

SOLUTION OF THE QUADRATICALLY HYPONORMAL COMPLETION PROBLEM

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ABSTRACT. For $m \geq 1$, let $\alpha : \alpha_0 < \dots < \alpha_m$ be a collection of $(m + 1)$ positive weights. The *Quadratically Hyponormal Completion Problem* seeks necessary and sufficient conditions on α to guarantee the existence of a quadratically hyponormal unilateral weighted shift W with α as initial segment of weights. We prove that α admits a quadratically hyponormal completion if and only if the self-adjoint $m \times m$ matrix

$$D_{m-1}(s) := \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{m-2} & \bar{r}_{m-2} \\ 0 & 0 & 0 & \dots & r_{m-2} & q_{m-1} \end{pmatrix}$$

is positive and invertible, where $q_k := u_k + |s|^2 v_k$, $r_k := s\sqrt{w_k}$, $u_k := \alpha_k^2 - \alpha_{k-1}^2$, $v_k := \alpha_k^2 \alpha_{k+1}^2 - \alpha_{k-1}^2 \alpha_{k-2}^2$, $w_k := \alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)^2$, and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. As a particular case, this result shows that a collection of *four* positive numbers $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ *always* admits a quadratically hyponormal completion. This provides a new qualitative criterion to distinguish quadratic hyponormality from 2-hyponormality.

1. INTRODUCTION

Let \mathcal{H} be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on \mathcal{H} . An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if $\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \dots, x_k \in \mathcal{H}$ ([2],[5, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$(1.1) \quad \begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).$$

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Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T , while subnormality requires the validity of (1.1) for all k . Let $[A, B] := AB - BA$ denote the commutator of two operators A and B , and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$(1.2) \quad M_k(T) := ([T^{*j}, T^i]_{i,j=1}^k)$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1.1). The Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is *k-hyponormal* for every $k \geq 1$ ([17]).

Recall ([1],[6],[17]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \dots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e.,

$$(1.3) \quad (M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \quad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \dots, \lambda_k \in \mathbb{C} \text{ ([17]).}$$

If $k = 2$ then T is said to be *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that *k-hyponormal* \Rightarrow *weakly k-hyponormal*, but the converse is not true in general.

The classes of (weakly) *k-hyponormal* operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([8],[9],[11],[12],[13],[15],[17],[20],[26]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle; in fact, even subnormality for Toeplitz operators has not yet been characterized (cf.[7],[22]). For weighted shifts, positive results appear in [8] and [13], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [18] and [19]).

Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \dots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2 . It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. Weighted shifts provide a fertile ground to analyze the relative position of subnormality, polynomial hyponormality, *k-hyponormality* and weak *k-hyponormality*. In [13], the first named author and L. Fialkow applied the solution of the Subnormal Completion Problem [12] to build the first example of a one-parameter family of non-2-hyponormal quadratically hyponormal weighted shifts of recursive type. Additional examples were later found in [4],[14],[15] and [25], in some cases providing qualitative indicators to separate quadratically hyponormality from 2-hyponormality.

In this paper we obtain a new qualitative indicator, as a corollary to the Quadratically Hyponormal Completion Criterion. Our main result is a complete solution of the Quadratically Hyponormal Completion Problem:

Given $\alpha : \alpha_0 < \alpha_1 < \dots < \alpha_m$, find necessary and sufficient conditions for the existence of a quadratically hyponormal weighted shift whose first $(m + 1)$ weights are those in α .

As a special case, we show that given four weights $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ there always exists a quadratically hyponormal completion, thus providing a new criterion to distinguish between quadratic hyponormality and 2-hyponormality.

Definition 1.1. A unilateral weighted shift W_α is *flat* (or briefly, α is flat) if $\alpha_1 = \alpha_2 = \alpha_3 = \dots$.

J. Stampfli [28] showed that for subnormal weighted shifts W_α , a *propagation* phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. Later, A. Joshi proved in [24] that the shift with weights $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = \dots = b$, $0 < a < b$, is *not* quadratically hyponormal, and P. Fan [21] established that for $a = 1$, $b = 2$, and $0 < s < \sqrt{5}/5$, $W_\alpha + sW_\alpha^2$ is *not* hyponormal. On the other hand, it was shown in [8, Theorem 2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: *If W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \dots$, i.e., W_α is subnormal.* Furthermore, in [8, Proposition 11] it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force flatness, thereby subnormality.

Theorem 1.2 (Propagation). *Let W_α be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$.*

- (i) ([28, Theorem 6]) *Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \dots$.*
- (ii) ([8, Corollary 6]) *Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*
- (iii) ([3, Theorem 1]) *Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.*

An immediate consequence of Theorem 1.2 is that, for purposes of testing quadratic hyponormality, we can decompose all nonsubnormal weighted shifts into two classes: those with $\alpha_0 = \alpha_1$ and those with strictly increasing weight sequences.

To study quadratic hyponormality, we consider the selfcommutator $[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]$. For $s \in \mathbb{C}$, we write $D(s) := [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]$ and we let

$$(1.4) \quad D_n(s) := P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n = \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix},$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \dots, e_n\}$,

$$(1.5) \quad \begin{cases} q_n := u_n + |s|^2 v_n \\ r_n := s\sqrt{w_n} \\ u_n := \alpha_n^2 - \alpha_{n-1}^2 \\ v_n := \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_{n-2}^2 \\ w_n := \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} := 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then d_n satisfies the following 2-step recursive formula:

$$(1.6) \quad d_0 = q_0, \quad d_1 = q_0 q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n.$$

If we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n + 1$, and if we write $d_n \equiv \sum_{i=0}^{n+1} c(n, i) t^i$, then the coefficients $c(n, i)$ satisfy a double-indexed recursive formula, namely

$$(1.7) \quad \begin{aligned} c(n+2, i) &= u_{n+2} c(n+1, i) + v_{n+2} c(n+1, i-1) - w_{n+1} c(n, i-1), \\ c(n, 0) &= u_0 \cdots u_n, \quad c(n, n+1) = v_0 \cdots v_n, \quad c(1, 1) = u_1 v_0 + v_1 u_0 - w_0 \end{aligned}$$

($n \geq 0, i \geq 1$). We say that W_α is *positively quadratically hyponormal* if $c(n, i) \geq 0$ for every $n \geq 0, 0 \leq i \leq n+1$ (cf. [10]). Positive quadratic hyponormality implies quadratic hyponormality, but the converse is false (cf. [4]).

The idea of the proof of Theorem 1.2 (iii) is based on the following observation: if W_α is quadratically hyponormal with $\alpha_1 = \alpha_2 = 1$, then a straightforward calculation shows that

$$d_4(t) = \alpha_0^2 \alpha_4^2 (\alpha_0^2 - 1) (\alpha_3^2 - 1)^3 t^2 + c(4, 3) t^3 + c(4, 4) t^4 + c(4, 5) t^5,$$

so

$$\lim_{t \rightarrow 0^+} \frac{d_4(t)}{t^2} = \alpha_0^2 \alpha_4^2 (\alpha_0^2 - 1) (\alpha_3^2 - 1)^3 \geq 0,$$

which forces $\alpha_0 = 1$ or $\alpha_3 = 1$, so that three equal weights are present and hence by [8, Theorem 2], flatness occurs.

Note that in Theorem 1.2 (iii) the condition “ $n \geq 1$ ” cannot be relaxed to “ $n \geq 0$ ”. For example, if

$$(1.8) \quad \alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n \geq 2),$$

then W_α is quadratically hyponormal (cf. [8, Proposition 7]) but not cubically hyponormal (and hence not subnormal); indeed, if we let

$$C_5(t) := \det \left(P_5 [(W_\alpha + t W_\alpha^2 + t^2 W_\alpha^3)^*, (W_\alpha + t W_\alpha^2 + t^2 W_\alpha^3)] P_5 \right)$$

then

$$\lim_{t \rightarrow 0^+} \frac{C_5(t)}{t^8} = -\frac{1}{2041200} < 0.$$

We conclude this section by recalling that when $\alpha_0 = \alpha_1 = 1$, quadratic hyponormality implies

$$(1.9) \quad \alpha_2 < \sqrt{2} \quad \text{and} \quad \alpha_3 \geq (2 - \alpha_2^2)^{-2} \quad (\text{cf. [10, p.78]}).$$

2. RECURSIVELY GENERATED SHIFTS IN THE STUDY OF COMPLETIONS

If W_α is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the *moments* of W_α are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ ($n \geq 0$) [27]; however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2$ ($n \geq 0$). A criterion for k -hyponormality can be given in terms of these moments ([8, Theorem 4]): if we build a $(k+1) \times (k+1)$ Hankel matrix $A(n; k)$ by

$$A(n; k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0),$$

then W_α is k -hyponormal if and only if $A(n; k) \geq 0$ ($n \geq 0$). C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [23], [5, III.8.16]) states that W_α is subnormal if and only if there exists a Borel probability measure μ supported in $[0, \|W_\alpha\|^2]$, with $\|W_\alpha\|^2 \in \text{supp } \mu$, such that $\gamma_n = \int t^n d\mu(t)$ for all $n \geq 0$. Given an initial segment of weights $\alpha : \alpha_0, \dots, \alpha_m$, a sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ ($i = 0, \dots, m$) is said to be *recursively generated* by α if there exist $r \geq 1$ and $\varphi_0, \dots, \varphi_{r-1} \in \mathbb{R}$ such that $\gamma_{n+r} = \varphi_0 \gamma_n + \dots + \varphi_{r-1} \gamma_{n+r-1}$ (all $n \geq 0$), where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ ($n \geq 1$); in this case $W_{\hat{\alpha}}$ is said to be *recursively generated*. If we let $g(t) := t^r - (\varphi_{r-1} t^{r-1} + \dots + \varphi_0)$, then g has r distinct real roots $0 \leq s_0 < \dots < s_{r-1}$ ([12, Theorem 3.9]). Let

$$V := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ s_0 & s_1 & \cdots & s_{r-1} \\ \vdots & \vdots & & \vdots \\ s_0^{r-1} & s_1^{r-1} & \cdots & s_{r-1}^{r-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix} := V^{-1} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

If the associated recursively generated weighted shift $W_{\hat{\alpha}}$ is subnormal, then its Berger measure is of the form $\mu := \rho_0 \delta_{s_0} + \dots + \rho_{r-1} \delta_{s_{r-1}}$. Let $\alpha : \alpha_0, \dots, \alpha_m$ ($m \geq 0$) be an initial segment of positive weights and let $\omega = \{\omega_n\}_{n=0}^\infty$ be a bounded sequence of positive numbers. We say that W_ω is a *completion* of α if $\omega_n = \alpha_n$ ($0 \leq n \leq m$), and we write $\alpha \subset \omega$. The *completion problem* for a property (P) entails finding necessary and sufficient conditions on α to ensure the existence of a weight sequence $\omega \supset \alpha$ such that W_ω satisfies (P) . In 1966, Stampfli [28] showed that for arbitrary $\alpha_0 < \alpha_1 < \alpha_2$ there always exists a subnormal weighted shift W_α whose first *three* weights are $\alpha_0, \alpha_1, \alpha_2$; he also proved that given *four* or more weights, it may not be possible to find a subnormal completion. In [12, Theorem 3.5], the following criterion was established.

Subnormal Completion Criterion. *If $\alpha : \alpha_0, \dots, \alpha_m$ ($m \geq 0$) is an initial segment of positive weights then the following are equivalent:*

- (i) α has a subnormal completion;
- (ii) α has a recursively generated subnormal completion;
- (iii) α has an $(\lfloor \frac{m}{2} \rfloor + 1)$ -hyponormal completion;
- (iv) the Hankel matrices

$$A(k) := \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{pmatrix} \quad \text{and} \quad B(\ell - 1) := \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_\ell \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{\ell+1} \\ \vdots & \vdots & & \vdots \\ \gamma_\ell & \gamma_{\ell+1} & \cdots & \gamma_{2\ell-1} \end{pmatrix}$$

are both positive ($k := \lfloor \frac{m+1}{2} \rfloor$ and $\ell := \lfloor \frac{m}{2} \rfloor + 1$) and the vector

$$\begin{pmatrix} \gamma_{k+1} \\ \vdots \\ \gamma_{2k+1} \end{pmatrix} \quad (\text{resp.} \quad \begin{pmatrix} \gamma_{k+1} \\ \vdots \\ \gamma_{2k} \end{pmatrix})$$

is in the range of $A(k)$ (resp. $B(\ell - 1)$) when m is even (resp. odd).

Thus the Subnormal Completion Problem reduces to the *Recursive Completion Problem*, which entails finding necessary and sufficient conditions on α to ensure that the recursively generated weight sequence $\hat{\alpha}$ is well-defined and bounded.

Also in [12, Proposition 3.19], the following criterion on k -hyponormal completions was established.

k -Hyponormal Completion Criterion. *If $\alpha : \alpha_0, \dots, \alpha_{2m}$ ($m \geq 1$) is an initial segment of positive weights then for $1 \leq k \leq m$ the following are equivalent:*

- (i) α has a k -hyponormal completion;
- (ii) the Hankel matrix

$$A(j, k) := \begin{pmatrix} \gamma_j & \cdots & \gamma_{j+k} \\ \vdots & & \vdots \\ \gamma_{j+k} & \cdots & \gamma_{j+2k} \end{pmatrix}$$

is positive for all j , $0 \leq j \leq 2m - 2k + 1$, and the vector

$$\begin{pmatrix} \gamma_{2m-k+2} \\ \vdots \\ \gamma_{2m+1} \end{pmatrix}$$

is in the range of $A(2m - 2k + 2, k - 1)$.

If α admits a k -hyponormal completion, then it admits a recursively generated one.

3. THE QUADRATICALLY HYPONORMAL COMPLETION CRITERION

We now formulate and solve the corresponding problem for *quadratic hyponormality*.

Quadratically Hyponormal Completion Problem. *Given $\alpha : \alpha_0 < \alpha_1 < \dots < \alpha_m$, find necessary and sufficient conditions for the existence of a quadratically hyponormal weighted shift whose first $(m + 1)$ weights are given by α .*

We pause to recall that, for a square matrix A , the notation $A > 0$ means $A \geq 0$ and A invertible.

Theorem 3.1 (Quadratically Hyponormal Completion Criterion). *Let $m \geq 2$ and let $\alpha : \alpha_0 < \alpha_1 < \dots < \alpha_m$ be an initial segment of positive weights. Then the following statements are equivalent:*

- (i) α has a quadratically hyponormal completion;
- (ii) $D_{m-1}(t) > 0$ for all $t \geq 0$.

Moreover, a quadratically hyponormal completion ω of α can be obtained in the following recursively generated fashion:

$$\omega : \alpha_0, \dots, \alpha_{m-2}, (\alpha_{m-1}, \alpha_m, \alpha_{m+1})^\wedge,$$

where α_{m+1} is chosen so that $\alpha_{m+1}^2 > \max\{\alpha_m^2, \frac{\alpha_{m-1}^2}{\alpha_m^2} [M(\alpha_m^2 - \alpha_{m-2}^2)^2 + \alpha_{m-2}^2]\}$ ($M := \max_{t \in [0, \infty)} \frac{d_{m-2}(t)}{d_{m-1}(t)}$).

Proof. We will use the notation in Section 1. First of all, note that $D_{m-1}(t) > 0$ for all $t \geq 0$ if and only if $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \dots, m-1$; this follows from the Nested Determinants Test (see [12, Remark 2.4]) or Choleski's Algorithm (see [12, Proposition 2.3]). A straightforward calculation gives

$$\begin{aligned} d_0(t) &= \alpha_0^2 + \alpha_0^2 \alpha_1^2 t \\ d_1(t) &= \alpha_0^2(\alpha_1^2 - \alpha_0^2) + \alpha_0^2 \alpha_1^2(\alpha_2^2 - \alpha_0^2)t + \alpha_0^2 \alpha_1^4 \alpha_2^2 t^2 \\ d_2(t) &= \alpha_0^2(\alpha_1^2 - \alpha_0^2)(\alpha_2^2 - \alpha_1^2) + \alpha_0^2 \alpha_2^2(\alpha_1^2 - \alpha_0^2)(\alpha_3^2 - \alpha_1^2)t \\ &\quad + \alpha_0^2 \alpha_1^2 \alpha_2^2 \left\{ \alpha_3^2(\alpha_2^2 - \alpha_0^2) - \alpha_1^2(\alpha_1^2 - \alpha_0^2) \right\} t^2 + \alpha_0^2 \alpha_1^4 \alpha_2^2 (\alpha_2^2 \alpha_3^2 - \alpha_1^2 \alpha_0^2) t^3, \end{aligned}$$

which shows that all coefficients of d_i ($i = 0, 1, 2$) are positive, so that $d_i(t) > 0$ for all $t \geq 0$ and $i = 0, 1, 2$.

Now suppose α has a quadratically hyponormal completion. Then evidently, $d_n(t) \geq 0$ for all $t \geq 0$ and all $n \geq 0$. In view of Theorem 1.2 (iii), $\{\alpha_n\}_{n=m}^\infty$ is strictly increasing. Thus $d_n(0) = u_0 \cdots u_n = \prod_{i=0}^n (\alpha_i^2 - \alpha_{i-1}^2) > 0$ for all $n \geq 0$. If $d_{n_0}(t_0) = 0$ for some $t_0 > 0$ and the first such $n_0 > 0$ ($3 \leq n_0 \leq m-1$), then (1.6) implies that $0 \leq d_{n_0+1}(t_0) = -|r_{n_0}(t_0)|^2 d_{n_0-1}(t_0) \leq 0$, which forces $r_{n_0}(t_0) = 0$, so that $\alpha_{n_0+1} = \alpha_{n_0-1}$, a contradiction. Therefore $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \dots, m-1$. This proves the implication (i) \Rightarrow (ii).

For the reverse implication, we must find a bounded sequence $\{\alpha_n\}_{n=m+1}^\infty$ such that $d_n(t) \geq 0$ for all $t \geq 0$ and all $n \geq 0$. Suppose $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \dots, m-1$. We now claim that there exists a constant $M_k > 0$ for which

$$(3.1) \quad \frac{d_{k-1}(t)}{d_k(t)} \leq M_k \quad \text{for all } t \geq 0 \text{ and for } k = 1, \dots, m-1.$$

Indeed, since $\frac{d_{k-1}(t)}{d_k(t)}$ is a continuous function of t on $[0, \infty)$, and $\deg(d_{k-1}) < \deg(d_k)$, it follows that

$$\max_{t \in [0, \infty)} \frac{d_{k-1}(t)}{d_k(t)} \leq \max \left\{ 1, \max_{t \in [0, \xi]} \frac{d_{k-1}(t)}{d_k(t)} \right\} =: M_k,$$

where ξ is the largest root of the equation $d_{k-1}(t) = d_k(t)$. This gives (3.1). Now a straightforward calculation shows that

$$\begin{aligned} d_m(t) &= q_m(t)d_{m-1}(t) - |r_{m-1}(t)|^2 d_{m-2}(t) \\ &= \left[u_m + \left(v_m - w_{m-1} \frac{d_{m-2}(t)}{d_{m-1}(t)} \right) t \right] d_{m-1}(t). \end{aligned}$$

So if we write $e_m(t) := v_m - w_{m-1} \frac{d_{m-2}(t)}{d_{m-1}(t)}$, then by (3.1), $e_m(t) \geq v_m - w_{m-1} M_{m-1}$. Now choose α_{m+1} so that $v_m - w_{m-1} M_{m-1} > 0$, i.e.,

$$\alpha_{m+1}^2 > \max \left\{ \alpha_m^2, \frac{\alpha_{m-1}^2}{\alpha_m^2} [M (\alpha_m^2 - \alpha_{m-2}^2)^2 + \alpha_{m-2}^2] \right\},$$

where $M := \max_{t \in [0, \infty)} \frac{d_{m-2}(t)}{d_{m-1}(t)}$. Then $e_m(t) \geq 0$ for all $t \geq 0$, so that

$$d_m(t) = (u_m + e_m(t)t) d_{m-1}(t) \geq u_m d_{m-1}(t) > 0.$$

Therefore, $d_{m-1}(t) \leq \frac{d_m(t)}{u_m}$. With α_{m+2} to be chosen later, we now consider d_{m+1} . We have

$$\begin{aligned} d_{m+1}(t) &= q_{m+1}(t) d_m(t) - |r_m(t)|^2 d_{m-1}(t) \\ &\geq \frac{1}{u_m} \left[u_m q_{m+1}(t) - |r_m(t)|^2 \right] d_m(t) \\ &= \frac{1}{u_m} \left[u_m u_{m+1} + (u_m v_{m+1} - w_m) t \right] d_m(t) \\ &= u_{m+1} d_m(t) + \frac{t}{u_m} (u_m v_{m+1} - w_m) d_m(t). \end{aligned}$$

Write $f_{m+1} := u_m v_{m+1} - w_m$. If we choose α_{m+2} such that $f_{m+1} \geq 0$, then $d_{m+1}(t) \geq 0$ for all $t > 0$. In particular we can choose α_{m+2} so that $f_{m+1} = 0$. i.e., $u_m v_{m+1} = w_m$, or

$$\alpha_{m+2}^2 := \frac{\alpha_m^2 (\alpha_{m+1}^2 - \alpha_{m-1}^2)^2 + \alpha_{m-1}^2 \alpha_m^2 (\alpha_m^2 - \alpha_{m-1}^2)}{\alpha_{m+1}^2 (\alpha_m^2 - \alpha_{m-1}^2)},$$

or equivalently,

$$\alpha_{m+2}^2 := \alpha_{m+1}^2 + \frac{\alpha_{m-1}^2 (\alpha_{m+1}^2 - \alpha_m^2)^2}{\alpha_{m+1}^2 (\alpha_m^2 - \alpha_{m-1}^2)}.$$

In this case, $d_{m+1}(t) \geq u_{m+1} d_m(t) \geq 0$. Repeating the argument (with α_{m+3} to be chosen later), we obtain

$$\begin{aligned} d_{m+2}(t) &= q_{m+2}(t) d_{m+1}(t) - |r_{m+1}(t)|^2 d_m(t) \\ &\geq \frac{1}{u_{m+1}} \left[u_{m+1} q_{m+2}(t) - |r_{m+1}(t)|^2 \right] d_{m+1}(t) \\ &= \frac{1}{u_{m+1}} \left[u_{m+1} u_{m+2} + (u_{m+1} v_{m+2} - w_{m+1}) t \right] d_{m+1}(t) \\ &= u_{m+2} d_{m+1}(t) + \frac{t}{u_{m+1}} (u_{m+1} v_{m+2} - w_{m+1}) d_{m+1}(t). \end{aligned}$$

Write $f_{m+2} := u_{m+1} v_{m+2} - w_{m+1}$. If we choose α_{m+3} such that $f_{m+2} = 0$, i.e.,

$$\alpha_{m+3}^2 := \alpha_{m+2}^2 + \frac{\alpha_m^2 (\alpha_{m+2}^2 - \alpha_{m+1}^2)^2}{\alpha_{m+2}^2 (\alpha_{m+1}^2 - \alpha_m^2)},$$

then $d_{m+2}(t) \geq u_{m+2} d_{m+1}(t) \geq 0$. Continuing this process with the sequence $\{\alpha_n\}_{n=m+2}^\infty$ defined recursively by

$$\varphi_1 := \frac{\alpha_m^2(\alpha_{m+1}^2 - \alpha_{m-1}^2)}{\alpha_m^2 - \alpha_{m-1}^2}, \quad \varphi_0 := -\frac{\alpha_{m-1}^2 \alpha_m^2 (\alpha_{m+1}^2 - \alpha_m^2)}{\alpha_m^2 - \alpha_{m-1}^2}$$

and

$$(3.2) \quad \alpha_{n+1}^2 := \varphi_1 + \frac{\varphi_0}{\alpha_n^2} \quad (n \geq m+1),$$

we obtain that $d_n(t) \geq 0$ for all $t > 0$ and all $n \geq m+2$. On the other hand, by an argument of Stampfli [28, Theorem 5], the sequence $\{\alpha_n\}_{n=m+2}^\infty$ is bounded. Therefore, a quadratically hyponormal completion $\{\alpha_n\}_{n=0}^\infty$ is obtained. The recursive relation (3.2) shows that the sequence $\{\alpha_n\}_{n=m+2}^\infty$ is obtained recursively from α_{m-1} , α_m and α_{m+1} , that is, $\{\alpha_n\}_{n=m-1}^\infty = (\alpha_{m-1}, \alpha_m, \alpha_{m+1})^\wedge$ (see [13], [28]). This completes the proof. \square

Given *four* weights $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, it may not be possible to find a 2-hyponormal completion. In fact, by the preceding criterion for subnormal and k -hyponormal completions, the following statements are equivalent:

- (i) α has a subnormal completion;
- (ii) α has a 2-hyponormal completion;
- (iii) $\det \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix} \geq 0$.

By contrast, a quadratically hyponormal completion *always* exists for *four* weights.

Corollary 3.2. *For arbitrary $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, there always exists a quadratically hyponormal completion ω of α .*

Proof. In the proof of Theorem 3.1, we showed that $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, 1, 2$. Thus the result immediately follows from Theorem 3.1. \square

Remark 3.3. To discuss the hypothesis $\alpha_0 < \alpha_1 < \dots < \alpha_m$ in Theorem 3.1, we consider the case where $\alpha : \alpha_0, \alpha_1, \dots, \alpha_m$ admits equal weights:

- (i) If $\alpha_0 < \alpha_1 = \dots = \alpha_m$ then there exists a trivial quadratically hyponormal completion (in fact, a subnormal completion) $\omega : \alpha_0 < \alpha_1 = \dots = \alpha_n = \alpha_{n+1} = \dots$.
- (ii) If $\{\alpha_n\}_{n=0}^m$ is such that $\alpha_j = \alpha_{j+1}$ for some $j = 1, 2, \dots, m-1$, and $\alpha_j \neq \alpha_k$ for some $1 \leq j, k \leq m$, then in view of Theorem 1.2 (iii), there does not exist any quadratically hyponormal completion of α .
- (iii) If $\alpha_0 = \alpha_1$, the conclusion of Theorem 3.1 may fail: for example, if $\alpha : 1, 1, 2, 3$ then $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, 1, 2$, whereas α admits no quadratically hyponormal completion because by (1.9) we must have $\alpha_2^2 < 2$.

Problem 3.4. *Given $\alpha : \alpha_0 = \alpha_1 < \alpha_2 < \dots < \alpha_m$, find necessary and sufficient conditions for the existence of a quadratically hyponormal completion ω of α .*

In [14], related to Problem 3.4, weighted shifts of the form $1, (1, \sqrt{b}, \sqrt{c})^\wedge$ have been studied and their quadratic hyponormality completely characterized in terms of b and c .

Remark 3.5. In Theorem 3.1, the recursively quadratically hyponormal completion requires a sufficiently large α_{m+1} . One might conjecture that if a quadratically hyponormal completion of $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$ exists, then

$$\omega : \alpha_0, \dots, \alpha_{m-3}, (\alpha_{m-2}, \alpha_{m-1}, \alpha_m)^\wedge$$

is such a completion. However, if $\alpha : \sqrt{\frac{9}{10}}, \sqrt{1}, \sqrt{2}, \sqrt{3}$ then $\omega : \sqrt{\frac{9}{10}}, (\sqrt{1}, \sqrt{2}, \sqrt{3})^\wedge$ is not quadratically hyponormal (by [13, Theorem 4.3], [25, Theorem 4.6]), even though by Corollary 3.2 a quadratically hyponormal completion does exist.

We conclude this section by establishing that for *five* or more weights, the gap between 2-hyponormal and quadratically hyponormal completions can be extremal.

Proposition 3.6. *For $a < b < c$, let $\eta : (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ be a recursively generated weight sequence, and consider $\alpha(x) : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{x}, \eta_4$ (five weights). Then*

- (i) α has a subnormal completion $\iff x = \eta_3^2$;
- (ii) α has a 2-hyponormal completion $\iff x = \eta_3^2$;
- (iii) α has a quadratically hyponormal completion $\iff c < x < \eta_4^2$.

Proof. Assertions (i) and (ii) follow from the argument used in the proof of [16, Theorem 3.2]. For assertion (iii), observe that by Theorem 3.1, α has a quadratically hyponormal completion if and only if $d_3(t) > 0$ for all $t \geq 0$. Without loss of generality, we write $a = 1$, $b = 1 + r$, $c = 1 + r + s$, and $x = 1 + r + s + u$ ($r > 0$, $s > 0$, $u > 0$). A straightforward calculation using *Mathematica* shows that the Maclaurin coefficients $c(3, i)$ of $d_3(t)$ are given by

$$c(3, 0) = rsu;$$

$$c(3, 1) = s^3(r + s)(1 + r + s + u)(r + r^2 + 2rs + s^2)^{-1};$$

$$c(3, 2) = (1 + r + s)(s^4 + rsu + 4r^2su + 5r^3su + 2r^4su + 2rs^2u + 7r^2s^2u + 5r^3s^2u + 2s^3u + 4rs^3u + 4r^2s^3u + s^4u + rs^4u + r^2u^2 + 2r^3u^2 + r^4u^2 + 3r^2su^2 + 3r^3su^2 + 2rs^2u^2 + 3r^2s^2u^2 + s^3u^2 + rs^3u^2)(r + r^2 + 2rs + s^2)^{-1};$$

$$c(3, 3) = (1 + r)(r + s)(1 + r + s)(1 + r + s + u)(r^2s^2 + r^3s^2 + s^3 + 2rs^3 + 2r^2s^3 + s^4 + rs^4 + r^2u + 2r^3u + r^4u + 3r^2su + 3r^3su + 2rs^2u + 3r^2s^2u + s^3u + rs^3u)(r^2 + r^3 + 2r^2s + rs^2)^{-1}; \text{ and}$$

$$c(3, 4) = (1 + r)^2(1 + r + s)(r + r^2 + 2s + 2rs + s^2 + u + ru + su)(r^2s + 2r^3s + r^4s + rs^2 + 4r^2s^2 + 3r^3s^2 + s^3 + 3rs^3 + 3r^2s^3 + s^4 + rs^4 + r^2u + 2r^3u + r^4u + 3r^2su + 3r^3su + 2rs^2u + 3r^2s^2u + s^3u + rs^3u)(r^2 + r^3 + 2r^2s + rs^2)^{-1}.$$

This readily shows that for $c < x < \alpha_4^2$, all Maclaurin coefficients of $d_3(t)$ are positive, so that $d_3(t) > 0$ for all $t \geq 0$. Moreover if $x = c$ or α_4^2 then Theorem 1.2 shows that no quadratically hyponormal completion exists. This proves assertion (iii). \square

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