k-HYPONORMALITY OF POWERS OF WEIGHTED SHIFTS VIA SCHUR PRODUCTS

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ABSTRACT. We characterize k-hyponormality and quadratic hyponormality of powers of weighted shifts using Schur product techniques.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. An operator $T\in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *subnormal* if T is the restriction of a normal operator (acting on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$) to an invariant subspace, and *hyponormal* if $T^*T \geq TT^*$.

The Bram-Halmos criterion for subnormality states that an operator is *subnormal* if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, x_2, \cdots, x_k \in \mathcal{H}$ ([Bra], [Con]). Using Choleski's Algorithm for operator matrices, it is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{1.1}$$

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. The notion of k-hyponormality has been introduced and studied in an attempt to bridge that gap ([Ath], [BEJ], [Cu2], [CMX], [JL], [McCP]). In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.1) for all k.

If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k-hyponormal* whenever the $k \times k$ operator matrix $M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$ is positive, or equivalently, the $(k + 1) \times (k + 1)$operator matrix (1.1) is positive, then the Bram-Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([CMX]).

Given a bounded sequence of positive numbers (called weights) $\alpha : \alpha_0, \alpha_1, \alpha_2, \alpha_3, \cdots$, the (unilateral) weighted shift W_α associated with α is the operator on

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$l^2(\mathbf{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $l^2(\mathbf{Z}_+)$. It is straightforward to check that W_α can never be normal, and that it is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The *moments* of α are usually defined by $\beta_0 := 1, \beta_{n+1} := \alpha_n \beta_n$ $(n \geq 0)$ ([Shi]); however, we will reserve this term for the sequence $\gamma_n := \beta_n^2$ $(n \geq 0)$. Berger's Theorem, which follows, states that W_α is subnormal if and only if the moments of α are the moments of a probability measure on $[0, \|W_\alpha\|^2]$.

Theorem 1.1. ***(Berger's Theorem*** [Con]*)* *W_α is subnormal if and only if there exists a Borel probability measure μ supported in $\left[0, \|W_\alpha\|^2\right]$, with $\|W_\alpha\|^2 \in \operatorname{supp} \mu$ and such that*

$$\gamma_n = \int t^n d\mu(t) \quad (\textit{all } n \geq 0).$$

In terms of k-hyponormality for weighted shifts, we will often use the following basic result.

Lemma 1.2. *(*[Cu1, Theorem 4]*)* *W_α is k-hyponormal if and only if the $(k+1) \times (k+1)$ Hankel matrices*

$$A_{n,k}(\alpha) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0) \tag{1.2}$$

are all nonnegative.

In this article we study k-hyponormality and quadratic hyponormality of powers of weighted shifts, using Schur product techniques. We characterize the k-hyponormality of powers of W_α in terms of the k-hyponormality of a finite collection of weighted shifts whose weight sequences are naturally derived from α. Similar techniques, when combined with the results in [BEJ], [Cu1], [CF2], [JP1] and [JP2], allow us to deal with back-step extensions of weighted shifts, and with weak k-hyponormality, including quadratic hyponormality.

2. k-HYPONORMALITY OF POWERS OF WEIGHTED SHIFTS

For matrices $A, B \in M_n(\mathbf{C})$, we let $A * B$ denote their Schur product. The followings result is well known.

Lemma 2.1. *(*[Pau]*)* *If $A \geq 0$ and $B \geq 0$, then $A * B \geq 0$.*

Definition 2.2. Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ and $\beta \equiv \{\beta_n\}_{n=0}^\infty$. The Schur product of α and β is defined by $\alpha\beta := \{\alpha_n \beta_n\}_{n=0}^\infty$.

Theorem 2.3. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ and $\beta \equiv \{\beta_n\}_{n=0}^\infty$ be two weight sequences, and assume that both W_α and W_β are k-hyponormal. Then $W_{\alpha\beta}$ is k-hyponormal.*

Proof. Let $\{\epsilon_n\}$ and $\{\eta_n\}$ be the moments of α and β, respectively. By hypothesis, $A_{n,k}(\alpha) \geq 0$ and $A_{n,k}(\beta) \geq 0$ (all $n \geq 0$). Since the corresponding moments γ_n of $\alpha\beta$ satisfy $\gamma_n = \epsilon_n \eta_n$ (all $n \geq 0$), it follows that $A_{n,k}(\alpha\beta) = A_{n,k}(\alpha) * A_{n,k}(\beta)$ (all $n \geq 0$). By Lemma 2.1, $A_{n,k}(\alpha\beta) \geq 0$ (all $n \geq 0$), so Lemma 1.2 now implies that $W_{\alpha\beta}$ is k-hyponormal. ◻

Corollary 2.4. *Let W_α and W_β be two weighted shifts, and assume that each is subnormal. Then $W_{\alpha\beta}$ is also subnormal.*

Proof. This is a straightforward application of the Bram-Halmos Criterion. $\square$

Definition 2.5. Given integers i and ℓ, with $\ell \geq 1$ and $0 \leq i \leq \ell - 1$, consider the decomposition $\mathcal{H} \equiv l^2(\mathbf{Z}_+) = \bigoplus_{j=0}^{\infty} \{e_j\}$, and define $\mathcal{H}_i := \bigoplus_{j=0}^{\infty} e_{\ell j+i}$. Moreover, for a weight sequence α let $\alpha(\ell : i) := \{ \prod_{m=0}^{\ell-1} \alpha_{\ell j+i+m} \}_{j=0}^{\infty}$. $\alpha(\ell : i)$ is the sequence of products of weights in adjacent packets of size ℓ, beginning with $\alpha_i \cdot ... \cdot \alpha_{i+\ell-1}$.

Example 2.6. Let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence. Then
(i) $\alpha(2:0) : \alpha_0\alpha_1, \alpha_2\alpha_3, \alpha_4\alpha_5, ...$
(ii) $\alpha(3:1) : \alpha_1\alpha_2\alpha_3, \alpha_4\alpha_5\alpha_6, \alpha_7\alpha_8\alpha_9, ...$
(iii) $\alpha(3:2) : \alpha_2\alpha_3\alpha_4, \alpha_5\alpha_6\alpha_7, \alpha_8\alpha_9\alpha_{10}, ...$

Proposition 2.7. *Let $\ell \geq 1$, let $0 \leq i \leq \ell - 1$, and let $\alpha(\ell : i)$ be as in Definition 2.5. Then $W_{\alpha(\ell:i)}$ is unitarily equivalent to $W_\alpha^\ell|_{\mathcal{H}_i}$. Therefore, $W_\alpha^{\,\ell}$ is unitarily equivalent to $\bigoplus_{i=0}^{\ell-1} W_{\alpha(\ell:i)}$.*

Proof. Since $W_\alpha^\ell e_{\ell j+i} = \prod_{m=0}^{\ell-1} \alpha_{\ell j+i+m} e_{\ell(j+1)+i}$, it is clear that $\mathcal{H}_i$ is an invariant subspace for W_α^ℓ. Moreover, $(W_\alpha^\ell)^* e_{\ell j+i} = \prod_{m=0}^{\ell-1} \alpha_{\ell(j-1)+i+m} e_{\ell(j-1)+i}$, so $\mathcal{H}_i$ is also invariant under $(W_\alpha^\ell)^*$. It follows that $\mathcal{H}_i$ is a reducing subspace for W_α^ℓ. If we now define a unitary operator $U : \mathcal{H} \longrightarrow \mathcal{H}_i$ by $U(e_j)=e_{\ell j+i}$, we see at once that $U^*(W_\alpha^\ell|_{\mathcal{H}_i})U=W_{\alpha(\ell:i)}$, as desired. $\square$

Corollary 2.8. *(a) W_α^ℓ is k-hyponormal $\Leftrightarrow$ $W_{\alpha(\ell:i)}$ is k-hyponormal for $0 \leq i \leq \ell - 1$.*
(b) W_α^ℓ is subnormal $\Leftrightarrow$ $W_{\alpha(\ell:i)}$ is subnormal for $0 \leq i \leq \ell - 1$.

Throughout the rest of this section, we assume that W_α is subnormal, with Berger measure μ. Observe that we can always write $\mu \equiv \nu + \rho\delta_0$ where $\nu(\{0\}) = 0$, and that W_α^ℓ is subnormal whenever W_α is subnormal. By Corollary 2.8, we know that each $W_{\alpha(\ell:i)}$ is subnormal, for $0 \leq i \leq \ell - 1$. We now seek to identify the Berger measures μ_i corresponding to each $W_{\alpha(\ell:i)}$.

Theorem 2.9. *(a) $d\mu_0(t) = d\mu(t^{1/\ell})$.*
(b) For $1 \leq i \leq \ell - 1$, $d\mu_i(t) = \frac{t^{i/\ell}}{\gamma_i} d\nu(t^{1/\ell})$.

Proof. Let γ_n be the moments of α $(n \geq 0)$. Then

$$\int t^n d\mu_0(t) = \gamma_{\ell n} = \int t^{\ell n} d\mu(t),$$

so $d\mu_0(t) = d\mu(t^{1/\ell})$. Similarly, for $1 \leq i \leq l - 1$,

$$\int t^n d\mu_i(t) = \frac{\gamma_{\ell n+i}}{\gamma_i} = \frac{1}{\gamma_i} \int t^{\ell n+i} d\nu(t),$$

so $d\mu_i(t) = \frac{t^{i/\ell}}{\gamma_i} d\nu(t^{1/\ell})$. $\square$

3. Back-step Extensions of Weighted Shifts

For a weight sequence α, we consider the *back-step extension* $\alpha(x) : x, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \cdots$ where $x > 0$.

Lemma 3.1. *Let W_α be a subnormal weighted shift with associated Berger measure μ.*
(a) (cf. [Cu1, Proposition 8]*) $W_{\alpha(x)}$ is subnormal if and only if (i) $\frac{1}{t} \in L^1(\mu)$ and (ii) $x^2 \leq (||\frac{1}{t}||_{L^1(\mu)})^{-1}$. In particular, $W_{\alpha(x)}$ is never subnormal when $\mu(\{0\}) > 0$.*
(b) if $x < (||\frac{1}{t}||_{L^1(\mu)})^{-1/2}$, the corresponding measure μ_x of $W_{\alpha(x)}$ satisfies $\mu_x(\{0\}) > 0$. In particular, $T := W_{\alpha(||\frac{1}{t}||^{-1/2}_{L^1(\mu)})}$ is the unique back-step extension of W_α with no mass at the origin.

Proof. (b) Let γ_n be the moments of T. Since T is subnormal, there exists a Berger measure ν such that

$$\gamma_n = \int t^n d\nu = \begin{cases} 1 & n = 0 \\ \frac{\int t^{n-1} d\mu}{\int \frac{1}{t} d\mu} & n \geq 1 \end{cases}.$$

Assume $x < (||\frac{1}{t}||_{L^1(\mu)})^{-1/2}$ and write $x = (1-\epsilon)(||\frac{1}{t}||_{L^1(\mu)})^{-1/2}$ for $0 < \epsilon < 1$. The moments η_n of $W_{\alpha(x)}$ are such that

$$\eta_n = \begin{cases} 1 & n = 0 \\ (1-\varepsilon)\gamma_n = \int t^n (1-\varepsilon) d\nu & n \geq 1 \end{cases}.$$

It follows that $\mu_x = (1-\epsilon)d\nu + \epsilon\delta_0$. □

Lemma 3.2. *Let W_α be a subnormal weighted shift, let $\ell \geq 1$, and let $k \geq 1$. The following statements are equivalent.*
(a) $W^\ell_{\alpha(x)}$ is k-hyponormal.
(b) $W_{\alpha(x)(\ell:0)}$ is k-hyponormal.

Proof. (a) $\Rightarrow$ (b). Straightforward from Corollary 2.8.
(b) $\Rightarrow$ (a). By Corollary 2.8(b), we know that $W_{\alpha(x)(\ell:i)} \equiv W_{\alpha(\ell:i-1)}$ is subnormal, and by Proposition 2.7, $W^\ell_{\alpha(x)} \cong \bigoplus_{i=0}^{\ell-1} W_{\alpha(x)(\ell:i)}$. It follows that for $1 \leq i \leq l-1$, $W_{\alpha(x)(\ell:i)}$ is k-hyponormal, which together with the assumption that $W_{\alpha(x)(\ell:0)}$ is k-hyponormal shows that $W^\ell_{\alpha(x)}$ is k-hyponormal. □

Theorem 3.3. *Let W_α be subnormal, with Berger measure $\mu \equiv \nu + \rho\delta_0$, and let $\ell \geq 1$. Then $W^\ell_{\alpha(x)}$ is subnormal if and only if $x \leq (||\frac{1}{t}||_{L^1(\nu)})^{-1/2}$. In particular, if $\rho = 0$, $W^\ell_{\alpha(x)}$ is subnormal if and only if $W_{\alpha(x)}$ is subnormal.*

Proof. It suffices to consider $W_{\alpha(x)(\ell:0)}$. Observe that $W_{\alpha(x)(\ell:0)}$ is a back-step extension of $W_{\alpha(\ell:\ell-1)}$. By Lemma 3.1, $W_{\alpha(x)(\ell:0)}$ is subnormal if and only if $x^2\gamma_{\ell-1} \leq (||\frac{1}{t}||_{L^1(\mu_{\ell-1})})^{-1} = \gamma_{\ell-1}(|||\frac{1}{t}||_{L^1(\nu)})^{-1}$. Therefore $W_{\alpha(x)(\ell:0)}$ is subnormal if and only if $x \leq (||\frac{1}{t}||_{L^1(\nu)})^{-1/2}$, as desired. □

Remark 3.4. Although for an operator T the subnormality of T^ℓ does not imply the subnormality of T, Theorem 3.3 shows that this is the case for back-step extensions of subnormal weighted shifts with Berger measures having no mass at the origin.

Theorem 3.5. *Let W_α be a subnormal weighted shift, with Berger measure μ. Then $W_{\alpha(x_n,x_{n-1},\cdots,x_1)}$ is subnormal if and only if*
(a) $\frac{1}{t^j} \in L^1(\mu)$ for all $1 \le j \le n$,
(b) $x_1 \cdots x_j = (\|\frac{1}{t^j}\|_{L^1(\mu)})^{-1/2}$ for $1 \le j \le n-1$ and $x_1 \cdots x_n \le (\|\frac{1}{t^n}\|_{L^1(\mu)})^{-1/2}$.

Proof. The case $n = 1$ was established in [Cu1, Proposition 8]. Here, and without loss of generality, we will only consider the case $n = 2$.
($\Rightarrow$) Assume that $W_{\alpha(x_2,x_1)}$ is subnormal. Since $W_{\alpha(x_1)}$ is a subnormal weighted shift possessing a subnormal extension (namely $W_{\alpha(x_2,x_1)}$), Lemma 3.1 implies that $x_1 = (\|\frac{1}{t}\|_{L^1(\mu)})^{-1/2}$. Moreover, since $W_{\alpha(x_2,x_1)}$ is subnormal, we must have $W^2_{\alpha(x_2,x_1)}$ subnormal, so Lemma 3.2 implies that $W_{\alpha(x_2,x_1)(2:0)} \equiv W_{\alpha(2:0)(x_2x_1)}$ is subnormal and $x_1x_2 \le \ (\|\frac{1}{t}\|_{L^1(\mu(t^{\frac{1}{2}}))})^{-1/2} = (\|\frac{1}{t^2}\|_{L^1(\mu)})^{-1/2}$.
($\Leftarrow$) Assume that (a) and (b) hold. Since $\frac{1}{t} \in L^1(\mu)$ and $x_1^2 = (\|\frac{1}{t}\|_{L^1(\mu)})$, we know that $W_{\alpha(x_1)}$ is subnormal with measure ν such that $\nu(\{0\}) = 0$. To check the subnormality of $W_{\alpha(x_2,x_1)} = W_{\alpha(x_1)(x_2)}$, by Theorem 3.3 it suffices to check the subnormality of $W^2_{\alpha(x_2,x_1)}$, and by Lemma 3.2, this reduces to verifying the subnormality of $W_{\alpha(x_2,x_1)(2:0)} = W_{\alpha(2:0)(x_2x_1)}$. If μ_1 denotes the Berger measure of $W_{\alpha(2:0)}$, that is, $d\mu_1(t) \equiv d\mu(t^{\frac{1}{2}})$, we know that $x_2x_1 \le (\|\frac{1}{t^2}\|_{L^1(\mu)})^{-1/2} = (\|\frac{1}{t}\|_{L^1(\mu_1)})^{-1/2}$. Therefore, we see that $W_{\alpha(2:0)(x_2x_1)}$ is subnormal, using Lemma 3.1. Thus, $W_{\alpha(x_2,x_1)(2:0)}$ is subnormal, as desired. □

4. Some Revealing Examples

Let $\alpha : \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ be a sequence of weights and let γ_n be the corresponding moments. For $x > 0$ let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be the associated back-step extension of α and assume that W_α is subnormal. It follows from [Cu1, Theorem 4] that $W_{\alpha(x)}$ is k-hyponormal if and only if

$$D_k := \begin{pmatrix} \frac{1}{x^2} & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-1} \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} \end{pmatrix} \ge 0.$$

Theorem 4.1. *For $\ell \ge 1$, $W^\ell_{\alpha(x)}$ is k-hyponormal if and only if*

$$D_{k;\ell} := \begin{pmatrix} \frac{1}{x^2} & \gamma_{\ell-1} & \gamma_{2\ell-1} & \cdots & \gamma_{k\ell-1} \\ \gamma_{\ell-1} & \gamma_{2\ell-1} & \gamma_{3\ell-1} & \cdots & \gamma_{(k+1)\ell-1} \\ \gamma_{2\ell-1} & \gamma_{3\ell-1} & \gamma_{4\ell-1} & \cdots & \gamma_{(k+2)\ell-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k\ell-1} & \gamma_{(k+1)\ell-1} & \gamma_{(k+2)\ell-1} & \cdots & \gamma_{2k\ell-1} \end{pmatrix} \ge 0.$$

Proof. It suffices to check that $W_{\alpha(x)(\ell:0)}$ is k-hyponormal. Now, the matrix detecting k-hyponormality for $W_{\alpha(x)(\ell:0)}$ is

$$D_k = x^2 \begin{pmatrix} \frac{1}{x^2} & \gamma_{\ell-1} & \gamma_{2\ell-1} & \cdots & \gamma_{k\ell-1} \\ \gamma_{\ell-1} & \gamma_{2\ell-1} & \gamma_{3\ell-1} & \cdots & \gamma_{(k+1)\ell-1} \\ \gamma_{2\ell-1} & \gamma_{3\ell-1} & \gamma_{4\ell-1} & \cdots & \gamma_{(k+2)\ell-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k\ell-1} & \gamma_{(k+1)\ell-1} & \gamma_{(k+2)\ell-1} & \cdots & \gamma_{2k\ell-1} \end{pmatrix} = x^2 D_{k;\ell},$$

so the result follows. ☐

Proposition 4.2. *For $\ell \geq 1$, let $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$.*
(1) $W^{\ell}_{\alpha(\sqrt{x})}$ is hyponormal $\Leftrightarrow$ $x \leq \frac{(\ell+1)^2}{2(2\ell+1)}$
(2) $W^{\ell}_{\alpha(\sqrt{x})}$ is 2-hyponormal $\Leftrightarrow$ $x \leq \frac{(\ell+1)^2(2\ell+1)^2}{2(3\ell+1)(4\ell^2+3\ell+1)}$.
(3) $W^{\ell}_{\alpha(\sqrt{x})}$ is subnormal $\Leftrightarrow$ $x \leq \frac{1}{2}$.

Proof. Observe that $\gamma_{k\ell-1} = \frac{2}{k\ell+1}$. Now consider

$$D_2 = \begin{pmatrix} \frac{1}{x} & \frac{2}{\ell+1} & \frac{2}{2\ell+1} \\ \frac{2}{\ell+1} & \frac{2}{2\ell+1} & \frac{2}{3\ell+1} \\ \frac{2}{2\ell+1} & \frac{2}{3\ell+1} & \frac{2}{4\ell+1} \end{pmatrix}.$$

By direct calculation we obtain

$$D_2 \geq 0 \iff x \leq \frac{(\ell+1)^2(2\ell+1)^2}{2(3\ell+1)(4\ell^2+3\ell+1)}.$$

Moreover, since W_α is subnormal, with measure $2tdt$ (in particular, with no mass at the origin), we see that $W^{\ell}_{\alpha(\sqrt{x})}$ is subnormal $\iff$ $W_{\alpha(\sqrt{x})}$ is subnormal. ☐

Corollary 4.3. *([Cu1, Proposition 7]) Let $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$.*
(1) $W_{\alpha(\sqrt{x})}$ is hyponormal $\Leftrightarrow$ $x \leq \frac{2}{3}$.
(2) $W_{\alpha(\sqrt{x})}$ is 2-hyponormal $\Leftrightarrow$ $x \leq \frac{9}{16}$.
(3) $W_{\alpha(\sqrt{x})}$ is subnormal $\Leftrightarrow$ $x \leq \frac{1}{2}$.

5. Quadratic hyponormality

We recall some terminology and notation from [Cu1], [CF2] and [CF3]. An operator T is said to be *quadratically hyponormal* if $T + sT^2$ is hyponormal for every $s \in \mathbf{C}$. Let W_α be a hyponormal weighted shift. For $s \in \mathbf{C}$, let $D(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]$, let P_n be the orthogonal projection onto $\bigvee_{i=0}^{n}\{e_i\}$, and let

$$D_n \equiv D_n(s) := P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n$$
$$= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix},$$

where $q_k := u_k + |s|^2 v_k$, $r_k := s\sqrt{w_k}$, $u_k := \alpha_k^2 - \alpha_{k-1}^2$, $v_k := \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2$, $w_k := \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2$ $(k \geq 0)$ and $\alpha_{-1} = \alpha_{-2} := 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbf{C}$ and every $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then it follows from [Cu1], [CF3] that $d_0 = q_0$, $d_1 = q_0q_1 - |r_0|^2$, and

$$d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2 d_n \qquad (n \geq 0),$$

and that d_n is actually a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin expansion $d_n(t) \equiv \sum_{i=0}^{n+1} c(n,i)t^i$. This gives at once the relations

$$\begin{array}{ccc} c(0,0) = u_0 & c(0,1) = v_0 & \\ c(1,0) = u_1 u_0 & c(1,1) = u_1 v_0 + u_0 v_1 - w_0 & c(1,2) = v_1 v_0 \end{array},$$

and

$$c(n+2,i) = u_{n+2}c(n+1,i) + v_{n+2}c(n+1,i-1) - w_{n+1}c(n,i-1)$$

$$(n \geq 0, 0 \leq i \leq n+1).$$

To detect the positivity of d_n, the following notion was introduced in [CF3].

Definition 5.1. We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$.

It is obvious that positive quadratic hyponormality implies quadratic hyponormality; moreover, quadratic hyponormality does not necessarily imply positive quadratic hyponormality [JP1].

Proposition 5.2. *With the above notation, assume that $u_{n+1}v_n \geq w_n$ $(n \geq 3)$. Then W_α is positively quadratically hyponormal if and only if $c(3,2) \geq 0$ and $c(4,3) \geq 0$.*

Proof. Immediate from [BEJ, Corollary 3.3 and Theorem 3.9]. □

Lemma 5.3. *Assume that W_α is subnormal and let $\ell \geq 1$, $k \geq 1$. The following statements are equivalent.*
(1) $W^\ell_{\alpha(x)}$ is weakly k-hyponormal.
(2) $W_{\alpha(x)(\ell:0)}$ is weakly k-hyponormal.

Proof. Imitate the proof of Lemma 3.2. □

Theorem 5.4. *Let $\alpha_n := \sqrt{\frac{n+2}{n+3}}$ $(n \geq 0)$, let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$, and let $\ell \geq 1$. Then $W_{\alpha(\sqrt{x})(\ell:0)}$ is positively quadratically hyponormal if and only if*

$$\left\{ \begin{array}{ll} x \leq \frac{(\ell+1)^2}{2(2\ell+1)} & \ell = 1,2 \\ x \leq \frac{(\ell+1)^2(1+7\ell+34\ell^2+44\ell^3)}{2(1+9\ell+45\ell^2+99\ell^3+94\ell^4)} & \ell \geq 3 \end{array} \right. . \tag{5.1}$$

Proof. Let $\beta_0 := \sqrt{\frac{2x}{\ell+1}}$ and $\beta_n := \sqrt{\frac{n\ell+1}{(n+1)\ell+1}}$ $(n \geq 1)$. Then $W_{\alpha(\sqrt{x})(\ell:0)} = W_\beta$. By direct calculation we see that

$$u_n = \beta_n^2 - \beta_{n-1}^2 = \frac{\ell^2}{((n+1)\ell+1)(n\ell+1)} \ \ (n \geq 2),$$

$$v_n = \beta_n^2\beta_{n+1}^2 - \beta_{n-1}^2\beta_{n-2}^2 = \frac{4\ell^2}{((n+2)\ell+1)(n\ell+1)} \ \ (n \geq 3)$$

and

$$w_n = \beta_n^2(\beta_{n+1}^2 - \beta_{n-1}^2)^2 = \frac{4\ell^2}{(n\ell+1)((n+1)\ell+1)((n+2)\ell+1)^2} \ \ (n \geq 2).$$

Since W_β has the property $u_{n+1}v_n \geq w_n \ \ (n \geq 3)$, by Proposition 5.2 it suffices to verify the nonnegativity of $c(3,2)$ and $c(4,3)$. By direct calculation,

$$c(3,2) \geq 0 \iff \frac{(\ell+1)^2(7+11\ell)}{4(3+10\ell+11\ell^2)}$$

and

$$c(4,3) \geq 0 \iff \frac{(\ell+1)^2(1+7\ell+34\ell^2+44\ell^3)}{2(1+9\ell+45\ell^2+99\ell^3+94\ell^4)}.$$

On the other hand, the hyponormality condition for W_β is $x \leq \frac{(\ell+1)^2}{2(2\ell+1)}$. Finally, observe that

$$\frac{(\ell+1)^2(7+11\ell)}{4(3+10\ell+11\ell^2)} \geq \frac{(\ell+1)^2}{2(2\ell+1)} \ \ (\text{all } \ell \geq 1),$$

$$\frac{(\ell+1)^2(1+7\ell+34\ell^2+44\ell^3)}{2(1+9\ell+45\ell^2+99\ell^3+94\ell^4)} \geq \frac{(\ell+1)^2}{2(2\ell+1)} \ \ (\text{if } \ell = 1,2),$$

and

$$\frac{(\ell+1)^2(1+7\ell+34\ell^2+44\ell^3)}{2(1+9\ell+45\ell^2+99\ell^3+94\ell^4)} \leq \frac{(\ell+1)^2}{2(2\ell+1)} \ \ (\text{if } \ell \geq 3).$$

This proves (5.1). □

Corollary 5.5. *Let* $\alpha_n := \sqrt{\frac{n+2}{n+3}}$ $(n \geq 0)$ *and* $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$.
(a) $W^2_{\alpha(\sqrt{x})}$ *is quadratically hyponormal* $\iff x \leq \frac{9}{10}$.
(b) If $\ell \geq 3$ *and* $x \leq \frac{(\ell+1)^2(1+7\ell+34\ell^2+44\ell^3)}{2(1+9\ell+45\ell^2+99\ell^3+94\ell^4)}$, *then* $W^\ell_{\alpha(\sqrt{x})}$ *is quadratically hyponormal.*

Remark 5.6. Let α be as in Corollary 5.5. Then $W^2_{\alpha(\sqrt{x})}$ is positively quadratically hyponormal if and only if $W^2_{\alpha(\sqrt{x})}$ is quadratically hyponormal if and only if $x \leq \frac{9}{10}$. Moreover, for $x = \frac{9}{10}$, $W^2_{\alpha(\sqrt{x})}$ has the first two weights equal, namely $\beta_0 = \beta_1 = \sqrt{\frac{3}{5}}$; this example resembles [Cu1, Proposition 7], where the first nontrivial quadratically hyponormal weighted shift with two equal weights appears. (For additional results along these lines, see [CJ].) Here we notice that for $x = \frac{9}{10}$, not only $W_{\alpha(\sqrt{x})}$ is quadratically hyponormal with two equal weights but also $W^2_{\alpha(\sqrt{x})}$ is quadratically hyponormal!

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