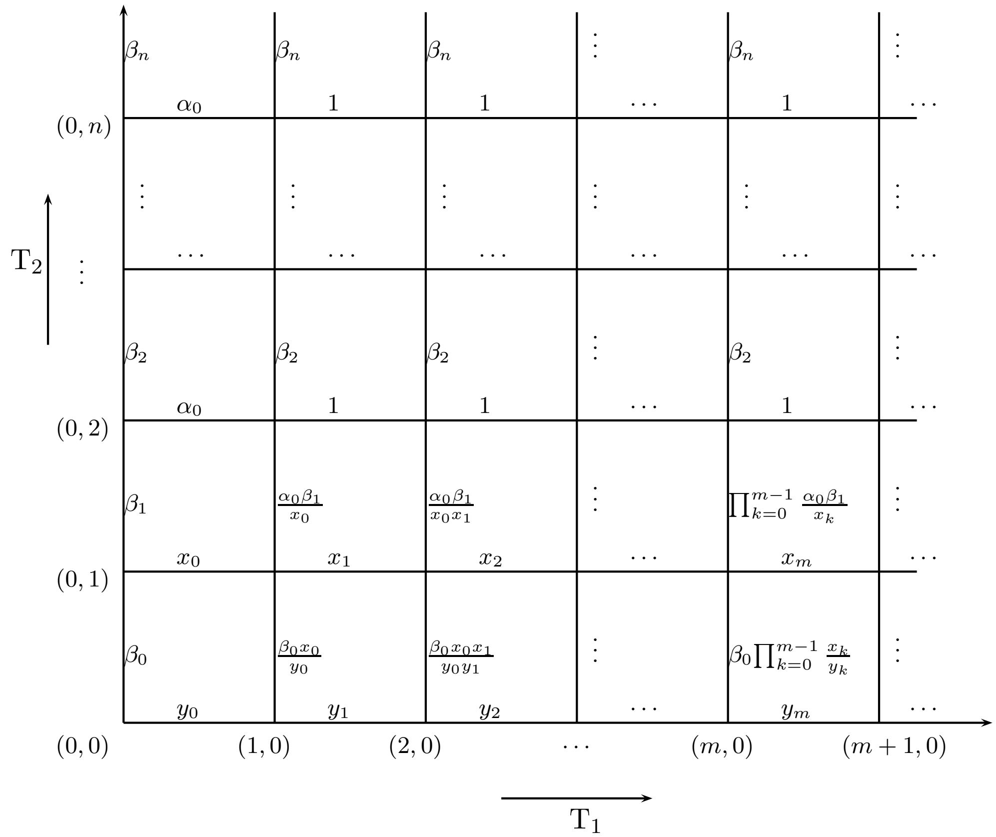

FIGURE 6. Weight diagram of the 2-variable weighted shift in Theorem 3.14

Subcase 1: $\mathbf{k} = (0,0)$. Here we have

$$M(0,0) \quad : \quad = \begin{pmatrix} x_1^2 - x_0^2 & \frac{\alpha_0^2 \beta_0}{x_0} - \beta_0 x_0 \\ \frac{\alpha_0^2 \beta_0}{x_0} - \beta_0 x_0 & \beta_1^2 - \beta_0^2 \end{pmatrix}$$

$$= \quad \begin{pmatrix} \frac{1}{6} & \frac{\beta_0}{x_0}(\alpha_0^2 - x_0^2) \\ \frac{\beta_0}{x_0}(\alpha_0^2 - x_0^2) & \beta_1^2 - \beta_0^2 \end{pmatrix} \geq 0$$

$$\Leftrightarrow \quad 6\beta_0^2(\alpha_0^2 - x_0^2)^2 \leq (\beta_1^2 - \beta_0^2) x_0^2, \tag{3.4}$$

which imposes a condition on x_0 and β_0.

Subcase 2: $\mathbf{k} = (m,0)$, with $m \geq 1$. Fix $m \geq 1$ and let $P_m := \prod_{k=0}^{m-1} x_k$. We then see that

$$M(m,0) \quad : \quad = \begin{pmatrix} x_{m+1}^2 - x_m^2 & \frac{\alpha_0 \beta_0}{x_m P_m} - x_m \frac{\alpha_0 \beta_0}{P_m} \\ \frac{\alpha_0 \beta_0}{x_m P_m} - x_m \frac{\alpha_0 \beta_0}{P_m} & \beta_1^2 - \frac{\alpha_0^2 \beta_0^2}{P_m^2} \end{pmatrix} \geq 0$$

$$\Leftrightarrow \quad x_m^2 (x_{m+1}^2 - x_m^2)(\beta_1^2 P_m^2 - \alpha_0^2 \beta_0^2) \geq \alpha_0^2 \beta_0^2 (1 - x_m^2)^2$$

$$\Leftrightarrow \quad x_m^2(\beta_1^2 P_m^2 - \alpha_0^2\beta_0^2) \geq (m+2)(m+3)\alpha_0^2\beta_0^2(1-x_m^2)^2$$

$$\Leftrightarrow \quad \frac{\beta_1^2 P_m^2}{\alpha_0^2\beta_0^2} \geq 1 + (m+2)(m+3)(\frac{1}{x_m} - x_m)^2. \tag{3.5}$$

We now let $p = 3$, so that $x_k^2 \geq 2$ (all $k \geq 0$) and therefore $P_m \geq 2^m$ (all $m \geq 1$). Since $\lim_{m\to\infty} \frac{2^m}{(m+2)(m+3)} = \infty$, it is clear that we can find β_0 sufficiently small so that both (3.4) and (3.5) hold.

Proof. From Subcases 1 and 2, it follows that $\mathbf{T}$ is jointly hyponormal.

Case 2: $k_2 = 2$. Here we let $p := 3$, $q := 18$, $\beta_2 := 4\beta_1$ and $\beta_1 := \frac{1}{\alpha_0}$, so that $\alpha_{m,1} \equiv x_m := \sqrt{p - \frac{1}{m+2}} \equiv \sqrt{3 - \frac{1}{m+2}}$ and $\alpha_{m,0} \equiv y_m := \sqrt{q - \frac{1}{m+2}} \equiv \sqrt{18 - \frac{1}{m+2}}$ $(m \geq 0)$. Since the restriction of (T_1, T_2) to $\bigvee\{e_{(k_1,k_2)} : k_2 \geq 2\}$ must be unitarily equivalent to $(S_{\alpha_0} \otimes I, I \otimes shift(\beta_2, \beta_3, \cdots))$, to guarantee the hyponormality of (T_1, T_2) it suffices to apply the Six-point Test (Lemma 3.2) to $\mathbf{k} = (m, n)$, with $m \geq 0$ and $0 \leq n \leq 1$.

Subcase 1: $\mathbf{k} = (0, 0)$. Here

$$M(0,0) := \begin{pmatrix} y_1^2 - y_0^2 & \frac{x_0^2\beta_0}{y_0} - \beta_0 y_0 \\ \frac{x_0^2\beta_0}{y_0} - \beta_0 y_0 & x_1^2 - x_0^2 \end{pmatrix} \geq 0$$

$$\Leftrightarrow y_0^2(y_1^2 - y_0^2)(x_1^2 - x_0^2) \geq \beta_0^2(x_0^2 - y_0^2)^2$$

$$\Leftrightarrow 225\beta_0^2 \leq \tfrac{35}{72},$$

so $M(0,0) \geq 0$ if and only if

$$\beta_0^2 \leq \frac{7}{3240} \cong 0.00216. \tag{3.6}$$

Subcase 2: $\mathbf{k} = (m, 0)$ (all $m \geq 1$). Fix $m \geq 1$ and let $P_m := \prod_{k=0}^{m-1} x_k$ and $Q_m := \prod_{k=0}^{m-1} y_k$. We have

$$M(m,0) \quad : \quad = \begin{pmatrix} y_{m+1}^2 - y_m^2 & x_m\beta_0\frac{x_m P_m}{y_m Q_m} - y_m\beta_0\frac{P_m}{Q_m} \\ x_m\beta_0\frac{x_m P_m}{y_m Q_m} - y_m\beta_0\frac{P_m}{Q_m} & \frac{\beta_1^2\alpha_0^2}{P_m^2} - \beta_0^2\frac{P_m^2}{Q_m^2} \end{pmatrix} \geq 0$$

$$\Leftrightarrow \quad y_m^2(y_{m+1}^2 - y_m^2)(\frac{\beta_1^2\alpha_0^2}{P_m^2} - \beta_0^2\frac{P_m^2}{Q_m^2}) \geq \beta_0^2(y_m^2 - x_m^2)^2\frac{P_m^2}{Q_m^2}$$

$$\Leftrightarrow \quad \frac{Q_m^2}{P_m^4} - \beta_0^2 \geq \frac{225(m+2)^2(m+3)}{18m+35}\beta_0^2$$

$$\Leftrightarrow \quad \frac{Q_m^2}{P_m^4} \geq (\frac{225(m+2)^2(m+3)}{18m+35} + 1)\beta_0^2.$$

It follows that $M(m,0) \geq 0$ (all $m \geq 1$) if and only if

$$\begin{aligned} \beta_0^2 \quad &\leq \quad f(m) := \frac{Q_m^2}{P_m^4(\frac{225(m+2)^2(m+3)}{18m+35} + 1)} \\ &= \quad \frac{18m+35}{225m^3 + 1575m^2 + 3618m + 2735}\prod_{k=0}^{m-1}\frac{18k^2 + 71k + 70}{9k^2 + 30k + 25} \ \ (\text{all } m \geq 1). \end{aligned}$$

Since f is an increasing function of m, we see that $M(m,0) \geq 0$ (all $m \geq 1$) if and only if

$$\beta_0^2 \leq f(1) = \frac{742}{40765} \cong 0.018. \tag{3.7}$$

Subcase 3: $\mathbf{k} = (0,1)$. Here we have

$$\begin{aligned} M(0,1) \quad : \quad &= \begin{pmatrix} x_1^2 - x_0^2 & \frac{\alpha_0^2\beta_1}{x_0} - \beta_1 x_0 \\ \frac{\alpha_0^2\beta_1}{x_0} - \beta_1 x_0 & \beta_2^2 - \beta_1^2 \end{pmatrix} \geq 0 \\ &= \begin{pmatrix} \frac{1}{6} & \frac{\beta_1}{x_0}(\alpha_0^2 - x_0^2) \\ \frac{\beta_1}{x_0}(\alpha_0^2 - x_0^2) & 15\beta_1^2 \end{pmatrix} \geq 0 \\ &\Leftrightarrow \quad (\alpha_0^2 - \frac{5}{2})^2 \leq \frac{25}{4}, \end{aligned} \tag{3.8}$$

which certainly holds, since $0 < \alpha_0 < 1$.

Subcase 4: $\mathbf{k} = (m,1)$, with $m \geq 1$. As in Subcase 2, fix $m \geq 1$ and let $P_m := \prod_{k=0}^{m-1} x_k$. We then see that

$$\begin{aligned} M(m,1) &:= \begin{pmatrix} x_{m+1}^2 - x_m^2 & \frac{\alpha_0\beta_1}{x_m P_m} - x_m \frac{\alpha_0\beta_1}{P_m} \\ \frac{\alpha_0\beta_1}{x_m P_m} - x_m \frac{\alpha_0\beta_1}{P_m} & \beta_2^2 - \frac{\alpha_0^2\beta_1^2}{P_m^2} \end{pmatrix} \geq 0 \\ &\Leftrightarrow x_m^2(x_{m+1}^2 - x_m^2)(\beta_2^2 P_m^2 - \alpha_0^2\beta_1^2) \geq \alpha_0^2\beta_1^2(1 - x_m^2)^2 \\ &\Leftrightarrow x_m^2(\beta_2^2 P_m^2 - 1) \geq (m+2)(m+3)(1 - x_m^2)^2 \\ &\Leftrightarrow \beta_2^2 \geq g(m) := \tfrac{1}{P_m^2}(1 + \tfrac{(m+3)(2m+3)^2}{3m+5}). \end{aligned} \tag{3.9}$$

It follows that $M(m,1) \geq 0$ (all $m \geq 1$) if and only if β_2 can be chosen to satisfy (3.9) for all $m \geq 1$. Since g is a decreasing function of m, it suffices to guarantee that $\beta_2^2 \geq g(1) = \frac{27}{5}$. If we now recall that $\beta_2 = 4\beta_1$ and that $\beta_1 = \frac{1}{\alpha_0}$, this condition is equivalent to $\alpha_0^2 \leq \frac{80}{27}$, which always holds, since $\alpha_0 < 1$.

Therefore, by Subcases 1, 2, 3 and 4, which yield the condition (3.6), we see that (T_1, T_2) is hyponormal if and only if $\beta_0^2 \leq \frac{7}{3240}$. Finally, and since we clearly have $\beta_0 < \beta_1 < \beta_2$, we can use the construction in [Sta] to define W_β, which incidentally has a 2-atomic Berger measure (cf. [CuFi]).

Case 3: $k_2 \geq 3$. Here we take p and q as in Case 2, to ensure that the restriction of $\mathbf{T}$ to the subspace associated with subindices (m,n) with $n \geq k_2 - 2$ is hyponormal. Once this is done, we use Lemma 3.13 to obtain r, so that the restriction of $\mathbf{T}$ to the subspace associated with subindices (m,n) with $n \geq k_2 - 3$ is hyponormal. Repeated application of Lemma 3.13 now completes the proof. □

Corollary 3.15. *Theorem 3.3 is optimal.*

4. Propagation in the Subnormal Case

In this section, we show that Theorem 3.3 can be improved if one of the T_i's is quadratically hyponormal and the other is subnormal. In Theorem 4.7 and Theorem 4.12, we consider horizontal flatness and optimality, and in Theorem 4.14, we show that a subnormal 2-variable weighted shift with two horizontally consecutive equal weights and two vertically consecutive equal weights must necessarily be flat. As in the previous section, we then establish that our result is optimal (see Example 5.13 below). We begin with some definitions and preliminary results.

Definition 4.1. *([CuYo1]) Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ on $X := \mathbb{R}_+$, if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on $\mathbb{R}_+$.*

Definition 4.2. *([CuYo1]) Let μ be a probability measure on $X \times Y$, and assume that $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (which is also a probability measure) on $X \times Y$ is given by*

$$d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\mu)}} d\mu(s,t).$$

Definition 4.3. *([CuYo1])Given a measure μ on $X \times Y$, the marginal measure μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. Observe that if μ is a probability measure, then so is μ^X.*

Lemma 4.4. *([CuYo1])(Subnormal backward extension of a 1-variable weighted shift) (cf [Cu2]) Let $T \equiv shift(\beta_0, \beta_1, \cdots)$ be a unilateral weighted shift whose restriction $T_{\mathcal{M}}$ to $\mathcal{M} := \vee\{e_1, e_2, \cdots\}$ is subnormal, with Berger measure $\eta_{\mathcal{M}}$. Then T is subnormal (with measure η) if and only if*
(i) $\frac{1}{t} \in L^1(\eta_{\mathcal{M}})$;
(ii) $\beta_0^2 \leq (\left\| \frac{1}{t} \right\|_{L^1(\eta_{\mathcal{M}})})^{-1}$.
In this case, $d\eta(t) = \frac{\beta_0^2}{t} d\eta_{\mathcal{M}}(t) + (1 - \beta_0^2 \left\| \frac{1}{t} \right\|_{L^1(\eta_{\mathcal{M}})}) d\delta_0(t)$, where δ_0 denotes the Dirac measure at 0. In particular, T is never subnormal when $\eta_{\mathcal{M}}(\{0\}) > 0$.

Lemma 4.5. *([CuYo1])(Subnormal backward extension of a 2-variable weighted shift) Consider the following 2-variable weighted shift (see Figure 7), and let $\mathcal{M}$ be the subspace associated to indices $\mathbf{k}$ with $k_2 \geq 1$. Assume that $\mathbf{T}|_{\mathcal{M}}$ is subnormal with measure $\mu_{\mathcal{M}}$ and that $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ is subnormal with measure ξ. Then $\mathbf{T}$ is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq (\left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \xi$.
Moreover, if $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \xi$. In the case when $\mathbf{T}$ is subnormal, the Berger measure μ of $\mathbf{T}$ is given by

$$d\mu(s,t) = \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + (d\xi(s) - \beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)) d\delta_0(t).$$

Lemma 4.6. *Let $\mathbf{T} \equiv (T_1, T_2)$, let $\mathcal{M}$ be as in Lemma 4.5, and assume that $\mathbf{T}|_{\mathcal{M}}$ is subnormal with Berger measure $\mu_{\mathcal{M}} \equiv \delta_1 \times \eta$. Assume further that $\beta_{00}^2 = (\left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$ and that $W_{\alpha^{(0)}} := shift(\alpha_{00}, \alpha_{10}, \alpha_{20}, \cdots)$ is subnormal. Then $\mathbf{T}$ is subnormal if and only if $\alpha_{i0} = 1$ (all $i \geq 0$), that is, $W_{\alpha^{(0)}}$ must necessarily be the (unweighted) unilateral shift U_+.*

Proof. Assume first that $\mathbf{T}$ is subnormal. Since $d\mu_{\mathcal{M}}(s,t) \equiv \delta_1(s) d\eta(t)$, we must have

$$\begin{aligned} d(\mu_{\mathcal{M}})_{ext}^X &= ((1 - \delta_0(t)) \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}} d\mu_{\mathcal{M}}(s,t))^X \\ &= d\delta_1(s) = d\xi_{\alpha^{(0)}}(s) \text{ (by Lemma 4.5)}, \end{aligned}$$

where $\xi_{\alpha^{(0)}}$ denotes the Berger measure of $W_{\alpha^{(0)}}$. It follows that $W_{\alpha^{(0)}} = U_+$.

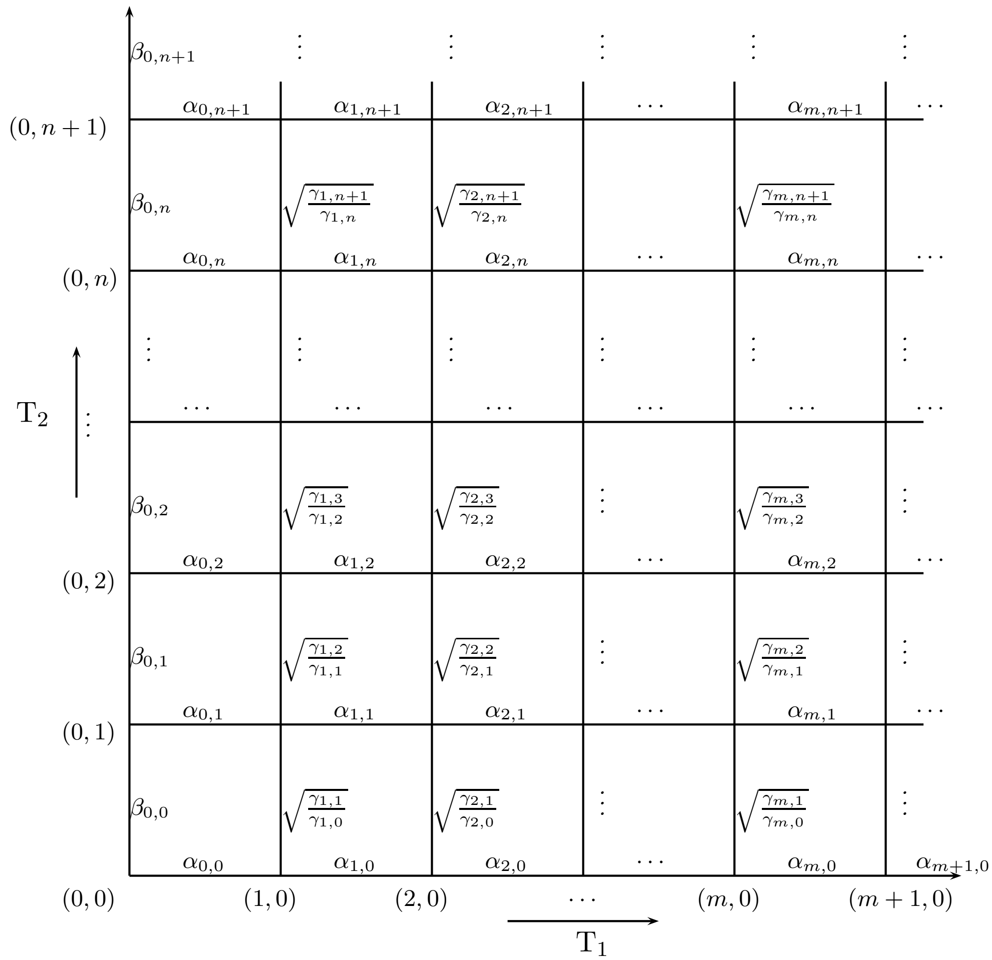

FIGURE 7. Weight diagram of the 2-variable weighted shift in Lemma 4.5

Conversely, assume that $W_{\alpha^{(0)}} = U_+$. By Lemma 4.4, $shift(\beta_{00}, \beta_{01}, \cdots)$ is subnormal, and we let $\tilde{\eta}$ denote its Berger measure. If we now let $\mu := \delta_1 \times \tilde{\eta}$, it easily follows that $\mathbf{T}$ is subnormal with Berger measure μ. $\square$

Theorem 4.7. *Let $\mathbf{T} \equiv (T_1, T_2)$ be commuting and hyponormal.*
(i) If T_1 is quadratically hyponormal, if T_2 is subnormal, and if $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 0$, then $\mathbf{T}$ is horizontally flat.
(ii) If, instead, T_1 is subnormal, T_2 is quadratically hyponormal, and if $\beta_{(k_1,k_2)+\varepsilon_2} = \beta_{(k_1,k_2)}$ for some $k_1, k_2 \geq 0$, then $\mathbf{T}$ is vertically flat.

Proof. Without loss of generality, we only consider the horizontally flat case, and we further assume $k_2 = 2$, that is, $\alpha_{k_1,2} = \alpha_{k_1+1,2}$ for some $k_1 \geq 0$. By Theorem 3.3 and Proposition 2.4, two equal weights occur at level 3, i.e., $\alpha_{k_1,3} = \alpha_{k_1+1,3}$. Moreover, for every $\ell \geq 2$ we have $\alpha_{k_1,2} = \alpha_{k_1,\ell}$ (all $k_1 \geq 1$). We now apply Corollary 1.2 to obtain $\beta_{(k_1,1)} = \beta_{(k_1,1)+\varepsilon_1}$ (all $k_1 \geq 1$). By the commuting property (1.1), it follows that

$$\alpha_{k_1,2} = \alpha_{k_1,1} = \alpha_{k_1+1,1} \text{ (all } k_1 \geq 1), \tag{4.1}$$

as desired. $\square$

Corollary 4.8. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a subnormal* 2*-variable weighted shift.*
(i) If $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ *for some* $k_1, k_2 \geq 0$*, then* $\mathbf{T}$ *is horizontally flat.*
(ii) If, instead, $\beta_{(k_1,k_2)+\varepsilon_2} = \beta_{(k_1,k_2)}$ *for some* $k_1, k_2 \geq 0$*, then* $\mathbf{T}$ *is vertically flat.*

Proof. Straightforward from Theorem 4.7. $\square$

Remark 4.9. Corollary 4.8 can also be obtained as a direct consequence of Lemma 4.5 and Lemma 4.6.

Theorem 4.7 is optimal in the following sense: propagation does not necessarily extend either to the left (0-th column) or down (0-th level). We will actually establish a stronger result, that is, the optimality of Corollary 4.8. We first review some basic facts.

Proposition 4.10. *(*[CuYo2]*) Let*

$$\alpha_k := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } k = 0 \\ \sqrt{\frac{2^k+\frac{1}{2}}{2^k+1}}, & \text{if } k \geq 1. \end{cases} \tag{4.2}$$

Then W_α *is subnormal with Berger measure* $\xi_\alpha := \frac{1}{3}\delta_0(s) + \frac{1}{3}\delta_{\frac{1}{2}}(s) + \frac{1}{3}\delta_1(s)$.

Proposition 4.11. *(*[CuYo2]*) Let*

$$\widehat{\alpha_k} := \begin{cases} \sqrt{2}, & \text{if } k = 0 \\ \sqrt{\frac{2^k+1}{2^k+\frac{1}{2}}}, & \text{if } k \geq 1 \end{cases}$$

then $\prod_{n=0}^{\infty} \widehat{\alpha_k} = \sqrt{3}$ *. (Observe that* $\widehat{\alpha_k} = \frac{1}{\alpha_k}$*, for* α_k *given by (4.2).)*

Theorem 4.12. *Consider the weighted shift* $\mathbf{T} \equiv (T_1, T_2)$ *with weight diagram given by Figure 8, where* $y \leq \frac{1}{\sqrt{3}}$*. Let* $W_0 := shift(\alpha_0, \alpha_1, \alpha_2, \cdots)$*, with* α_k *given by (4.2). Then* $\mathbf{T}$ *is subnormal.*

Proof. To check subnormality, we use Lemma 4.5. Since

$$\xi_0 = \frac{1}{3}(\delta_0 + \delta_{\frac{1}{2}} + \delta_1)$$

and

$$d\mu_{\mathcal{M}}(s,t) = (d\delta_0(s) + d\delta_1(s))tdt,$$

we get

$$\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X = y^2(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1).$$

Thus, $y^2(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) \leq \frac{1}{6}(\delta_0 + \delta_1) \leq \xi_0$. Lemma 4.5 now implies that $\mathbf{T}$ is subnormal. $\square$

Corollary 4.13. *Theorem 4.7 is optimal.*

Proof. Straightforward from Theorem 4.12. $\square$

Theorem 4.14. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a subnormal* 2*-variable weighted shift, and assume that* $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ *and* $\beta_{(\ell_1,\ell_2)+\varepsilon_2} = \beta_{(\ell_1,\ell_2)}$ *for some* $k_1, k_2, \ell_1, \ell_2 \geq 0$*. Then* $\mathbf{T}$ *is flat.*

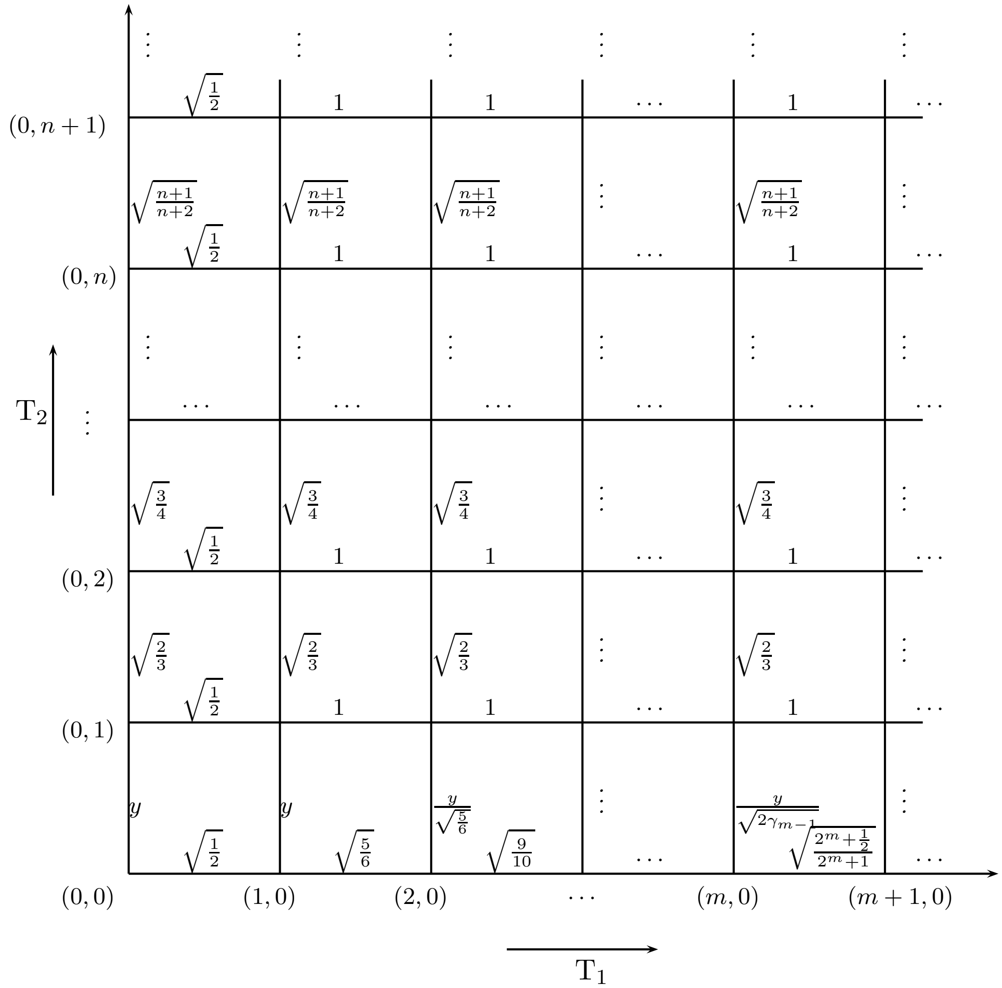

FIGURE 8. Weight diagram of the 2-variable weighted shift in Theorem 4.12

Proof. Straightforward from Theorem 4.7. □

Corollary 4.15. *Theorem 4.14 is optimal.*

Proof. Straightforward from Example 5.13 below. □

5. Symmetrically flat 2-variable weighted shifts

Recall that a 2-variable weighted shift $\mathbf{T}$ is *flat* if $\mathbf{T}$ is horizontally and vertically flat, and *symmetrically flat* if $\mathbf{T}$ is flat and $\alpha_{11} = \beta_{11}$ (cf. Definition 3.1). In ([CuYo1, Theorem 2.12]), we produced an example of a symmetrically flat, contractive, 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ (that is, $\alpha_{11} = \beta_{11} = 1$, and $\|T_1\| \leq 1$ and $\|T_2\| \leq 1$) with T_1, T_2 subnormal, such that $\mathbf{T}$ is hyponormal but not subnormal. In this section, we study the class $\mathcal{SFC}$ of symmetrically flat, contractive, 2-variable weighted shifts, with T_1 and T_2 subnormal, and we give a complete characterization of hyponormality and subnormality in $\mathcal{SFC}$; our main result is Corollary 5.6, with a concrete criterion for hyponormality and subnormality.

Symmetrically flat 2-variable weighted shifts are determined by three main parts: (i) a subnormal shift in the 0-th row ($shift(x_0, x_1, x_2 \cdots)$, with Berger measure ξ); (ii) a subnormal shift in the 0-th column ($shift(y_0, y_1, y_2, \cdots)$, with Berger measure η); and (iii) a positive number a (the α_{01} weight) (cf. Figure 9). By [CuYo2, Theorem 3.3], the measures ξ and η can be written as

$$\begin{aligned} \xi &\equiv p\delta_0 + q\delta_1 + [1-(p+q)]\rho \\ \eta &\equiv u\delta_0 + v\delta_1 + [1-(u+v)]\sigma, \end{aligned} \tag{5.1}$$

where $0 < p, q, u, v < 1$, $p + q \leq 1$, $u + v \leq 1$, and ρ, σ are probability measures with $\rho(\{0\} \cup \{1\}) = \sigma(\{0\} \cup \{1\}) = 0$. The following lemma is essential to detect joint hyponormality in the presence of flatness.

Lemma 5.1. *(*[CuYo1, Theorem 5.2]*) Let* $\mathbf{T} \equiv (T_1, T_2)$*, let* $\mathcal{M}$ *be the subspace associated to indices* $\mathbf{k}$ *with* $k_2 \geq 1$*, and assume that* $\mathbf{T}|_{\mathcal{M}}$ *is subnormal with Berger measure* $\delta_1 \times \delta_1$*. Assume further that* T_1 *and* T_2 *are contractions, that* $W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with Berger measure* ξ*, and that* T_2 *is subnormal. Then* $\mathbf{T}$ *is subnormal.*

Remark 5.2. Lemma 5.1 (together with its proof [CuYo1, Theorem 5.2]) reveals that for the 2-variable weighted shift given by Figure 4, the hyponormality of T_2 is equivalent to the subnormality of $\mathbf{T}$, which in turn is equivalent to the hyponormality of $\mathbf{T}$.

Theorem 5.3. *Let* $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{SFC}$ *be given by Figure 9. Then* $\mathbf{T}$ *is hyponormal if and only if*

$$y_0 \leq h := \sqrt{\frac{x_0^2 y_1^2 (x_1^2 - x_0^2)}{x_0^2 (x_1^2 - x_0^2) + (a^2 - x_0^2)^2}}. \tag{5.2}$$

Proof. By Lemma 5.1 and Remark 5.2, the subnormality of T_1 (resp. T_2) implies the subnormality of $\mathbf{T}|_{\mathcal{N}}$ (resp. $\mathbf{T}|_{\mathcal{M}}$), where $\mathcal{N}$ (resp. $\mathcal{M}$) is the subspace associated to indices $\mathbf{k}$ with $k_1 \geq 1$ (resp. indices $\mathbf{k}$ with $k_2 \geq 1$). Thus, to verify the hyponormality of $\mathbf{T}$ it suffices to apply the Six-point Test (Lemma 3.2) to $\mathbf{k} = (0, 0)$. We have

$$\begin{aligned} &\begin{pmatrix} x_1^2 - x_0^2 & \frac{a^2 y_0}{x_0} - x_0 y_0 \\ \frac{a^2 y_0}{x_0} - x_0 y_0 & y_1^2 - y_0^2 \end{pmatrix} \geq 0 \\ &\Leftrightarrow x_0^2 (y_1^2 - y_0^2)(x_1^2 - x_0^2) \geq y_0^2 (a^2 - x_0^2)^2 \\ &\Leftrightarrow y_0 \leq \sqrt{\tfrac{x_0^2 y_1^2 (x_1^2 - x_0^2)}{x_0^2 (x_1^2 - x_0^2) + (a^2 - x_0^2)^2}} = h. \end{aligned} \tag{5.3}$$

It follows that $\mathbf{T}$ is hyponormal if and only if $y_0 \leq h$, as desired. $\square$

We next consider joint subnormality for 2-variable weighted shifts in $\mathcal{SFC}$. We recall Berger's Theorem in the 2-variable case and the notion of moment of order $\mathbf{k}$ for a pair (α, β) satisfying (1.1). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \begin{cases} 1, & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

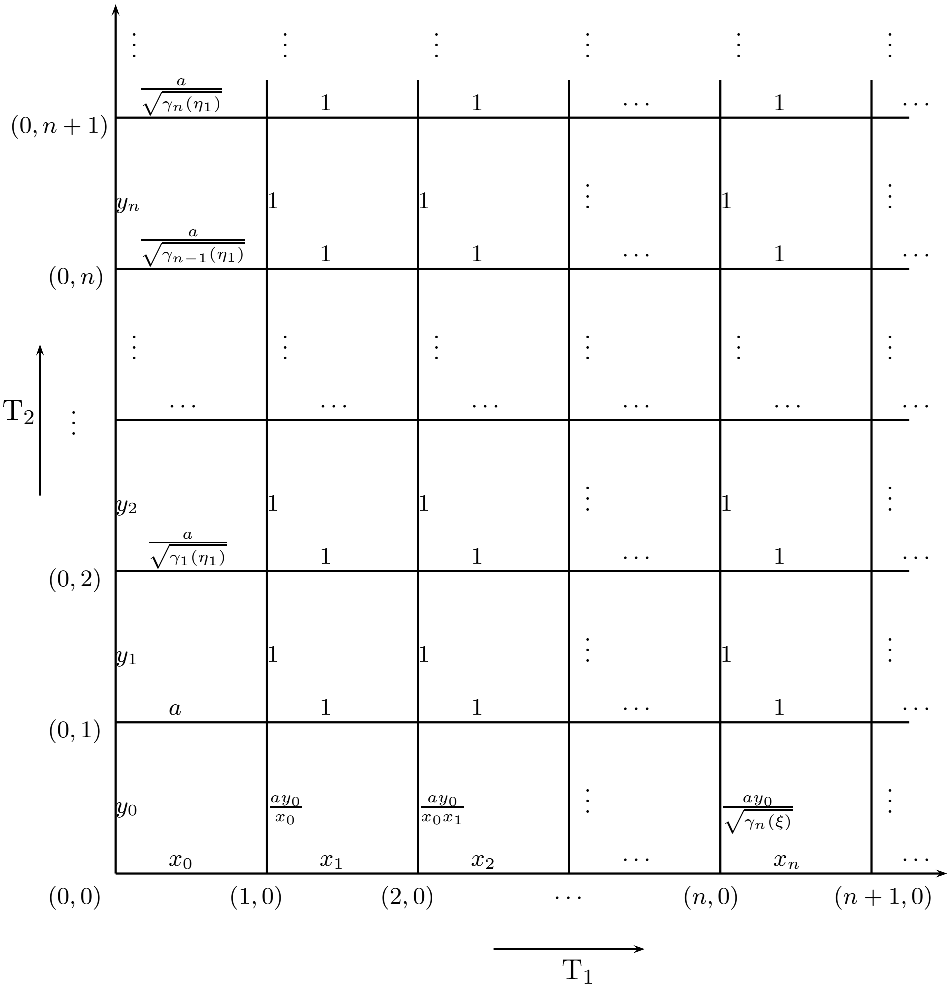

Figure 9. Weight diagram of a general symmetrically flat, contractive, 2-variable weighted shift; η_1 denotes the Berger measure of $shift(y_1, y_2, \cdots)$.

We remark that, due to the commutativity condition (1.1), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to (k_1, k_2).

Lemma 5.4. *(Berger's Theorem, 2-variable case) (*[JeLu]*) A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a regular Borel probability measure μ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ ($a_i := \|T_i\|^2$) such that $\gamma_{\mathbf{k}} = \iint_R \mathbf{t}^{\mathbf{k}} d\mu(\mathbf{t}) := \iint_R s^{k_1} t^{k_2} d\mu(s,t)$ (all $\mathbf{k} \in \mathbb{Z}_+^2$).*

Theorem 5.5. *Let $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{SFC}$ be given by Figure 9. Then $\mathbf{T}$ is subnormal if and only if*

$$y_0 \leq s := \min \left\{ \sqrt{\frac{q}{a^2}}, \sqrt{\frac{p}{\left\| \frac{1}{t} \right\|_{L^1(\eta_1)} - a^2}} \right\}.$$

Proof. Consider the subspaces $\mathcal{M} := \{\mathbf{k} \in \mathbb{Z}_+^2 : k_2 \geq 1\}$ and $\mathcal{P} := \{\mathbf{k} \in \mathbb{Z}_+^2 : k_1 \geq 1 \text{ and } k_2 \geq 1\}$, let $\mathbf{T}|_{\mathcal{M}}$ and $\mathbf{T}|_{\mathcal{P}}$ denote the restrictions of $\mathbf{T}$ to $\mathcal{M}$ and $\mathcal{P}$, and let η_1 denote the Berger measure of $shift(y_1, y_2, \cdots)$. Since T_2 is subnormal, and since $\mathbf{T}|_{\mathcal{P}}$ is the restriction of $\mathbf{T}|_{\mathcal{M}}$ to the subspace $\mathcal{P}$, we can apply Lemma 4.5 to $\mathbf{T}|_{\mathcal{M}}$ and the subspace $\mathcal{P}$ (therefore using as initial data the measures $\delta_1 \times \delta_1$ and η_1) to show that the subnormality of T_2 implies $a^2\delta_1 \leq \eta_1$, which in turn gives the subnormality of $\mathbf{T}|_{\mathcal{M}}$. The Berger measure of $\mathbf{T}|_{\mathcal{M}}$, $\mu_{\mathcal{M}}$, is then given by

$$\mu_{\mathcal{M}} = a^2\delta_1 \times \delta_1 + \delta_0 \times (\eta_1 - a^2\delta_1). \tag{5.4}$$

Once we know this, we apply Lemma 4.5 again, this time to the 2-variable weighted shift $\mathbf{T}$ and the subspace $\mathcal{M}$. First, observe that $\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = \left\|\frac{1}{t}\right\|_{L^1(\eta_1)}$ and from (5.4) we have

$$\begin{aligned} d(\mu_{\mathcal{M}})_{ext}(s,t) &= d(a^2\delta_1 \times \delta_1 + \delta_0 \times (\eta_1 - a^2\delta_1))_{ext}(s,t) \\ &= (1-\delta_0(t))\frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}}\{a^2 d\delta_1(s) d\delta_1(t) \\ &\quad + d\delta_0(s)(d\eta_1(t) - a^2 d\delta_1(t))\} \\ &= \frac{1}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}}\left\{a^2 d\delta_1(s)\frac{d\delta_1(t)}{t} + d\delta_0(s)(\frac{d\eta_1(t)}{t} - a^2\frac{d\delta_1(t))}{t}\right\} \end{aligned}$$

and therefore

$$\begin{aligned} (\mu_{\mathcal{M}})_{ext}^X &= \frac{1}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}}\left\{a^2\delta_1 + \delta_0(\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - a^2\right\} \\ &= \left(1 - \frac{a^2}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}}\right)\delta_0 + \left(\frac{a^2}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}}\right)\delta_1. \end{aligned}$$

If we now apply Lemma 4.5 and recall (5.1), we see that the necessary and sufficient condition for $\mathbf{T}$ to be subnormal is

$$y_0^2\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}\left((1 - \frac{a^2}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}})\delta_0 + \frac{a^2}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}}\delta_1\right) \leq p\delta_0 + q\delta_1 + [1-(p+q)]\rho,$$

or equivalently,

$$\begin{cases} y_0^2\left(\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - a^2\right) & \leq \quad p \\ y_0^2 a^2 & \leq \quad q. \end{cases}$$

It follows at once that $\mathbf{T}$ is subnormal if and only if $y_0 \leq s$, as desired. $\square$

We summarize Theorems 5.3 and 5.5 as follows.

Corollary 5.6. *The commuting subnormal pair $\mathbf{T} \equiv (T_1, T_2)$ in Figure 9 is jointly hyponormal and not subnormal if and only if*

$$s < y_0 \leq h.$$

Of course we know that $s \leq h$, but a priori we cannot tell whether the inequality can be strict. We will now exhibit a large collection of 2-variable weighted shifts $\mathbf{T} \in \mathcal{SFC}$ such that $\mathbf{T}$ is hyponormal but not subnormal; we will do this by describing a collection of values for x_0, x_1, y_0 and a for which $s < h$. To avoid a trivial case, we shall assume $y_1 < 1$. We begin with

Lemma 5.7. *For $\xi \equiv p\delta_0 + q\delta_1 + (1-p-q)\rho$ as above, we have*
(i) $\int s \, d\xi(s) \geq q$
and
(ii) $\int (1-s) \, d\xi(s) \geq p$.
In each case, strict inequality holds if and only if $p+q<1$.

Proof. Straightforward from the form of ξ. □

Proposition 5.8. *Let $x_0\sqrt{\frac{1-x_1^2}{1-x_0^2}} < a < x_0$. Then $s < h$.*

Proof. We first observe that a straightforward calculation reveals that

$$P := x_0^2 x_1^2 + a^2 - a^2 x_0^2 - x_0^2 > 0 \tag{5.5}$$

whenever $x_0\sqrt{\frac{1-x_1^2}{1-x_0^2}} < a$. Now consider

$$\begin{aligned} \frac{h^2}{y_1^2} - \frac{1-x_0^2}{1-a^2} &= \frac{x_0^2(x_1^2-x_0^2)}{x_0^2(x_1^2-x_0^2)+(a^2-x_0^2)^2} - \frac{1-x_0^2}{1-a^2} \\ &= \frac{(x_0^2-a^2)P}{(1-a^2)\left[x_0^2(x_1^2-x_0^2)+(a^2-x_0^2)^2\right]} > 0. \end{aligned} \tag{5.6}$$

Next, we calculate

$$1 - x_0^2 \equiv \int (1-s) d\xi(s) \geq p \ \text{ (by Lemma 5.7(ii))}. \tag{5.7}$$

Thirdly, we recall that

$$\begin{aligned} 1 &= \left(\int d\eta_1(t)\right)^2 \leq \int t d\eta_1(t) \int \frac{1}{t} d\eta_1(t) \\ &\quad \text{(using Cauchy-Schwartz in } L^2(\eta_1)) \\ &= y_1^2 \left\|\frac{1}{t}\right\|_{L^1(\eta_1)} < \left\|\frac{1}{t}\right\|_{L^1(\eta_1)}. \end{aligned} \tag{5.8}$$

Finally, we have

$$\frac{p}{\left(\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - a^2\right)} < \frac{p}{(1-a^2)\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}} \ \text{(since } \left\|\frac{1}{t}\right\|_{L^1(\eta_1)} > 1). \tag{5.9}$$

We then have

$$\begin{aligned} s^2 &\leq \frac{p}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - a^2} < \frac{p}{(1-a^2)\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}} \leq \frac{1-x_0^2}{(1-a^2)\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}} \ \text{(by (5.7) and (5.9))} \\ &\leq \frac{(1-x_0^2)y_1^2}{1-a^2} < h^2 \ \text{(by (5.8) and (5.6))}, \end{aligned}$$

as desired. □

Proposition 5.9. *Let $x_0 = a$, and assume that $p+q<1$. Then $s<h$.*

Proof. First, observe that $h = y_1$ when $x_0 = a$; cf. (5.2). Then

$$\begin{aligned} s^2 &\equiv \min\left\{\frac{q}{a^2}, \frac{p}{\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - a^2}\right\} \\ &< \frac{1-x_0^2}{(1-a^2)\left\|\frac{1}{t}\right\|_{L^1(\eta_1)}} = y_1^2 \text{ (by Lemma 5.7, (5.7) and (5.8))} \qquad (5.10) \\ &= h^2, \end{aligned}$$

as desired. □

We summarize the above facts in the following result.

Theorem 5.10. *Let* $x_0\sqrt{\frac{1-x_1^2}{1-x_0^2}} < a \le x_0$*, assume that* $p+q<1$*, and choose* y_0 *in the (nonempty!) interval* $(s,h]$*. Then the 2-variable weighted shift* $\mathbf{T} \equiv \mathbf{T}(x_0, x_1, y_0, a)$ *is hyponormal but not subnormal.*

We conclude this section by describing a class of numerical examples that illustrates Theorem 5.10. Consider the 2-variable weighted shift whose weight diagram is given by Figure 10).

To analyze this shift, we will need the following auxiliary results, of independent interest.

Lemma 5.11. *(cf.* [CLY]*) For* $0 < r \le 1$ *let*

$$\beta_n(r) := \begin{cases} \sqrt{\frac{3}{4}}r, & \text{if } n=0 \\ \sqrt{\frac{(n+1)(n+3)}{(n+2)^2}}, & \text{if } n \ge 1. \end{cases} \qquad (5.11)$$

Then $W_{\beta(r)}$ *is subnormal.*

Proof. On $[0,1]$, consider the probability measure

$$d\eta(t) := (1-r^2)d\delta_0(t) + \frac{r^2}{2}dt + \frac{r^2}{2}d\delta_1(t). \qquad (5.12)$$

For $n \ge 1$ we have

$$\begin{aligned} \gamma_n(\beta(r)) &= r^2\frac{3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdot\frac{3\cdot 5}{4^2}\cdot\ldots\cdot\frac{n(n+2)}{(n+1)^2} \\ &= \frac{(n+2)r^2}{2(n+1)} = \frac{r^2}{2}\cdot\frac{1}{n+1} + \frac{r^2}{2} = \int t^n d\eta(t). \end{aligned}$$

Thus, η is the Berger measure of $W_{\beta(r)}$, so $W_{\beta(r)}$ is subnormal (all $r \in (0,1]$). □

Lemma 5.12. *Let*

$$\widehat{\beta_n} := \sqrt{\frac{(n+2)^2}{(n+3)(n+1)}} \ (n \ge 1).$$

Then $\prod_{n=1}^{\infty} \widehat{\beta_n} = \sqrt{\frac{3}{2}}$*. (Observe that* $\widehat{\beta_n} = \frac{1}{\beta_n}$ *(all* $n \ge 1$*), if* β_n *is given by (5.11).)*

Proof. Observe that

$$\prod_{n=1}^{k}(\widehat{\beta_n})^2 = \prod_{n=1}^{k}\frac{(n+2)^2}{(n+3)(n+1)} = \frac{3(k+2)}{2(k+3)}$$

which converge to $\frac{3}{2}$ as $k \to \infty$. □

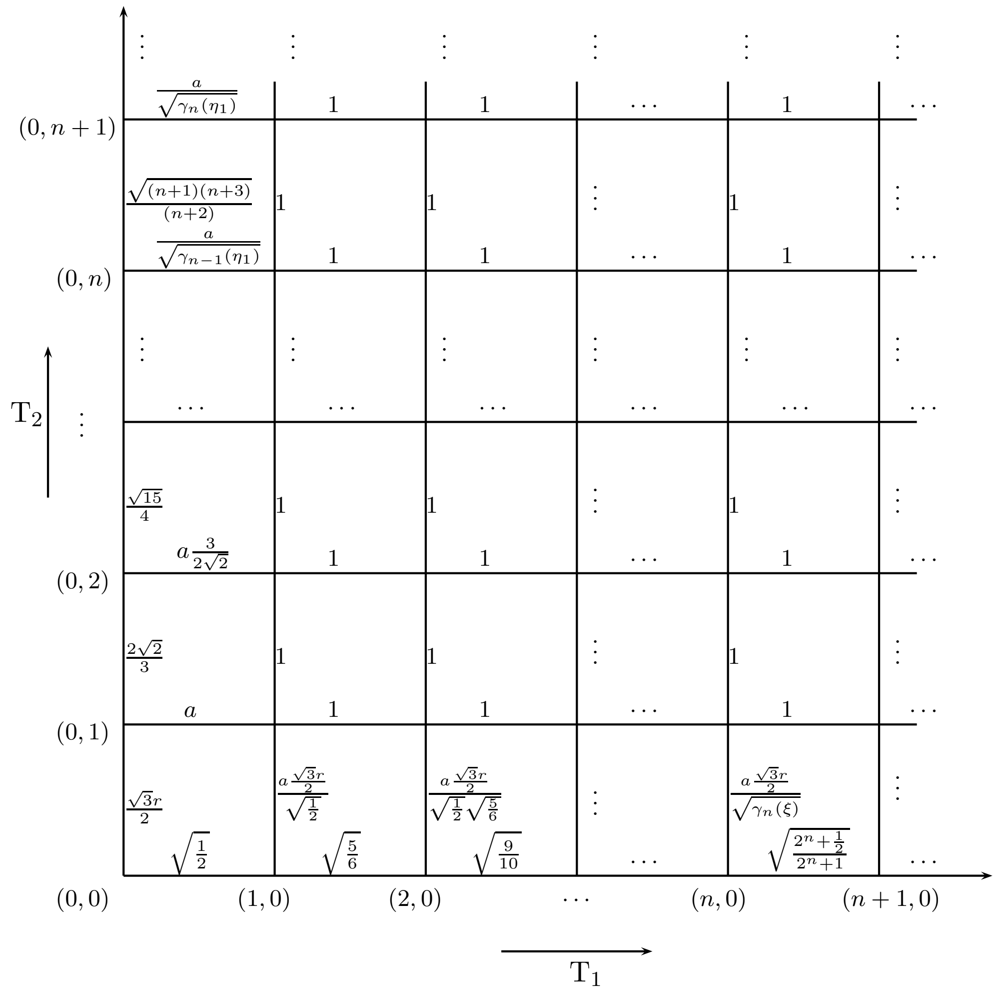

FIGURE 10. Weight diagram of the 2-variable weighted shift in Example 5.13

Example 5.13. (Illustration of Theorem 5.10) We first recall the three assembly parts needed for a 2-variable weighted shift T to be in $\mathcal{SFC}$: (i) a subnormal shift in the 0-th row ($shift(x_0, x_1, x_2 \cdots)$, with Berger measure ξ); (ii) a subnormal shift in the 0-th column ($shift(y_0, y_1, y_2, \cdots)$, with Berger measure η); and (iii) a positive number a (the α_{01} weight). Toward (ii) we shall use the shift in Lemma 5.11, with Berger measure given by (5.12); toward (i) we shall use the measure

$$\xi := \frac{1}{3}(\delta_0 + \delta_{\frac{1}{2}} + \delta_1)$$

on $[0, 1]$, so that $p = q = \frac{1}{3}$; finally, toward (iii) we will keep a as a parameter. The resulting 2-variable weighted shift will be denoted $\mathbf{T}(a; r)$. We will now specify the values of a and r that make $\mathbf{T}(a; r)$ contractive, hyponormal, and not subnormal. To guarantee that $\mathbf{T}(a; r)$ is a pair of contractions, and using Lemma 5.12, it is easy to see that we need $a \leq \sqrt{\frac{2}{3}}$. Next, we observe that $x_0 = \sqrt{\frac{1}{2}}$, $x_1 = \sqrt{\frac{5}{6}}$, and $d\eta_1(t) = \frac{2}{3}[tdt + d\delta_1(t)]$ $(t \in [0, 1])$, so $\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} = \frac{4}{3}$ and $y_1 = \sqrt{\frac{8}{9}}$

. Moreover, $x_0\sqrt{\frac{1-x_1^2}{1-x_0^2}} = \sqrt{\frac{1}{6}}$. By Theorem 5.10, we need to keep $a \in (\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}]$. Thus, for $a \in (\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}]$ we calculate

$$h \equiv \sqrt{\frac{x_0^2 y_1^2 (x_1^2 - x_0^2)}{x_0^2(x_1^2 - x_0^2) + (a^2 - x_0^2)^2}} = \frac{2\sqrt{2}}{3\sqrt{1 + 6(a^2 - \frac{1}{2})^2}}$$

and

$$s \equiv \min\left\{\sqrt{\frac{q}{a^2}}, \sqrt{\frac{p}{\left(\left\|\frac{1}{t}\right\|_{L^1(\eta_1)} - a^2\right)}}\right\} = \min\{\frac{1}{\sqrt{3a^2}}, \sqrt{\frac{1}{4-3a^2}}\} = \sqrt{\frac{1}{4-3a^2}}.$$

Thus, for $a \in (\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}]$ we can then choose $y_0 \equiv \sqrt{\frac{3}{4}}r$ in the interval $(\frac{1}{2\sqrt{1-a^2}}, \frac{2\sqrt{2}}{3\sqrt{1+6(a^2-\frac{1}{2})^2}}]$ and ensure that $\mathbf{T}(a;r)$ is hyponormal and not subnormal (cf. Figure 11). $\square$

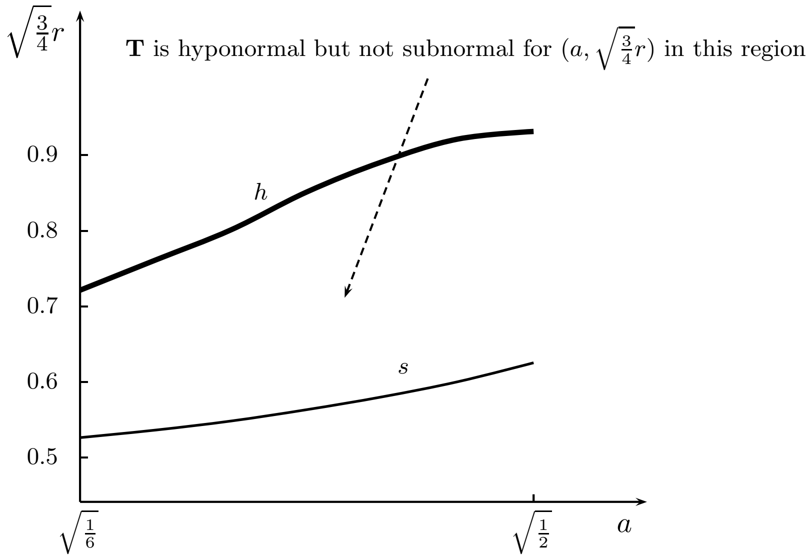

Figure 11. Graphs of h and s on the interval $[\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}]$.

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