PROPAGATION PHENOMENA
FOR HYPONORMAL 2-VARIABLE WEIGHTED SHIFTS

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Abstract. We study the class of hyponormal 2-variable weighted shifts with two consecutive equal weights in the weight sequence of one of the coordinate operators. We show that under natural assumptions on the coordinate operators, the presence of consecutive equal weights leads to horizontal or vertical flatness, in a way that resembles the situation for 1-variable weighted shifts. In 1-variable, it is well known that flat weighted shifts are necessarily subnormal (with finitely atomic Berger measures). By contrast, we exhibit a large collection of flat (i.e., horizontally and vertically flat) 2-variable weighted shifts which are hyponormal but not subnormal. Moreover we completely characterize the hyponormality and subnormality of symmetrically flat contractive 2-variable weighted shifts.

1. Introduction

The Lifting Problem for Commuting Subnormals (LPCS) asks for necessary and sufficient conditions on a pair of commuting subnormal operators on Hilbert space to admit a joint normal extension. In previous work we have proved that the (joint) hyponormality of the pair, while being a necessary condition, is by no means sufficient ([CuYo1], [CuYo2]). We have also established that in a very special situation, hyponormality is indeed sufficient [CuYo1, Theorem 5.2 and Remark 5.3]. This involves 2-variable weighted shifts with weight sequences which are constant except for the 0-th row in the index set $\mathbb{Z}_+^2$. One is then tempted to claim that a similar result might be true for weight sequences which are constant in a slightly smaller domain of indices, e.g., those indices $k \in \mathbb{Z}_+^2$ with $k_1, k_2 \geq 1$. However, in this paper we show that such is not the case, that is, hyponormality and subnormality are quite different even in those cases.

For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ a bounded sequence of positive real numbers (called weights), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_k := \alpha_k e_{k+1}$ (all $k \geq 0$), where $\{e_k\}_{k=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. A quadratically hyponormal weighted shift $W_\alpha$ with $\alpha_{k+1} = \alpha_k$ for some $k \geq 1$ must necessarily be (i) flat (i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$), and (ii) subnormal. For 2-variable weighted shifts associated with weight sequences $\{\alpha_k\}, \{\beta_k\} \in \ell^\infty(\mathbb{Z}_+^2)$, we first establish the correct analogue of (i) (Theorem 3.3), and we then show that there is a rich family of sequences $\{\alpha_k\}, \{\beta_k\}$ giving rise to flat, non-subnormal, hyponormal 2-variable weighted shifts; this is in sharp contrast with the 1-variable situation. The optimality of Theorem 3.3 is established through an elaborate construction which uses Bergman-like weighted shifts (Theorem 3.14). Finally, in Section 5 we completely characterize the hyponormality and subnormality of symmetrically flat contractive 2-variable weighted shifts, which sheds new light on the relationship between flatness and subnormality.

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Recall that a bounded linear operator \( T \in \mathcal{B}(\mathcal{H}) \) on a complex Hilbert space \( \mathcal{H} \) is normal if \( T^*T = TT^* \), subnormal if \( T = N|_{\mathcal{H}} \), where \( N \) is normal and \( N(\mathcal{H}) \subseteq \mathcal{H} \), and hyponormal if \( T^*T \geq TT^* \). For \( k \geq 1 \) and \( T \in \mathcal{B}(\mathcal{H}) \), \( T \) is \( k \)-hyponormal if \( (I, T, \ldots, T^k) \) is (jointly) hyponormal. Additionally, \( T \) is weakly \( k \)-hyponormal if \( p(T) \) is hyponormal for every polynomial \( p \) of degree at most \( k \). Thus, if \( T \) is \( k \)-hyponormal then \( T \) is weakly \( k \)-hyponormal, and “hyponormality” “1-hyponormality” and “weak 1-hyponormality” are all identical notions ([Ath]). On the other hand, results in ([CMX]), ([Cu2]) and ([McCP]) show that if \( T \) is weakly 2-hyponormal (also called quadratically hyponormal), then \( T \) need not be 2-hyponormal. The Bram-Halmos characterization of subnormality ([Con, III.1.9]) can be paraphrased as follow: \( T \) is subnormal if and only if \( T \) is \( k \)-hyponormal for every \( k \geq 1 \). The converse implication, whether \( T \) polynomially hyponormal \( \Rightarrow \) \( T \) subnormal, was settled in the negative in ([CuPu]); indeed, it was shown that there exists a polynomially hyponormal operator which is not 2-hyponormal. Previously, S. McCullough and V. Paulsen had established ([McCP]) that one can find a non-subnormal polynomially hyponormal operator if and only if one can find a unilateral weighted shift with the same property. Thus, although the existence proof in ([CuPu]) is abstract, by combining the results in ([CuPu]) and ([McCP]) we now know that there exists a polynomially hyponormal unilateral weighted shift which is not subnormal.

For \( S, T \in \mathcal{B}(\mathcal{H}) \) we let \([ST] := ST - TS\). We say that a commuting \( n \)-tuple \( T = (T_1, \ldots, T_n) \) of operators on \( \mathcal{H} \) is (jointly) hyponormal if the operator matrix

\[
[T^*, T] := \begin{pmatrix}
[T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\
[T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\
& \ddots & \ddots & \ddots \\
[T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n]
\end{pmatrix}
\]

is positive on the direct sum of \( n \) copies of \( \mathcal{H} \) (cf. [Ath], [CMX]). The \( n \)-tuple \( T \) is said to be normal if \( T \) is commuting and each \( T_i \) is normal, and \( T \) is subnormal if \( T \) is the restriction of a normal \( n \)-tuple to a common invariant subspace. Clearly, normal \( \Rightarrow \) subnormal \( \Rightarrow \) hyponormal.

For \( \alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^\infty(\mathbb{Z}_+) \) and \( W_\alpha \) the associated unilateral weighted shift, the moments of \( \alpha \) are given as

\[
\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 
1 & \text{if } k = 0 \\
\alpha_0 \cdot \alpha_1 \cdots \alpha_{k-1} \cdot \alpha_k & \text{if } k > 0 
\end{cases}
\]

It is easy to see that \( W_\alpha \) is never normal, and that it is hyponormal if and only if \( \alpha_0 \leq \alpha_1 \leq \cdots \). If \( \alpha_{k+1} = \alpha_k \) for all \( k \geq 1 \), \( W_\alpha \) is called flat. On occasion, we will write \( shift(\alpha_0, \alpha_1, \alpha_2, \cdots) \) to denote the weighted shift with weight sequence \( \{\alpha_k\}_{k=0}^{\infty} \). We also denote by \( U_+ := shift(1, 1, 1, \cdots) \) the (unweighted) unilateral shift, and for \( 0 < a < 1 \) we let \( S_a := shift(a, 1, 1, \cdots) \); the shift \( S_a \) is the prototypical flat weighted shift, and it is subnormal.

Similarly, consider double-indexed positive bounded sequences \( \{\alpha_k\}, \{\beta_k\} \in \ell^\infty(\mathbb{Z}_+^2), k \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+ \) and let \( \ell^2(\mathbb{Z}_+^2) \) be the Hilbert space of square-summable complex sequences indexed by \( \mathbb{Z}_+^2 \). (Recall that \( \ell^2(\mathbb{Z}_+^2) \) is canonically isometrically isomorphic to \( \ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+) \).) We define the 2-variable weighted shift \( T \) by

\[
T_1 e_k := \alpha_{k+\varepsilon_1}, \\
T_2 e_k := \beta_{k+\varepsilon_2},
\]

where \( \varepsilon_1 := (1, 0) \) and \( \varepsilon_2 := (0, 1) \). Clearly,

\[
T_1 T_2 = T_2 T_1 \iff \beta_{k+\varepsilon_1} \alpha_k = \alpha_{k+\varepsilon_2} \beta_k \quad (\text{all } k).
\]  

(1.1)
In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts $W_\alpha$ and $W_\beta$ gives rise to a 2-variable weighted shift $T \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1} \triangledown \beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+^2$). In this case, $T$ is subnormal (resp. hyponormal) if and only if so are $T_1$ and $T_2$; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+^2) \otimes \ell^2(\mathbb{Z}_+)$, $T_1 \cong W_\alpha \otimes I$ and $T_2 \cong I \otimes W_\beta$, and $T$ is also doubly commuting. For this reason, we do not focus attention on shifts of this type, and use them only when the above mentioned triviality is desirable or needed.

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [Con, III.8.16]): $W_\alpha$ is subnormal if and only if there exists a probability measure $\xi$ supported in $[0, \|W_\alpha\|]$ (called the Berger measure of $W_\alpha$) such that $\gamma_k(\alpha) := \alpha^2 \cdots \alpha^2_{k-1} = \int t^k d\xi(t)$ ($k \geq 1$). For instance, the Berger measures of $U_+$ and $S_\alpha$ are $\delta_1$ and $(1-a^2)\delta_0 + a^2 \delta_1$, respectively, where $\delta_x$ denotes the point-mass probability measure with support the singleton $\{x\}$.

If $W_\alpha$ is subnormal, and if for $h \geq 1$ we let $M_h := \vee\{e_k : k \geq h\}$ denote the invariant subspace obtained by removing the first $h$ vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{M_h}$ is $\frac{1}{t^n}t^h d\xi(t)$. For $h = 2$, one can use this to prove the following result.

Lemma 1.1. Let $T \equiv shift(\beta_0, \beta_1, \cdots)$ be a subnormal weighted shift, with Berger measure $\eta$, and let $T_M$ be its restriction to $M := \vee\{e_2, e_3, \cdots\}$. Then $\beta^2_1 = (\|\frac{1}{t}\|_{L^1(\eta)})^{-1}$. 

Proof. We have 
\[ \|\frac{1}{t}\|_{L^1(\eta)} = \int \frac{1}{t} (\frac{1}{t})^2 \eta(t) = \int \frac{\gamma_1}{\gamma_2} t d\eta(t) = \frac{\gamma_1}{\gamma_2} = \frac{1}{\beta^2_1}, \]
as desired. \qed

Corollary 1.2. Let $T \equiv (T_1, T_2)$ be a commuting 2-variable weighted shift, assume that $T_2$ is subnormal, and assume that there exists $k_1 \geq 0$ such that $\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2)} + \epsilon_2$ for all $k_2 \geq 2$. Then $\beta_{(k_1,1)} = \beta_{(k_1,1)} + \epsilon_1$.

Proof. Consider $S := shift(\beta_{(k_1,2)}, \beta_{(k_1,3)}, \cdots)$ and $S' := shift(\beta_{(k_1+1,2)}, \beta_{(k_1+1,3)}, \cdots)$. Since $T_2$ is subnormal, we know that both $S$ and $S'$ are subnormal, with Berger measures $\eta$ and $\eta'$, respectively. Since $\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2)} + \epsilon_2$, the commuting property (1.1) readily implies that $\beta_{(k_1,k_2)} = \beta_{(k_1,k_2)} + \epsilon_1$ for all $k_2 \geq 2$, that is $S = S'$, that is, $\eta = \eta'$. By Lemma 1.1, we must have 
\[ \beta^2_{(k_1,1)} = (\|\frac{1}{t}\|_{L^1(\eta)})^{-1} = (\|\frac{1}{t}\|_{L^1(\eta')})^{-1} = \beta^2_{(k_1,1)} + \epsilon_1, \]
as desired. \qed

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2. Propagation Phenomena for 1-Variable Weighted Shifts

In this section, we review some basic propagation phenomena for 1-variable weighted shifts, and we then develop the results for the 2-variable case in Sections 3 and 4. J. Stampfli showed in [Sta] that for a subnormal weighted shift $W_\alpha$, a propagation phenomenon occurs which forces the flatness of $W_\alpha$ whenever two equal weights are present.

Proposition 2.1. (Subnormality, One-variable Case) [Sta] Let $W_\alpha$ be a subnormal weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then $W_\alpha$ is flat.
The first named author showed that in the presence of 2-hyponormality (resp. quadratic hyponormality) of weighted shifts, a propagation phenomenon also occurs which forces the flatness of $W_\alpha$ whenever two equal weights (resp. three equal weights) are present.

**Proposition 2.2.** (2-hyponormality, One-variable Case) ([Cu2]) Let $W_\alpha$ be a 2-hyponormal weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then $W_\alpha$ is flat.

**Proposition 2.3.** (Quadratic Hyponormality, One-variable Case) ([Choi]) Let $W_\alpha$ be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$, and assume that $W_\alpha$ is quadratically hyponormal. If $\alpha_k = \alpha_{k+1} = \alpha_{k+2}$ for some $k \geq 0$, then $W_\alpha$ is flat.

Y. Choi later improved Proposition 2.3, as follows.

**Proposition 2.4.** (Quadratic Hyponormality, One-variable Case, Improved Version) ([Choi]) Let $W_\alpha$ be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$, and assume that $W_\alpha$ is quadratically hyponormal. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 1$, then $W_\alpha$ is flat.

Moreover, Y. Choi showed that, in the presence of polynomially hyponormality, two consecutive equal weights again force flatness.

**Proposition 2.5.** (Polynomially hyponormality) ([Choi]) Let $W_\alpha$ be a unilateral weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$, and assume that $W_\alpha$ is polynomially hyponormal. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then $W_\alpha$ is flat.

### 3. Propagation in the 2-variable Hyponormal Case

In this section, we show that if a commuting, (jointly) hyponormal pair $T \equiv (T_1, T_2)$ with $T_1$ quadratically hyponormal satisfies $\alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2)}$ for some $k_1,k_2 \geq 1$, then $(T_1, T_2(I \otimes U_{k_2}^{k_2-1}))$ is horizontally flat (see Definition 3.1 below); this is the content of Theorem 3.3. We also prove that Theorem 3.3 is optimal in the following sense: the propagation does not extend either to the left (0-th column) or down (below $k_2$-th level).

We begin with:

**Definition 3.1.** A 2-variable weighted shift $T \equiv (T_1, T_2)$ is horizontally flat (resp. vertically flat) if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$). We say that $T$ is flat if $T$ is horizontally and vertically flat (cf. Figure 1), and we say that $T$ is symmetrically flat if $T$ is flat and $\alpha_{11} = \beta_{11}$.

**Lemma 3.2.** ([Cu1]) (Six-point Test) Let $T \equiv (T_1, T_2)$ be a 2-variable weighted shift, with weight sequences $\alpha$ and $\beta$. Then

$$[T^*, T] \geq 0 \iff \left(\left(\begin{array}{cc}
\alpha_k^2 - \alpha_k^2 & \alpha_k \alpha_{k+1}^2 \\
\alpha_k \alpha_{k+1}^2 & \beta_{k+1} \beta_k^2
\end{array}\right) \geq 0 \text{ for all } k \in \mathbb{Z}_+^2\right)$$

**Theorem 3.3.** Let $T \equiv (T_1, T_2)$ be a commuting, hyponormal 2-variable weighted shift.

(i) If $T_1$ is quadratically hyponormal and $\alpha_{(k_1,k_2)+\epsilon_1} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1, T_2(I \otimes U_{k_2}^{k_2-1}))$ is horizontally flat.

(ii) If, instead, $T_2$ is quadratically hyponormal and $\beta_{(k_1,k_2)+\epsilon_2} = \beta_{(k_1,k_2)}$ for some $k_1, k_2 \geq 1$, then $(T_1(U_{k_2}^{k_2-1} \otimes I), T_2)$ is vertically flat.
Figure 1. Weight diagram of a flat 2-variable weighted shift (with round dots for horizontal flatness, triangular dots for vertical flatness)

Proof. Without loss of generality, we only prove (i). Consider the restricted weight diagram based at \((k_1, k_2)\) (see Figure 3).

Recall that, by joint hyponormality, we have

\[
\begin{pmatrix}
\alpha_{(k_1, k_2), + \varepsilon_1}^2 - \alpha_{(k_1, k_2)}^2 & \alpha_{(k_1, k_2), + \varepsilon_1} \beta_{(k_1, k_2), + \varepsilon_1} - \beta_{(k_1, k_2), + \varepsilon_1} \alpha_{(k_1, k_2)} \\
\alpha_{(k_1, k_2), + \varepsilon_1} \beta_{(k_1, k_2), + \varepsilon_1} - \beta_{(k_1, k_2), + \varepsilon_1} \alpha_{(k_1, k_2)} & \beta_{(k_1, k_2), + \varepsilon_1}^2 - \beta_{(k_1, k_2), + \varepsilon_1}^2
\end{pmatrix} \geq 0.
\]

Since \(\alpha_{(k_1, k_2), + \varepsilon_1} = \alpha_{(k_1, k_2)}\), it follows that

\[
\alpha_{(k_1, k_2), + \varepsilon_1} \beta_{(k_1, k_2), + \varepsilon_1} = \beta_{(k_1, k_2)} \alpha_{(k_1, k_2)}. \tag{3.1}
\]

By the commuting property (1.1),

\[
\alpha_{(k_1, k_2), + \varepsilon_1} = \alpha_{(k_1, k_2), + \varepsilon_2} \beta_{(k_1, k_2), + \varepsilon_1} = \alpha_{(k_1, k_2), + \varepsilon_2} \beta_{(k_1, k_2)}. \tag{3.2}
\]
Therefore
\[ \alpha^{2}(k_1, k_2) + \varepsilon_2 \beta(k_1, k_2) = \alpha(k_1, k_2) + \varepsilon_2 \left( \alpha(k_1, k_2) + \varepsilon_2 \beta(k_1, k_2) \right) = \alpha(k_1, k_2) + \varepsilon_2 \beta(k_1, k_2) + \varepsilon_1 \] (by (3.2))
\[ = \alpha(k_1, k_2) \left( \beta(k_1, k_2) + \varepsilon_2 \right) = \alpha(k_1, k_2) \beta(k_1, k_2) \alpha(k_1, k_2) \] (by (3.1)).

Thus, \[ \alpha^{2}(k_1, k_2) + \varepsilon_2 \beta(k_1, k_2) = \alpha(k_1, k_2) \beta(k_1, k_2) \alpha(k_1, k_2) \], which implies that \( \alpha(k_1, k_2) + \varepsilon_2 = \alpha(k_1, k_2) \). We now recall Theorem 2.4, which says that flatness can be propagated to the right, that is, \( \alpha(k_1, k_2) + \varepsilon_1 = \alpha(k_1, k_2) + 2 \varepsilon_1 \). It follows that \( \alpha(k_1, k_2) + \varepsilon_1 + \varepsilon_2 = \alpha(k_1, k_2) + \varepsilon_2 \), and then two equal weights occurs at level \( k_2 + 1 \), which then implies \( \alpha(k_1, k_2) + 2 \varepsilon_2 = \alpha(k_1, k_2) + \varepsilon_2 = \alpha(k_1, k_2) \). It is now easy to see that for every level \( \ell \geq k_2 \) we must have \( \alpha(k_1, \ell) = \alpha(k_1, k_2) \) (all \( k_1 \geq 1 \)). Using Theorem 2.4 to propagate these equalities to the left, we eventually conclude that
\[ \alpha(k_1, \ell) = \alpha(1, k_2) \] (\( k_1 \geq 1, \ell \geq k_2 \)).
We thus obtain that $(T_1, T_2)\big|_{\{e(k_1, \ell): k_1 \geq 1, \ell \geq k_2\}}$ is unitarily equivalent to $(\alpha_{(1,k_2)} U_+ \otimes I, I \otimes W_\eta)$, where $\eta_k := \beta_{1,k+k_2}(k \geq 0)$. This can be rephrased as saying that $(T_1, T_2(I \otimes U_k^{k_2-1}))$ is horizontally flat, as desired. □

**Remark 3.4.** The proof of Theorem 3.3 shows that for $T \equiv (T_1, T_2)$ commuting and hyponormal, and for $k_1, k_2 \geq 0$, 

$$\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} \Rightarrow \beta_{(k_1,k_2)} = \beta_{(k_1,k_2)+\varepsilon_1}$$

(by (3.1) and (3.2)). Moreover, if $k_2 \geq 1$, 

$$\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)} \quad \text{and} \quad \alpha_{(k_1,k_2)+\varepsilon_1-\varepsilon_2} = \alpha_{(k_1,k_2)-\varepsilon_2} \Rightarrow \alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)+\varepsilon_1-\varepsilon_2}.$$

**Remark 3.5.** The proof of Theorem 3.3 also reveals that asking $T \equiv (T_1, T_2)$ to be jointly hyponormal is significantly stronger than asking both $T_1$ and $T_2$ to be hyponormal. For, consider the 2-variable weighted shift whose weight diagram is given by Figure 4. In [CuYo1, Theorem 5.2], we established that in the case when $\|W_\alpha\| \leq 1$, $T$ is subnormal if and only if $T$ is hyponormal. Thus, a necessary condition for the hyponormality of $T$ is the subnormality of $W_0 := \text{shift}(\alpha_{00}, \alpha_{10}, \cdots)$. For $0 < a < 1$, let $x_0 \equiv x_1 := a$ and let $x_k := 1$ ($k \geq 2$). Clearly $W_0$ is hyponormal and not subnormal, and if we take $0 < y \leq a^2$ we can guarantee that each of $T_1$ and $T_2$ is hyponormal, yet $T$ is not. An alternative way to see this is observe that if $T$ were hyponormal then $\alpha_{01}$ would equal $a$, since $\alpha_{00} = \alpha_{10}$.

We will now show that Theorem 3.3 is optimal in the following sense: the propagation does not necessarily extend either to the left (0-th column) or down (below $k_2$-th level). To demonstrate this optimality, we first introduce the class of Bergman-like weighted shifts.

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**Figure 4.** Weight diagram of the 2-variable weighted shift in Remark 3.5
Definition 3.6. For \( \ell \geq 1 \), the Bergman-like weighted shift on \( \ell^2(\mathbb{Z}_+) \) is \( B^{(\ell)}_+ := \text{shift}(\{ \sqrt{\ell - \frac{1}{k+2}} : k \geq 0 \}) \); that is,

\[
B^{(\ell)}_+ e_k := \sqrt{\ell - \frac{1}{k+2}} e_{k+1} \quad (k \geq 0).
\]

In particular, \( B^{(1)}_+ \equiv B_+ := \text{shift}(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots) \) is the Bergman shift.

Remark 3.7. (i) \( B_+ \) is subnormal with Berger measure \( d\xi(s) := ds \) on \([0,1]\).
(ii) \( (\text{[CuYo3]} \) \( B^{(2)}_+ \) is subnormal with Berger measure \( d\xi(s) := \frac{s}{\pi \sqrt{2s-s^2}} \) on \([0,2]\).

Our next step is to show that \( B^{(\ell)}_+ \) (\( \ell \geq 1 \)) is always 2-hyponormal. To this end, we need two preliminary results.

Lemma 3.8. (Nested Determinants Test; Special Case) If \( a > 0 \) and \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0 \), then

\[
\begin{pmatrix} a & b \\ b & c \end{pmatrix} \geq 0 \Leftrightarrow \det \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & e \end{pmatrix} \geq 0.
\]

Proof. Straightforward from Choleski’s Algorithm (\cite{Atk}).

Lemma 3.9. \( (\text{[Cu2]} \) Let \( W_+ e_k = \alpha_k e_{k+1} \) (\( k \geq 0 \)) be a hyponormal weighted shift. The following statements are equivalent:
(i) \( W_+ \) is 2-hyponormal.
(ii) The matrix

\[
(\begin{pmatrix} W^j_+ & W^i_+ \end{pmatrix} e_{k+j}, e_{k+i})^2_{i,j=1}
\]

is positive semi-definite for all \( k \geq -1 \).
(iii) The matrix

\[
(\gamma_k \gamma_{k+i+j} - \gamma_{k+i} \gamma_{k+j})^2_{i,j=1}
\]

is positive semi-definite for all \( k \geq 0 \), where as usual \( \gamma_0 = 1 \), \( \gamma_n = \alpha_0^2 \cdots \alpha_{n-1}^2 \) (\( n \geq 1 \)).
(iv) The Hankel matrix

\[
H(2;k) := (\gamma_{k+i+j-2})^3_{i,j=1}
\]

is positive semi-definite for all \( k \geq 0 \).

We now use symbolic manipulation to prove the following result.

Theorem 3.10. All Bergman-like weighted shifts \( B^{(\ell)}_+ \) (\( \ell \geq 1 \)) are 2-hyponormal.

Proof. By Lemma 3.8 and Lemma 3.9, to prove that \( B^{(\ell)}_+ \) is 2-hyponormal it suffices to see that \( \det H(2;k) > 0 \) for all \( k \geq 0 \). Now,

\[
\det H(2;k) = \gamma_k^3 \det \begin{pmatrix}
1 & \alpha_k^2 & \alpha_k^2 \alpha_{k+1} \\
\alpha_k^2 & \alpha_k^2 \alpha_{k+1} & \alpha_k^2 \alpha_{k+1} \alpha_{k+2} \\
\alpha_k^2 \alpha_{k+1} & \alpha_k^2 \alpha_{k+1} \alpha_{k+2} & \alpha_k^2 \alpha_{k+1} \alpha_{k+2} \alpha_{k+3} \\
\end{pmatrix}
\]

\[
= \gamma_k^3 \frac{2(\ell+1)((k+2)\ell - 1)^2((k+3)\ell - 1)}{(k+2)^2(k+3)^2(k+4)^2(k+5)} > 0,
\]

as desired. \( \square \)
Corollary 3.11. For every \( \ell \geq 1 \), the Bergman-like weighted shift \( B_+^{(\ell)} \) is quadratically hyponormal.

Remark 3.12. In ([CPY]), we prove a much stronger result: all Bergman-like weighted shifts \( B_+^{(\ell)} \) (all \( \ell \geq 1 \)) are subnormal.

Theorem 3.14 below says that the amount of propagation provided by Theorem 3.3 is maximum; briefly, we say that Theorem 3.3 is optimal. Observe that for the 2-variable weighted shift in Figure 6, we have \( \alpha_{(k_1,k_2) + 1} = \alpha_{(k_1,k_2)} \) (all \( k_1 \geq 1, k_2 \geq 2 \)), yet \( \alpha_{(k_1,k_2)} < \alpha_{(k_1,k_2) + 1} \) for all \( k_1 \geq 0 \) and \( k_2 = 0,1 \) and \( \alpha_{(0,k_2)} < \alpha_{(1,k_2)} \) for all \( k_2 \geq 0 \). In other words, the trivial weight structure present in the subspace \( \bigvee \{ e_{(k_1,k_2)} : k_1 \geq 1, k_2 \geq 2 \} \) cannot be expanded either to the left (0th column) or down (first row). First, we need an auxiliary result, of independent interest.

Lemma 3.13. Consider the 2-variable weighted shift \( T \equiv (T_1, T_2) \) given by Figure 5, where \( \text{shift}(x_0, x_1, x_2, \cdots) \) and \( \text{shift}(y_0, y_1, y_2, \cdots) \) are Bergman-like weighted shifts. Assume that \( (T_1, T_2)|_\mathcal{M} \) is jointly hyponormal, where \( \mathcal{M} \) is the subspace associated to indices \( k \) with \( k_2 \geq 1 \). Then there exists a Bergman-like weighted shift \( \text{shift}(z_0, z_1, z_2, \cdots) \) and a hyponormal weighted shift \( W_\beta := \text{shift}(\beta_0, \beta_1, \beta_2, \cdots) \) (\( \beta_n < \beta_{n+1} \) for all \( n \geq 0 \)) such that \( T \) is jointly hyponormal.
Proof. Let

\[ \text{shift}(x_0, x_1, x_2, \cdots) \equiv \text{shift}(\{ p - \frac{1}{n+2} : n \geq 0 \}) \]
\[ \text{shift}(y_0, y_1, y_2, \cdots) \equiv \text{shift}(\{ q - \frac{1}{n+2} : n \geq 0 \}) \]
\[ \text{shift}(z_0, z_1, z_2, \cdots) \equiv \text{shift}(\{ r - \frac{1}{n+2} : n \geq 0 \}) \] for some integers \( p < q < r \).

Since the restriction of \((T_1, T_2)\) to \( \sqrt{\{ e(k_1, k_2) : k_2 \geq 1 \} \} \) is jointly hyponormal, it suffices to apply the Six-point Test (Lemma 3.2) to \( k = (n, 0) \), with \( n \geq 0 \).

**Case 1:** \( k = (0, 0) \). Here

\[ M(0, 0) := \begin{pmatrix} z_0^2 - z_0^2 & \beta_0 \frac{y_0}{y_0} - \beta_0 \cdot z_0 \\ \end{pmatrix} \geq 0 \]
\[ \iff z_0^2(z_1^2 - z_0^2)(\beta_1^2 - \beta_0^2) \geq \beta_0^2(z_0^2 - y_0^2) \]
\[ \iff (r - \frac{1}{2})(\beta_1^2 - \beta_0^2) \geq 6\beta_0^2(r - q)^2. \]

If we choose \( \beta_0 \) such that \( \beta_0 \leq \beta_1 \) and \( \beta_1^2 \geq \frac{12(r-q)^2+2r-1}{27} \beta_0^2 \), we obtain \( M(0, 0) \geq 0 \).

**Case 2:** \( k = (n, 0) \) \((n \geq 1)\). Here

\[ M(n, 0) := \begin{pmatrix} z_{n+1}^2 - z_n^2 & \beta_0 \frac{y_0}{y_0} \prod_{k=0}^{n-1} \frac{y_k}{y_k} - z_n \beta_0 \prod_{k=0}^{n-1} \frac{y_k}{y_k} \\ \end{pmatrix} \geq 0 \]
\[ \iff (z_{n+1}^2 - z_n^2)(\beta_1^2 \prod_{k=0}^{n-1} \frac{y_k}{y_k})^2 - \beta_0^2 \prod_{k=0}^{n-1} \frac{y_k}{y_k} \geq \beta_0^2(y_0 \prod_{k=0}^{n-1} \frac{y_k}{y_k} - z_n \prod_{k=0}^{n-1} \frac{y_k}{y_k})^2 \]
\[ \iff z_n^2(z_{n+1}^2 - z_n^2)(\beta_1^2 \prod_{k=0}^{n-1} \frac{y_k}{y_k})^2 - \beta_0^2 \prod_{k=0}^{n-1} \frac{y_k}{y_k} \geq \beta_0^2(y_0 \prod_{k=0}^{n-1} \frac{y_k}{y_k} - z_n \prod_{k=0}^{n-1} \frac{y_k}{y_k})^2 \]
\[ \iff z_n^2(z_{n+1}^2 - z_n^2)(\beta_1^2 \prod_{k=0}^{n-1} \frac{y_k}{y_k})^2 - \beta_0^2 \geq \beta_0^2(y_n \prod_{k=0}^{n-1} \frac{y_k}{y_k})^2 \]

If we choose \( p, q \) and \( r \) such that \( \frac{y_{k+1}}{y_k} \geq 3 \), then \( \frac{r(n+2)-1}{(n+2)^2(n+3)}(\frac{12(r-q)^2+2r-1}{27}) \prod_{k=0}^{n-1} \left(\frac{y_{k+1}}{y_k}\right)^2 - 1 \geq (r-q)^2 \)

(all \( n \geq 1 \)), which implies \( M(n, 0) \geq 0 \) \((n \geq 1)\).

By Cases 1 and 2, it follows that \((T_1, T_2)\) is jointly hyponormal.

\[ \square \]

**Theorem 3.14.** For every \( k_2 \geq 1 \) and \( 0 < \alpha_0 < 1 \) there exist

(i) a family \( \{ B_{n+1}^{(k_2)} \}_{k_2=0}^{k_2-1} \) of Bergman-like weighted shifts, and

(ii) a subnormal weighted shift \( W_\beta := \text{shift}(\beta_0, \beta_1, \beta_2, \cdots) \) (with \( \beta_n < \beta_{n+1} \) for all \( n \geq 0 \)),

such that the commuting 2-variable weighted shift \( T \) with a weight diagram whose first \( k_2 \) rows are \( B_{n+1}^{(k_2)} \), \( \cdots \), \( B_{n+1}^{(k_2-1)} \), whose remaining rows are \( S_{\alpha_0} \), and whose 0-th column is given by \( W_\beta \), is (jointly) hyponormal (see Fig. 6 for the case \( k_2 = 2 \)).

**Proof.** We divide the proof into three cases, according to the value of \( k_2 \).

**Case 1:** \( k_2 = 1 \). For \( p \geq 1 \) let \( x_{m, 0} \equiv x_{m, 0} := \sqrt{p - \frac{1}{m+2}} \) \((m \geq 0)\). Since the restriction of \((T_1, T_2)\) to \( \sqrt{\{ e(k_1, k_2) : k_2 \geq 1 \} \} \) must be unitarily equivalent to \( (S_{\alpha_0} \otimes I) (I \otimes \text{shift}(\beta_1, \beta_2, \cdots)) \), to guarantee the hyponormality of \((T_1, T_2)\) it suffices to apply the Six-point Test (Lemma 3.2) to \( k = (m, 0) \), with \( m \geq 0 \).

\[ \square \]
Subcase 1: $k = (0, 0)$. Here we have

$$M(0, 0) : = \begin{pmatrix}
\alpha_0^2 \beta_0 - \beta_0 x_0 \\
\beta_0^2 x_0 - x_0
\end{pmatrix}$$

$$= \begin{pmatrix}
\frac{1}{6} \\
\frac{\beta_0}{x_0} (\alpha_0^2 - x_0^2)
\end{pmatrix} \geq 0$$

$$\Leftrightarrow 6\beta_0^2 (\alpha_0^2 - x_0^2)^2 \leq (\beta_1^2 - \beta_0^2) x_0^2,$$  \hspace{1cm} (3.4)

which imposes a condition on $x_0$ and $\beta_0$.

Subcase 2: $k = (m, 0)$, with $m \geq 1$. Fix $m \geq 1$ and let $P_m := \prod_{k=0}^{m-1} x_k$. We then see that

$$M(m, 0) : = \begin{pmatrix}
\alpha_0 \beta_0^2 x_0 - x_0 \\
\alpha_0 \beta_0^2 x_0 - x_0
\end{pmatrix} \geq 0$$

$$\Leftrightarrow x_m^2 (x_{m+1}^2 - x_m^2) (\beta_1^2 P_m - \alpha_0^2 \beta_0^2) \geq \alpha_0^2 \beta_0^2 (1 - x_m^2)^2.$$
From Subcases 1 and 2, it follows that \( \alpha_m \) hold.

Subcase 2: \( k = (m, n) \). Here we let \( P_m = \prod_{k=0}^{m-1} x_k \) and \( Q_m = \prod_{k=0}^{m-1} y_k \). We have

\[
M(m, 0) := \left( \frac{y_{m+1}^2 - y_m^2}{x_{m+1}^2 - x_m^2} - \beta_0 y_0 \right) \geq 0
\]

\[
\Leftrightarrow y_0^2 (y_{m+1}^2 - y_m^2) (x_{m+1}^2 - x_m^2) \geq \beta_0^2 (x_0^2 - y_0^2)^2
\]

\[
\Leftrightarrow 225 \beta_0^2 \leq \frac{35}{12}
\]

so \( M(m, 0) \geq 0 \) if and only if

\[
\beta_0^2 \leq \frac{7}{3240} \approx 0.00216.
\]

Subcase 2: \( k = (m, 0) \) (all \( m \geq 1 \)). Fix \( m \geq 1 \) and let \( P_m = \prod_{k=0}^{m-1} x_k \) and \( Q_m = \prod_{k=0}^{m-1} y_k \). We have

\[
M(m, 0) := \left( \frac{y_{m+1}^2 - y_m^2}{x_{m+1}^2 - x_m^2} - \beta_0 y_0 \right) \geq 0
\]

\[
\Leftrightarrow y_0^2 (y_{m+1}^2 - y_m^2) (x_{m+1}^2 - x_m^2) \geq \beta_0^2 (x_0^2 - y_0^2)^2
\]

\[
\Leftrightarrow 225 \beta_0^2 \leq \frac{35}{12}
\]

It follows that \( M(m, 0) \geq 0 \) (all \( m \geq 1 \)) if and only if

\[
\beta_0^2 \leq f(m) := \frac{Q_m^2}{P_m^4 \frac{225(m+2)^2(m+3)}{18m+35} + 1}
\]

\[
= \frac{18m + 35}{225m^3 + 1575m^2 + 3618m + 2735} \prod_{k=0}^{m-1} \frac{18k^2 + 71k + 70}{9k^2 + 30k + 25} \quad \text{(all } m \geq 1\).\]
Since $f$ is an increasing function of $m$, we see that $M(m,0) \geq 0$ (all $m \geq 1$) if and only if
\[
\beta_0^2 \leq f(1) = \frac{742}{40765} \approx 0.018. \tag{3.7}
\]

**Subcase 3: $k = (0,1)$**. Here we have
\[
M(0,1) := \left( \frac{x_1^2 - x_0^2}{\alpha_0^2 x_0} - \frac{\beta_1 x_0}{\beta_2 - \beta_1^2} \right) \geq 0
\]
\[
= \left( \frac{1}{6} \frac{\beta_1 (\alpha_0^2 - x_0^2)}{x_0 (\alpha_0^2 - x_0^2)} \right) \geq 0
\]
\[
\iff (\alpha_0^2 - \frac{5}{2})^2 \leq \frac{25}{4}, \tag{3.8}
\]
which certainly holds, since $0 < \alpha_0 < 1$.

**Subcase 4: $k = (m,1)$, with $m \geq 1$**. As in Subcase 2, fix $m \geq 1$ and let $P_m := \prod_{k=0}^{m-1} x_k$. We then see that
\[
M(m,1) := \left( \frac{x_{m+1}^2 - x_m^2}{\alpha_0^2 x_m} - \frac{\alpha_0 x_m}{\alpha_0 x_m} \frac{\beta_2}{\beta_2 - \beta_1^2} \right) \geq 0
\]
\[
\iff x_m^2 (x_{m+1}^2 - x_m^2) (\beta_2^2 P_m^2 - \alpha_0^2 \beta_1^2) \geq \alpha_0^2 \beta_1^2 (1 - x_m^2)^2 \tag{3.9}
\]
\[
\iff x_m^2 (\beta_2^2 P_m^2 - 1) \geq (m+2)(m+3)(1-x_m^2)^2
\]
\[
\iff \beta_2^2 \geq g(m) := \frac{1}{P_m^2} \left( 1 + \frac{(m+3)(2m+3)^2}{3m+5} \right).
\]
It follows that $M(m,1) \geq 0$ (all $m \geq 1$) if and only if $\beta_2$ can be chosen to satisfy (3.9) for all $m \geq 1$. Since $g$ is a decreasing function of $m$, it suffices to guarantee that $\beta_2^2 \geq g(1) = \frac{27}{5}$. If we now recall that $\beta_2 = 4\beta_1$ and that $\beta_1 = \frac{1}{\alpha_0}$, this condition is equivalent to $\beta_0^2 \leq \frac{39}{11}$, which always holds, since $\alpha_0 < 1$.

Therefore, by Subcases 1, 2, 3 and 4, which yield the condition (3.6), we see that $(T_1, T_2)$ is hyponormal if and only if $\beta_0^2 \leq \frac{7}{40765}$. Finally, and since we clearly have $\beta_0 < \beta_1 < \beta_2$, we can use the construction in [Sta] to define $W_\beta$, which incidentally has a 2-atomic Berger measure (cf. [CuFi]).

**Case 3: $k_2 \geq 3$**. Here we take $p$ and $q$ as in Case 2, to ensure that the restriction of $T$ to the subspace associated with subindices $(m, n)$ with $n \geq k_2 - 2$ is hyponormal. Once this is done, we use Lemma 3.13 to obtain $r$, so that the restriction of $T$ to the subspace associated with subindices $(m, n)$ with $n \geq k_2 - 3$ is hyponormal. Repeated application of Lemma 3.13 now completes the proof. \qed

**Corollary 3.15.** Theorem 3.3 is optimal.

### 4. Propagation in the Subnormal Case

In this section, we show that Theorem 3.3 can be improved if one of the $T_i$’s is quadratically hyponormal and the other is subnormal. In Theorem 4.7 and Theorem 4.12, we consider horizontal flatness and optimality, and in Theorem 4.14, we show that a subnormal 2-variable weighted shift with two horizontally consecutive equal weights and two vertically consecutive equal weights must necessarily be flat. As in the previous section, we then establish that our result is optimal (see Example 5.13 below). We begin with some definitions and preliminary results.
Definition 4.1. ([CuYo1]) Let $\mu$ and $\nu$ be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ on $X := \mathbb{R}_+$, if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq \mathbb{R}_+$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on $\mathbb{R}_+$.

Definition 4.2. ([CuYo1]) Let $\mu$ be a probability measure on $X \times Y$, and assume that $\frac{1}{t} \in L^1(\mu)$. The extremal measure $\mu_{\text{ext}}$ (which is also a probability measure) on $X \times Y$ is given by

$$d\mu_{\text{ext}}(s, t) := \frac{1}{t \|1\|_{L^1(\mu)}} d\mu(s, t).$$

Definition 4.3. ([CuYo1]) Given a measure $\mu$ on $X \times Y$, the marginal measure $\mu^X$ is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto $X$. Thus, $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$. Observe that if $\mu$ is a probability measure, then so is $\mu^X$.

Lemma 4.4. ([CuYo1]) (Subnormal backward extension of a 1-variable weighted shift) Let $T \equiv \text{shift}(\beta_0, \beta_1, \cdots)$ be a unilateral weighted shift whose restriction $T_M$ to $M := \vee \{e_1, e_2, \cdots\}$ is subnormal, with Berger measure $\eta_M$. Then $T$ is subnormal (with measure $\eta$) if and only if

(i) $\frac{1}{t} \in L^1(\eta_M)$;

(ii) $\beta_0^2 \leq (\| \frac{1}{t} \|_{L^1(\eta_M)})^{-1}$.

In this case, $d\eta(t) = \frac{\beta_0^2}{t} d\eta_M(t) + (1 - \beta_0^2 \| \frac{1}{t} \|_{L^1(\eta_M)}) d\delta_0(t)$, where $\delta_0$ denotes the Dirac measure at 0. In particular, $T$ is never subnormal when $\eta_M(\{0\}) > 0$.

Lemma 4.5. ([CuYo1]) (Subnormal backward extension of a 2-variable weighted shift) Consider the following 2-variable weighted shift (see Figure 7), and let $M$ be the subspace associated to indices $k$ with $k_2 \geq 1$. Assume that $T|_M$ is subnormal with measure $\mu_M$ and that $W_0 := \text{shift}(\alpha_0, \alpha_{10}, \cdots)$ is subnormal with measure $\xi$. Then $T$ is subnormal if and only if

(i) $\frac{1}{t} \in L^1(\mu_M)$;

(ii) $\beta_{00}^2 \leq (\| \frac{1}{t} \|_{L^1(\mu_M)})^{-1}$;

(iii) $\beta_{00}^2 \| \frac{1}{t} \|_{L^1(\mu_M)} (\mu_M)_{\text{ext}} \leq \xi$.

Moreover, if $\beta_{00}^2 \| \frac{1}{t} \|_{L^1(\mu_M)} = 1$, then $(\mu_M)_{\text{ext}} = \xi$. In the case when $T$ is subnormal, the Berger measure $\mu$ of $T$ is given by

$$d\mu(s, t) = \beta_{00}^2 \| \frac{1}{t} \|_{L^1(\mu_M)} d(\mu_M)_{\text{ext}}(s, t) + (d\xi(s) - \beta_{00}^2 \| \frac{1}{t} \|_{L^1(\mu_M)}) d(\mu_M)_{\text{ext}}(s))d\delta_0(t).$$

Lemma 4.6. Let $T \equiv (T_1, T_2)$, let $M$ be as in Lemma 4.5, and assume that $T|_M$ is subnormal with Berger measure $\mu_M \equiv \delta_1 \times \eta$. Assume further that $\beta_{00}^2 = (\| \frac{1}{t} \|_{L^1(\mu_M)})^{-1}$ and that $W_0 := \text{shift}(\alpha_0, \alpha_{10}, \alpha_{20}, \cdots)$ is subnormal. Then $T$ is subnormal if and only if $\alpha_i = 1$ (all $i \geq 0$), that is, $W_0$ must necessarily be the (unweighted) unilateral shift $U_+$.

Proof. Assume first that $T$ is subnormal. Since $d\mu_M(s, t) \equiv \delta_1(s)d\eta(t)$, we must have

$$d(\mu_M)_{\text{ext}}^X = ((1 - \delta_0(t)) \frac{1}{t \| \frac{1}{t} \|_{L^1(\mu_M)}} d\mu_M(s, t))^X$$

$$= d\delta_1(s) = d\xi_{\alpha(0)}(s) \text{ (by Lemma 4.5)},$$

where $\xi_{\alpha(0)}$ denotes the Berger measure of $W_{\alpha(0)}$. It follows that $W_{\alpha(0)} = U_+$. 

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Conversely, assume that $W_{\alpha(0)} = U_+$. By Lemma 4.4, $shift(\beta_{00}, \beta_{01}, \cdots)$ is subnormal, and we let $\tilde{\eta}$ denote its Berger measure. If we now let $\mu := \delta_1 \times \tilde{\eta}$, it easily follows that $T$ is subnormal with Berger measure $\mu$. \hfill \square

**Theorem 4.7.** Let $T = (T_1, T_2)$ be commuting and hyponormal.

(i) If $T_1$ is quadratically hyponormal, if $T_2$ is subnormal, and if $\alpha_{(k_1,k_2)+\varepsilon_1} = \alpha_{(k_1,k_2)}$ for some $k_1, k_2 \geq 0$, then $T$ is horizontally flat.

(ii) If, instead, $T_1$ is subnormal, $T_2$ is quadratically hyponormal, and if $\beta_{(k_1,k_2)+\varepsilon_2} = \beta_{(k_1,k_2)}$ for some $k_1, k_2 \geq 0$, then $T$ is vertically flat.

**Proof.** Without loss of generality, we only consider the horizontally flat case, and we further assume $k_2 = 2$, that is, $\alpha_{k_1,2} = \alpha_{k_1+1,2}$ for some $k_1 \geq 0$. By Theorem 3.3 and Proposition 2.4, two equal weights occur at level 3, i.e., $\alpha_{k_1,3} = \alpha_{k_1+1,3}$. Moreover, for every $\ell \geq 2$ we have $\alpha_{k_1,2} = \alpha_{k_1,\ell}$ (all $k_1 \geq 1$). We now apply Corollary 1.2 to obtain $\beta_{(k_1,1)} = \beta_{(k_1,1)+\varepsilon_1}$ (all $k_1 \geq 1$). By the commuting

\[
\begin{array}{cccccc}
\beta_{0,n+1} & \cdots & \beta_{0,n} & \cdots & \beta_{0,2} & \cdots \\
\beta_{1,n+1} & \cdots & \beta_{1,n} & \cdots & \beta_{1,2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{m,n+1} & \cdots & \beta_{m,n} & \cdots & \beta_{m,2} & \cdots \\
\end{array}
\]

**Figure 7.** Weight diagram of the 2-variable weighted shift in Lemma 4.5

\[
\begin{array}{cccccc}
\alpha_{0,n+1} & \cdots & \alpha_{0,n} & \cdots & \alpha_{0,2} & \cdots \\
\alpha_{1,n+1} & \cdots & \alpha_{1,n} & \cdots & \alpha_{1,2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{m,n+1} & \cdots & \alpha_{m,n} & \cdots & \alpha_{m,2} & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccc}
\sqrt{\frac{n+1}{n}} & \cdots & \sqrt{\frac{n+1}{n}} & \cdots & \sqrt{\frac{n+1}{n}} & \cdots \\
\sqrt{\frac{n+1}{n}} & \cdots & \sqrt{\frac{n+1}{n}} & \cdots & \sqrt{\frac{n+1}{n}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sqrt{\frac{n+1}{n}} & \cdots & \sqrt{\frac{n+1}{n}} & \cdots & \sqrt{\frac{n+1}{n}} & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccc}
\alpha_{0,2} & \cdots & \alpha_{0,1} & \cdots & \alpha_{0,0} & \cdots \\
\alpha_{1,2} & \cdots & \alpha_{1,1} & \cdots & \alpha_{1,0} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{m,2} & \cdots & \alpha_{m,1} & \cdots & \alpha_{m,0} & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccc}
\sqrt{\frac{1}{1}} & \cdots & \sqrt{\frac{1}{1}} & \cdots & \sqrt{\frac{1}{1}} & \cdots \\
\sqrt{\frac{1}{1}} & \cdots & \sqrt{\frac{1}{1}} & \cdots & \sqrt{\frac{1}{1}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sqrt{\frac{1}{1}} & \cdots & \sqrt{\frac{1}{1}} & \cdots & \sqrt{\frac{1}{1}} & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccc}
\alpha_{0,1} & \cdots & \alpha_{0,0} & \cdots & \alpha_{0,-1} & \cdots \\
\alpha_{1,1} & \cdots & \alpha_{1,0} & \cdots & \alpha_{1,-1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{m,1} & \cdots & \alpha_{m,0} & \cdots & \alpha_{m,-1} & \cdots \\
\end{array}
\]
property (1.1), it follows that
\[ \alpha_{k,2} = \alpha_{k,1} = \alpha_{k+1,1} \quad \text{(all $k_1 \geq 1$)}, \]  
(4.1)
as desired.

**Corollary 4.8.** Let $T \equiv (T_1, T_2)$ be a subnormal 2-variable weighted shift.

(i) If $\alpha(k_1, k_2) + \varepsilon_1 = \alpha(k_1, k_2)$ for some $k_1, k_2 \geq 0$, then $T$ is horizontally flat.

(ii) If, instead, $\beta(k_1, k_2) + \varepsilon_2 = \beta(k_1, k_2)$ for some $k_1, k_2 \geq 0$, then $T$ is vertically flat.

**Proof.** Straightforward from Theorem 4.7.

**Remark 4.9.** Corollary 4.8 can also be obtained as a direct consequence of Lemma 4.5 and Lemma 4.6.

Theorem 4.7 is optimal in the following sense: propagation does not necessarily extend either to the left (0-th column) or down (0-th level). We will actually establish a stronger result, that is, the optimality of Corollary 4.8. We first review some basic facts.

**Proposition 4.10.** ([CuYo2]) Let
\[ \alpha_k := \begin{cases} \sqrt{\frac{1}{2}}, & \text{if } k = 0 \\ \sqrt{\frac{k^2 + 1}{2 + 1}}, & \text{if } k \geq 1. \end{cases} \] (4.2)
Then $W_\alpha$ is subnormal with Berger measure $\xi_\alpha := \frac{1}{3}\delta_0(s) + \frac{1}{3}\delta_1(s) + \frac{1}{3}\delta_1(s)$.

**Proposition 4.11.** ([CuYo2]) Let
\[ \hat{\alpha}_k := \begin{cases} \sqrt{\frac{2}{3}}, & \text{if } k = 0 \\ \sqrt{\frac{k^2 + 1}{2 + 1}}, & \text{if } k \geq 1. \end{cases} \] Then $\prod_{n=0}^\infty \hat{\alpha}_k = \sqrt{3}$. (Observe that $\hat{\alpha}_k = \frac{1}{\alpha_k}$, for $\alpha_k$ given by (4.2).)

**Theorem 4.12.** Consider the weighted shift $T \equiv (T_1, T_2)$ with weight diagram given by Figure 8, where $y \leq \frac{1}{\sqrt{3}}$. Let $W_0 := \text{shift}(\alpha_0, \alpha_1, \alpha_2, \cdots)$, with $\alpha_k$ given by (4.2). Then $T$ is subnormal.

**Proof.** To check subnormality, we use Lemma 4.5. Since
\[ \xi_0 = \frac{1}{3}(\delta_0 + \delta_2 + \delta_1) \]
and
\[ d\mu_M(s,t) = (d\delta_0(s) + d\delta_1(s))tdt, \]
we get
\[ \beta_0^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_M)}^{X} (\mu_M)_{ext}^X = y^2 \left( \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \right). \]
Thus, $y^2 \left( \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1 \right) \leq \frac{1}{6}(\delta_0 + \delta_1) \leq \xi_0$. Lemma 4.5 now implies that $T$ is subnormal.

**Corollary 4.13.** Theorem 4.7 is optimal.

**Proof.** Straightforward from Theorem 4.12.

**Theorem 4.14.** Let $T \equiv (T_1, T_2)$ be a subnormal 2-variable weighted shift, and assume that $\alpha(k_1, k_2) + \varepsilon_1 = \alpha(k_1, k_2)$ and $\beta(\ell_1, \ell_2) + \varepsilon_2 = \beta(\ell_1, \ell_2)$ for some $k_1, k_2, \ell_1, \ell_2 \geq 0$. Then $T$ is flat.
Proof. Straightforward from Theorem 4.7. \square

Corollary 4.15. Theorem 4.14 is optimal.

Proof. Straightforward from Example 5.13 below. \square

5. Symmetrically flat 2-variable weighted shifts

Recall that a 2-variable weighted shift \( T \) is flat if \( T \) is horizontally and vertically flat, and symmetrically flat if \( T \) is flat and \( \alpha_{11} = \beta_{11} \) (cf. Definition 3.1). In ([CuYo1, Theorem 2.12]), we produced an example of a symmetrically flat, contractive, 2-variable weighted shift \( T \equiv (T_1, T_2) \) (that is, \( \alpha_{11} = \beta_{11} = 1 \), and \( \|T_1\| \leq 1 \) and \( \|T_2\| \leq 1 \)) with \( T_1, T_2 \) subnormal, such that \( T \) is hyponormal but not subnormal. In this section, we study the class \( SFC \) of symmetrically flat, contractive, 2-variable weighted shifts, with \( T_1 \) and \( T_2 \) subnormal, and we give a complete characterization of
Lemma 5.1. Let $T \equiv (T_1, T_2)$ be given by Figure 9. Then $T$ is hyponormal if and only if
\[
y_0 \leq h := \sqrt{\frac{x_0^2 y_1^2 (x_1^2 - x_0^2)}{x_0^2 (x_1^2 - x_0^2) + (a^2 - x_0^2)^2}}.
\]  

Proof. By Lemma 5.1 and Remark 5.2, the subnormality of $T_1$ (resp. $T_2$) implies the subnormality of $T_{|\mathcal{N}}$ (resp. $T_{|\mathcal{M}}$), where $\mathcal{N}$ (resp. $\mathcal{M}$) is the subspace associated to indices $k$ with $k_1 \geq 1$ (resp. indices $k$ with $k_2 \geq 1$). Thus, to verify the hyponormality of $T,$ it suffices to apply the Six-point Test (Lemma 3.2) to $k = (0,0).$ We have
\[
x_0^2 y_1^2 (x_1^2 - x_0^2) \geq y_0^2 (a^2 - x_0^2)^2
\]
and
\[
y_0 \leq \sqrt{\frac{x_0^2 y_1^2 (x_1^2 - x_0^2)}{x_0^2 (x_1^2 - x_0^2) + (a^2 - x_0^2)^2}} = h.
\]

It follows that $T$ is hyponormal if and only if $y_0 \leq h,$ as desired. 

We next consider joint subnormality for 2-variable weighted shifts in $\mathcal{SFC}.$ We recall Berger's Theorem in the 2-variable case and the notion of moment of order $k$ for a pair $(\alpha, \beta)$ satisfying (1.1). Given $k \in \mathbb{Z}_+^2,$ the moment of $(\alpha, \beta)$ of order $k$ is
\[
\gamma_k \equiv \gamma_k(\alpha, \beta) := \begin{cases} 1, & \text{if } k = 0 \\ \alpha^2_{(0,0)} \cdots \alpha^2_{(k_1-1,0)} \beta^2_{(0,0)} \cdots \beta^2_{(0,k_2-1)} \alpha^2_{(0,0)} \cdots \alpha_{(k_1-1,0)} \beta^2_{(k_1,0)} \cdots \beta^2_{(k_1,k_2-1)}, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \alpha^2_{(0,0)} \cdots \alpha_{(k_1-1,0)} \beta^2_{(k_1,0)} \cdots \beta^2_{(k_1,k_2-1)}, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha^2_{(0,0)} \cdots \alpha_{(k_1-1,0)} \beta^2_{(k_1,0)} \cdots \beta^2_{(k_1,k_2-1)}, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}
\]
We remark that, due to the commutativity condition (1.1), $\gamma_k$ can be computed using any nondecreasing path from $(0, 0)$ to $(k_1, k_2)$.

**Lemma 5.4.** (Berger’s Theorem, 2-variable case) ([JeLu]) A 2-variable weighted shift $T \equiv (T_1, T_2)$ admits a commuting normal extension if and only if there is a regular Borel probability measure $\mu$ defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ ($a_i := \|T_i\|_1$) such that $\gamma_k = \iint_{R} t^k d\mu(t) := \iint_{R} s^{k_1} t^{k_2} d\mu(s, t)$ (all $k \in \mathbb{Z}_+^2$).

**Theorem 5.5.** Let $T \equiv (T_1, T_2) \in \mathcal{SFC}$ be given by Figure 9. Then $T$ is subnormal if and only if

$$y_0 \leq s := \min \left\{ \sqrt{\frac{q}{a^2}}, \sqrt{\frac{p}{\|T\|_{L^1(\eta_1)} - a^2}} \right\},$$
Proof. Consider the subspaces $\mathcal{M} := \{ k \in \mathbb{Z}_+^2 : k_2 \geq 1 \}$ and $\mathcal{P} := \{ k \in \mathbb{Z}_+^2 : k_1 \geq 1 \text{ and } k_2 \geq 1 \}$, let $T|_\mathcal{M}$ and $T|_\mathcal{P}$ denote the restrictions of $T$ to $\mathcal{M}$ and $\mathcal{P}$, and let $\eta$ denote the Berger measure of shift $(y_1, y_2, \cdots)$. Since $T_2$ is subnormal, and since $T|_\mathcal{P}$ is the restriction of $T|_\mathcal{M}$ to the subspace $\mathcal{P}$, we can apply Lemma 4.5 to $T|_\mathcal{M}$ and the subspace $\mathcal{P}$ (therefore using as initial data the measures $\delta_1 \times \delta_1$ and $\eta$) to show that the subnormality of $T_2$ implies $a^2 \delta_1 \leq \eta$, which in turn gives the subnormality of $T|_\mathcal{M}$. The Berger measure of $T|_\mathcal{M}$, $\mu$, is then given by

$$\mu = a^2 \delta_1 \times \delta_1 + \delta_0 \times (\eta - a^2 \delta_1). \quad (5.4)$$

Once we know this, we apply Lemma 4.5 again, this time to the 2-variable weighted shift $T$ and the subspace $\mathcal{M}$. First, observe that $\|\frac{1}{t}\|_{L^1(\mu)} = \|\frac{1}{t}\|_{L^1(\eta)}$ and from (5.4) we have

$$d(\mu)_{\text{ext}}(s, t) = d(a^2 \delta_1 \times \delta_1 + \delta_0 \times (\eta - a^2 \delta_1))_{\text{ext}}(s, t)$$

$$= (1 - \delta_0(t)) \frac{1}{t} \|\frac{1}{t}\|_{L^1(\mu)} \{ a^2 d\delta_1(s) d\delta_1(t)$$

$$+ d\delta_0(s) d\eta_1(t) - a^2 d\delta_1(t) \}$$

and therefore

$$(\mu)_{\text{ext}} = \frac{1}{\|\frac{1}{t}\|_{L^1(\eta)}} \left\{ a^2 \delta_1 \times \delta_1 + \delta_0 \left( \|\frac{1}{t}\|_{L^1(\eta)} - a^2 \right) \right\}$$

$$= (1 - \|\frac{1}{t}\|_{L^1(\eta)}) \delta_0 + \left( \|\frac{1}{t}\|_{L^1(\eta)} - a^2 \right) \delta_1.$$

If we now apply Lemma 4.5 and recall (5.1), we see that the necessary and sufficient condition for $T$ to be subnormal is

$$y_0^2 \|\frac{1}{t}\|_{L^1(\eta)} \left( 1 - \frac{a^2}{\|\frac{1}{t}\|_{L^1(\eta)}} \right) \delta_0 + \frac{a^2}{\|\frac{1}{t}\|_{L^1(\eta)}} \delta_1 \leq p \delta_0 + q \delta_1 + [1 - (p + q)] \rho,$$

or equivalently,

$$\begin{cases} y_0^2 \left( \|\frac{1}{t}\|_{L^1(\eta)} - a^2 \right) \leq p \\ y_0^2 a^2 \leq q. \end{cases}$$

It follows at once that $T$ is subnormal if and only if $y_0 \leq s$, as desired. \hfill $\Box$

We summarize Theorems 5.3 and 5.5 as follows.

**Corollary 5.6.** The commuting subnormal pair $T \equiv (T_1, T_2)$ in Figure 9 is jointly hyponormal and not subnormal if and only if

$$s < y_0 \leq h.$$
Lemma 5.7. For $\xi \equiv p\delta_0 + q\delta_1 + (1 - p - q)\rho$ as above, we have
(i) $\int s \, d\xi(s) \geq q$
and
(ii) $\int (1 - s) \, d\xi(s) \geq p$.
In each case, strict inequality holds if and only if $p + q < 1$.

Proof. Straightforward from the form of $\xi$. □

Proposition 5.8. Let $x_0\sqrt{\frac{1 - x_0^2}{1 - x_1^2}} < a < x_0$. Then $s < h$.

Proof. We first observe that a straightforward calculation reveals that
\[ P := x_0^2x_1^2 + a^2 - a^2x_0^2 - x_0^2 > 0 \tag{5.5} \]
whenever $x_0\sqrt{\frac{1 - x_1^2}{1 - x_0^2}} < a$. Now consider
\[ \frac{h^2}{y_1^2} - \frac{1 - x_0^2}{1 - a^2} = \frac{x_0^2(x_1^2 - x_0^2)}{x_0^2(x_1^2 - x_0^2) + (a^2 - x_0^2)^2} - \frac{1 - x_0^2}{1 - a^2} = \frac{(x_0^2 - a^2)P}{(1 - a^2)[x_0^2(x_1^2 - x_0^2) + (a^2 - x_0^2)^2]} > 0. \tag{5.6} \]
Next, we calculate
\[ 1 - x_0^2 \equiv \int (1 - s) \, d\xi(s) \geq p \ 	ext{(by Lemma 5.7(ii))}. \tag{5.7} \]
Thirdly, we recall that
\[ 1 = \int d\eta_1(t)^2 \leq \int t \, d\eta_1(t) \int \frac{1}{t} \, d\eta_1(t) \]
(\text{using Cauchy-Schwartz in } L^2(\eta_1))
\[ = y_1^2 \left\| \frac{1}{t} \right\|_{L^1(\eta_1)} < \left\| \frac{1}{t} \right\|_{L^1(\eta_1)}. \tag{5.8} \]
Finally, we have
\[ \frac{p}{\left\| \frac{1}{t} \right\|_{L^1(\eta_1)} - a^2} < \frac{p}{(1 - a^2) \left\| \frac{1}{t} \right\|_{L^1(\eta_1)}} \ 	ext{(since } \left\| \frac{1}{t} \right\|_{L^1(\eta_1)} > 1). \tag{5.9} \]
We then have
\[ s^2 \leq \frac{p}{\left\| \frac{1}{t} \right\|_{L^1(\eta_1)} - a^2} < \frac{p}{(1 - a^2) \left\| \frac{1}{t} \right\|_{L^1(\eta_1)}} \leq \frac{1 - x_0^2}{(1 - a^2) \left\| \frac{1}{t} \right\|_{L^1(\eta_1)}} \text{ (by (5.7) and (5.9))} \]
\[ \leq \frac{(1 - x_0^2)y_1^2}{1 - a^2} < h^2 \ 	ext{(by (5.8) and (5.6))}, \]
as desired. □

Proposition 5.9. Let $x_0 = a$, and assume that $p + q < 1$. Then $s < h$. 21
Proof. First, observe that \( h = y_1 \) when \( x_0 = a \); cf. (5.2). Then

\[
    s^2 \equiv \min \left\{ \frac{q}{a^2}, \frac{p}{\| \frac{1}{1} \| L^1(y_1) - a^2} \right\} < \frac{1 - x_0^2}{(1 - a^2)} \frac{1}{\| \frac{1}{1} \| L^1(y_1)} = y_1^2 \quad \text{(by Lemma 5.7, (5.7) and (5.8))} \quad (5.10)
\]

as desired. \( \Box \)

We summarize the above facts in the following result.

**Theorem 5.10.** Let \( x_0 \sqrt{\frac{1-x_0^2}{1-x_0}} < a \leq x_0 \), assume that \( p + q < 1 \), and choose \( y_0 \) in the (nonempty!) interval \( (s, h) \). Then the 2-variable weighted shift \( T \equiv T(x_0, x_1, y_0, a) \) is hyponormal but not sub-normal.

We conclude this section by describing a class of numerical examples that illustrates Theorem 5.10. Consider the 2-variable weighted shift whose weight diagram is given by Figure 10.

To analyze this shift, we will need the following auxiliary results, of independent interest.

**Lemma 5.11.** (cf. [CLY]) For \( 0 < r \leq 1 \) let

\[
    \beta_n(r) := \begin{cases} 
    \sqrt{\frac{3}{4} r}, & \text{if } n = 0 \\
    \sqrt{\frac{(n+2)(n+3)}{(n+2)^2}}, & \text{if } n \geq 1.
    \end{cases} \quad (5.11)
\]

Then \( W_{\beta(r)} \) is subnormal.

**Proof.** On \([0, 1]\), consider the probability measure

\[
    d\eta(t) := (1 - r^2)d\delta_0(t) + \frac{r^2}{2} dt + \frac{r^2}{2} d\delta_1(t). \quad (5.12)
\]

For \( n \geq 1 \) we have

\[
    \gamma_n(\beta(r)) = \frac{r^2}{2^2} \cdot \frac{3}{2} \cdot 4 \cdot \frac{3}{4^2} \cdot 5 \cdot \frac{4}{(n+1)^2} \cdot \ldots \cdot \frac{n(n+2)}{(n+1)^2}
\]

\[
    = \frac{(n+2)r^2}{2(n+1)} = \frac{r^2}{2} \cdot \frac{1}{n+1} + \frac{r^2}{2} = \int t^n d\eta(t).
\]

Thus, \( \eta \) is the Berger measure of \( W_{\beta(r)} \), so \( W_{\beta(r)} \) is subnormal (all \( r \in (0, 1] \)). \( \Box \)

**Lemma 5.12.** Let

\[
    \widehat{\beta}_n := \sqrt{\frac{(n+2)^2}{(n+3)(n+1)}} \quad (n \geq 1).
\]

Then \( \prod_{n=1}^{\infty} \widehat{\beta}_n = \sqrt{\frac{3}{2}} \). (Observe that \( \widehat{\beta}_n = \frac{1}{\beta_n} \) (all \( n \geq 1 \), if \( \beta_n \) is given by (5.11)).

**Proof.** Observe that

\[
    \prod_{n=1}^{k} (\widehat{\beta}_n)^2 = \prod_{n=1}^{k} \frac{(n+2)^2}{(n+3)(n+1)} = \frac{3(k+2)}{2(k+3)}
\]

which converge to \( \frac{3}{2} \) as \( k \to \infty \). \( \Box \)
Example 5.13. (Illustration of Theorem 5.10) We first recall the three assembly parts needed for a 2-variable weighted shift $T$ to be in $\mathcal{SFC}$: (i) a subnormal shift in the 0-th row ($\text{shift}(x_0, x_1, x_2 \ldots)$, with Berger measure $\xi$); (ii) a subnormal shift in the 0-th column ($\text{shift}(y_0, y_1, y_2, \ldots)$, with Berger measure $\eta$); and (iii) a positive number $a$ (the $\alpha_{01}$ weight). Toward (ii) we shall use the shift in Lemma 5.11, with Berger measure given by (5.12); toward (i) we shall use the measure
\[ \xi := \frac{1}{3}(\delta_0 + \delta_1) \]
on [0, 1], so that $p = q = \frac{1}{3}$; finally, toward (iii) we will keep $a$ as a parameter. The resulting 2-variable weighted shift will be denoted $T(a; r)$. We will now specify the values of $a$ and $r$ that make $T(a; r)$ contractive, hyponormal, and not subnormal. To guarantee that $T(a; r)$ is a pair of contractions, and using Lemma 5.12, it is easy to see that we need $a \leq \sqrt{\frac{2}{3}}$. Next, we observe that $x_0 = \sqrt{\frac{1}{2}}$, $x_1 = \sqrt{\frac{5}{6}}$, and $d\eta_1(t) = \frac{2}{3}[tdt + d\delta_1(t)]$ ($t \in [0, 1]$), so $\|T\|_{L^1(\eta_1)} = \frac{4}{3}$ and $y_1 = \sqrt{\frac{5}{3}}$. 

\[ \begin{array}{cccccc}
(0, n + 1) & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sqrt{\gamma_n(\eta_1)} & a & 1 & 1 & \cdots & 1 \\
\sqrt{(n+1)(n+3)} & 1 & 1 & \vdots & \vdots & \vdots \\
a \sqrt{\gamma_{n-1}(\eta_1)} & 1 & 1 & \cdots & 1 \\
(0, n) & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(0, 1) & \sqrt{\frac{\pi}{2}} & \frac{\sqrt{\pi}}{2} & \frac{\sqrt{\pi}}{2} & \frac{\sqrt{\pi}}{2} & \vdots \\
(0, 2) & \frac{\sqrt{\pi}}{3} & 1 & 1 & \vdots & \vdots \\
(0, 0) & \sqrt{\frac{1}{2}} & \sqrt{\frac{5}{6}} & \sqrt{\frac{5}{10}} & \sqrt{\frac{5}{14}} & \cdots & \sqrt{\frac{5}{\pi + 1}} \\
(1, 0) & \sqrt{\frac{1}{2}} & \sqrt{\frac{5}{6}} & \sqrt{\frac{5}{10}} & \sqrt{\frac{5}{14}} & \cdots & \sqrt{\frac{5}{\pi + 1}} \\
(2, 0) & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(n, 0) & \sqrt{\pi} & \sqrt{\frac{\pi}{2}} & \sqrt{\frac{\pi}{2}} & \sqrt{\frac{\pi}{2}} & \vdots \\
(n + 1, 0) & \frac{\sqrt{\pi}}{3} & 1 & 1 & \vdots & \vdots \\
\end{array} \]

Figure 10. Weight diagram of the 2-variable weighted shift in Example 5.13
Moreover, \( x_0 \sqrt{\frac{1-x_1^2}{1-x_0^2}} = \sqrt{\frac{1}{6}} \). By Theorem 5.10, we need to keep \( a \in (\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}) \). Thus, for \( a \in (\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}) \) we calculate

\[
h \equiv \frac{x_0^2 y_1^2 (a_1^2 - x_0^2)}{x_0^2 (x_1^2 - x_0^2) + (a_1^2 - x_0^2)^2} = \frac{2\sqrt{2}}{3 \sqrt{1 + 6(a_1^2 - \frac{1}{4})^2}}
\]

and

\[
s \equiv \min \left\{ \sqrt{\frac{q}{a^2}}, \sqrt{\frac{p}{\| T \|_{L^1(y_1)} - a^2}} \right\} = \min \left\{ \frac{1}{\sqrt{3a^2}}, \frac{1}{\sqrt{4 - 3a^2}} \right\} = \sqrt{\frac{1}{4 - 3a^2}}.
\]

Thus, for \( a \in (\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}) \) we can then choose \( y_0 \equiv \sqrt{\frac{3}{4}} r \) in the interval \( (\sqrt{\frac{1}{2}}, \sqrt{\frac{2\sqrt{2}}{3 \sqrt{1 + 6(a_1^2 - \frac{1}{4})^2}}}) \) and ensure that \( T(a; r) \) is hyponormal and not subnormal (cf. Figure 11). 

\[\Box\]

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**Figure 11.** Graphs of \( h \) and \( s \) on the interval \( [\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{2}}] \).

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**References**


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