k-HYPONORMALITY OF FINITE RANK PERTURBATIONS OF UNILATERAL WEIGHTED SHIFTS

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Abstract. In this paper we explore finite rank perturbations of unilateral weighted shifts W_α. First, we prove that the subnormality of W_α is never stable under nonzero finite rank pertrubations unless the perturbation occurs at the zeroth weight. Second, we establish that 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n\,[(W_\alpha + sW_\alpha^2)^*,\, W_\alpha + sW_\alpha^2]\,P_n$ are nonnegative, for every $n \geq 0$, where P_n denotes the orthogonal projection onto the basis vectors $\{e_0, \cdots, e_n\}$. Finally, for α strictly increasing and W_α 2-hyponormal, we show that for a small finite-rank perturbation α' of α, the shift $W_{\alpha'}$ remains quadratically hyponormal.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2],[4, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad \text{(all } k \geq 1). \tag{1.1}$$

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires

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the validity of (1.1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{1.2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1.1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([16]).

Recall ([1],[16],[5]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([16])

$$(M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}. \tag{1.3}$$

If $k = 2$ then T is said to be *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal ⇒ weakly k-hyponormal, but the converse is not true in general.

The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([7],[8],[10], [11],[12],[14],[16], [19],[22]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle; in fact, even subnormality for Toeplitz operators has not been characterized (cf.[20], [6]). For weighted shifts, positive results appear in [7] and [12], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [17] and [18]).

In the present paper we renew our efforts to help describe the above mentioned gap between subnormality and hyponormality, with particular emphasis on polynomial hyponormality. We focus on the class of unilateral weighted shifts, and initiate a study of how the above mentioned notions behave under finite perturbations of the weight sequence. We first obtain three concrete results:

(i) the subnormality of W_α is never stable under nonzero finite rank perturbations unless the perturbation is confined to the zeroth weight (Theorem 2.1);

(ii) 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n \left[(W_\alpha + sW_\alpha^2)^*, \, W_\alpha + sW_\alpha^2 \right] P_n$ are nonnegative, for every $n \geq 0$, where P_n denotes the orthogonal projection onto the basis vectors $\{e_0, \cdots, e_n\}$ (Theorem 2.2); and

(iii) if α is strictly increasing and W_α is 2-hyponormal then for α' a small perturbation of α, the shift $W_{\alpha'}$ remains positively quadratically hyponormal (Theorem 2.3).

Along the way we establish two related results, each of independent interest:

(iv) an integrality criterion for a subnormal weighted shift to have an n-step subnormal extension (Theorem 6.1); and

(v) a proof that the sets of k-hyponormal and weakly k-hyponormal operators are closed in the strong operator topology (Proposition 6.7).

2. Statement of Main Results

C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [21], [4, III.8.16]) states that W_α is subnormal if and only if there exists a Borel probability measure μ supported in $[0,||W_\alpha||^2]$, with $||W_\alpha||^2 \in \operatorname{supp}\mu$, such that

$$\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \geq 0.$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots \alpha_m$, the sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ $(i = 0, \cdots, m)$ is said to be *recursively generated* by α if there exist $r \geq 1$ and $\varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R}$ such that

$$\gamma_{n+r} = \varphi_0\gamma_n + \cdots + \varphi_{r-1}\gamma_{n+r-1} \quad (\text{all } n \geq 0), \tag{2.1}$$

where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$. In this case $W_{\hat{\alpha}}$ with weights $\hat{\alpha}$ is said to be *recursively generated.* If we let

$$g(t) := t^r - \left(\varphi_{r-1}t^{r-1} + \cdots + \varphi_0\right), \tag{2.2}$$

then g has r distinct real roots $0 \leq s_0 < \cdots < s_{r-1}$ ([11, Theorem 3.9]). Let

$$V := \begin{pmatrix} 1 & 1 & \dots & 1 \\ s_0 & s_1 & \dots & s_{r-1} \\ \vdots & \vdots & & \vdots \\ s_0^{r-1} & s_1^{r-1} & \dots & s_{r-1}^{r-1} \end{pmatrix}$$

and let

$$\begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix} := V^{-1} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

If the associated recursively generated weighted shift $W_{\hat{\alpha}}$ is subnormal then its Berger measure is of the form

$$\mu := \rho_0\delta_{s_0} + \cdots + \rho_{r-1}\delta_{r-1}.$$

For example, given $\alpha_0 < \alpha_1 < \alpha_2$, $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is the recursive weighted shift whose weights are calculated according to the recursive relation

$$\alpha_{n+1}^2 = \varphi_1 + \varphi_0\frac{1}{\alpha_n^2}, \tag{2.3}$$

where

$$\varphi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \varphi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}. \tag{2.4}$$

In this case, $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is subnormal with 2–atomic Berger measure. Let $W_{x\,(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ denote the weighted shift whose weight sequence consists of the initial weight x followed by the weight sequence of $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$.

By the Density Theorem ([11, Theorem 4.2 and Corollary 4.3]), we know that if W_α is a subnormal weighted shift with weights $\alpha = \{\alpha_n\}$ and $\epsilon > 0$, then there exists a nonzero compact operator K with $||K|| < \epsilon$ such that $W_\alpha + K$ is a recursively generated subnormal weighted shift; in fact $W_\alpha + K = W_{\widehat{\alpha^{(m)}}}$ for some $m \geq 1$, where $\alpha^{(m)} : \alpha_0, \cdots, \alpha_m$. The following result shows that K cannot generally be taken to be finite rank.

Theorem 2.1 (Finite Rank Perturbations of Subnormal Shifts). *If W_α is a subnormal weighted shift then there exists no nonzero finite rank operator $F (\neq cP_{\{e_0\}})$ such that $W_\alpha + F$ is a subnormal weighted shift. Concretely, suppose W_α is a subnormal weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and assume $\alpha' = \{\alpha'_n\}$ is a nonzero perturbation of α in a finite number of weights except the initial weight; then $W_{\alpha'}$ is not subnormal.*

We next consider the selfcommutator $[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$. Let W_α be a hyponormal weighted shift. For $s \in \mathbb{C}$, we write

$$D(s) := [(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$$

and we let

$$D_n(s) := P_n[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]P_n = \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix}, \tag{2.5}$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$,

$$\begin{cases} q_n := u_n + |s|^2 v_n \\ r_n := s\sqrt{w_n} \\ u_n := \alpha_n^2 - \alpha_{n-1}^2 \\ v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2 \\ w_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases} \tag{2.6}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then d_n satisfies the following 2–step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0 q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n. \tag{2.7}$$

If we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$, and if we write $d_n \equiv \sum_{i=0}^{n+1} c(n,i) t^i$, then the coefficients $c(n,i)$ satisfy a double-indexed recursive formula, namely

$$\begin{aligned} c(n+2,i) &= u_{n+2}\,c(n+1,i) + v_{n+2}\,c(n+1,i-1) - w_{n+1}\,c(n,i-1), \\ c(n,0) &= u_0 \cdots u_n, \quad c(n,n+1) = v_0 \cdots v_n, \quad c(1,1) = u_1 v_0 + v_1 u_0 - w_0 \end{aligned} \tag{2.8}$$

($n \geq 0$, $i \geq 1$). We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for every $n \geq 0$, $0 \leq i \leq n+1$ (cf. [9]). Evidently, positively quadratically hyponormal $\Longrightarrow$ quadratically hyponormal. The converse, however, is not true in general (cf. [3]).

The following theorem establishes a useful relation between 2-hyponormality and positive quadratic hyponormality.

Theorem 2.2. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence and assume that W_α is 2-hyponormal. Then W_α is positively quadratically hyponormal. More precisely, if W_α is 2-hyponormal then*

$$c(n,i) \ \geq \ v_0 \cdots v_{i-1} u_i \cdots u_n \qquad (n \geq 0,\ 0 \leq i \leq n+1). \tag{2.9}$$

In particular, if α is strictly increasing and W_α is 2-hyponormal then the Maclaurin coefficients of $d_n(t)$ are positive for all $n \geq 0$.

If W_α is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$, then the *moments* of W_α are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ $(n \geq 0)$ [23]; however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2$ $(n \geq 0)$. A criterion for k-hyponormality can be given in terms of these moments ([7, Theorem 4]): if we build a $(k+1) \times (k+1)$ Hankel matrix $A(n;k)$ by

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0), \tag{2.10}$$

then

$$W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k) \geq 0 \quad (n \geq 0). \tag{2.11}$$

In particular, for α strictly increasing, W_α is 2-hyponormal if and only if

$$\det \begin{pmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} \\ \gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\ \gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4} \end{pmatrix} \geq 0 \quad (n \geq 0). \tag{2.12}$$

One might conjecture that if W_α is a k-hyponormal weighted shift whose weight sequence is strictly increasing then W_α remains weakly k-hyponormal under a small perturbation of the weight sequence. We will show below that this is true for $k = 2$ (Theorem 2.3).

In [12, Theorem 4.3], it was shown that the gap between 2-hyponormality and quadratic hyponormality can be detected by unilateral shifts with a weight sequence $\alpha : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$. In particular, there exists a maximum value $H_2 \equiv H_2(a,b,c)$ of x that makes $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ 2-hyponormal; H_2 is called the modulus of 2-hyponormality (cf. [12]). Any value of $x > H_2$ yields a non-2-hyponormal weighted shift. However, if $x - H_2$ is small enough, $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is still quadratically hyponormal. The following theorem shows that, more generally, for finite rank perturbations of weighted shifts with strictly increasing weight sequences, there always exists a gap between 2-hyponormality and quadratic hyponormality.

Theorem 2.3 (Finite Rank Perturbations of 2-hyponormal Shifts). *Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a strictly increasing weight sequence. If W_α is 2-hyponormal then W_α remains positively quadratically hyponormal under a small nonzero finite rank perturbation of α.*

3. PROOF OF THEOREM 2.1

Proof of Theorem 2.1. It suffices to show that if T is a weighted shift whose restriction to $\bigvee\{e_n, e_{n+1}, \cdots\}$ $(n \geq 2)$ is subnormal then there is at most one α_{n-1} for which T is subnormal.

Let $W := T|_{\bigvee\{e_{n-1},e_n,e_{n+1},\cdots\}}$ and $S := T|_{\bigvee\{e_n,e_{n+1},\cdots\}}$, where $n \geq 2$. Then W and S have weights $\alpha_k(W) := \alpha_{k+n-1}$ and $\alpha_k(S) := \alpha_{k+n}$ $(k \geq 0)$. Thus the corresponding moments are related by the equation

$$\gamma_k(S) = \alpha_n^2 \cdots \alpha_{n+k-1}^2 = \frac{\gamma_{k+1}(W)}{\alpha_{n-1}^2}.$$

We now adapt the proof of [7, Proposition 8]. Suppose S is subnormal with associated Berger measure μ. Then $\gamma_k(S) = \int_0^{||T||^2} t^k \, d\mu$. Thus W is subnormal if and only if there exists a probability measure ν on $[0, ||T||^2]$ such that

$$\frac{1}{\alpha_{n-1}^2} \int_0^{||T||^2} t^{k+1} \, d\nu(t) = \int_0^{||T||^2} t^k \, d\mu(t) \quad \text{for all } k \geq 0,$$

which readily implies that $t \, d\nu = \alpha_{n-1}^2 \, d\mu$. Thus W is subnormal if and only if the formula

$$d\nu := \lambda \cdot \delta_0 + \frac{\alpha_{n-1}^2}{t} d\mu \tag{3.1}$$

defines a probability measure for some $\lambda \geq 0$, where δ_0 is the point mass at the origin. In particular $\frac{1}{t} \in L^1(\mu)$ and $\mu(\{0\}) = 0$ whenever W is subnormal. If we repeat the above argument for W and $V := T|_{\bigvee\{e_{n-2},e_{n-1},\cdots\}}$, then we should have that $\nu(\{0\}) = 0$ whenever V is subnormal. Therefore we can conclude that if V is subnormal then $\lambda = 0$, and hence

$$d\nu = \frac{\alpha_{n-1}^2}{t} d\mu. \tag{3.2}$$

Thus we have

$$1 = \int_0^{||T||^2} d\nu(t) = \alpha_{n-1}^2 \int_0^{||T||^2} \frac{1}{t} d\mu(t),$$

so that

$$\alpha_{n-1}^2 = \left(\int_0^{||T||^2} \frac{1}{t} d\mu(t) \right)^{-1}, \tag{3.3}$$

which implies that α_{n-1} is determined uniquely by $\{\alpha_n, \alpha_{n+1}, \cdots\}$ whenever T is subnormal. This completes the proof. $\square$

Theorem 2.1 says that a nonzero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at the initial weight. However, this is not the case for k-hyponormality. To see this we use a close relative of the Bergman shift B_+ (whose weights are given by $\alpha = \{\sqrt{\frac{n+1}{n+2}}\}_{n=0}^\infty$); it is well known that B_+ is subnormal.

Example 3.1. *For $x > 0$, let T_x be the weighted shift whose weights are given by*

$$\alpha_0 := \sqrt{\frac{1}{2}}, \quad \alpha_1 := \sqrt{x}, \quad \textit{and} \quad \alpha_n := \sqrt{\frac{n+1}{n+2}} \ (n \geq 2).$$

Then we have:

(i) *T_x is subnormal $\Longleftrightarrow x = \frac{2}{3}$;*

(ii) *T_x is 2-hyponormal $\Longleftrightarrow \frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35}$.*

Proof. Assertion (i) follows from Theorem 2.1. For assertion (ii) we use (2.12): T_x is 2-hyponormal if and only if

$$\det \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2}x \\ \frac{1}{2} & \frac{1}{2}x & \frac{3}{8}x \\ \frac{1}{2}x & \frac{3}{8}x & \frac{3}{10}x \end{pmatrix} \geq 0 \quad \text{and} \quad \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2}x & \frac{3}{8}x \\ \frac{1}{2}x & \frac{3}{8}x & \frac{3}{10}x \\ \frac{3}{8}x & \frac{3}{10}x & \frac{1}{4}x \end{pmatrix} \geq 0,$$

or equivalently, $\frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35}$. □

For perturbations of recursive subnormal shifts of the form $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, subnormality and 2-hyponormality coincide.

Theorem 3.2. *Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be recursively generated by $\sqrt{a}, \sqrt{b}, \sqrt{c}$. If T_x is the weighted shift whose weights are given by $\alpha_x : \alpha_0, \cdots, \alpha_{j-1}, \sqrt{x}, \alpha_{j+1}, \cdots$, then we have*

$$T_x \textit{ is subnormal} \Longleftrightarrow T_x \textit{ is 2-hyponormal} \Longleftrightarrow \begin{cases} x = \alpha_j^2 & \textit{if } j \geq 1; \\ x \leq a & \textit{if } j = 0. \end{cases}$$

Proof. Since α is recursively generated by $\sqrt{a}, \sqrt{b}, \sqrt{c}$, we have that $\alpha_0^2 = a,\ \alpha_1^2 = b,\ \alpha_2^2 = c$,

$$\alpha_3^2 = \frac{b(c^2 - 2ac + ab)}{c(b-a)}, \quad \text{and} \quad \alpha_4^2 = \frac{bc^3 - 4abc^2 + 2ab^2c + a^2bc - a^2b^2 + a^2c^2}{(b-a)(c^2 - 2ac + ab)}. \tag{3.4}$$

Case 1 ($j = 0$): It is evident that T_x is subnormal if and only if $x \leq a$. For 2-hyponormality observe by (2.12) that T_x is 2-hyponormal if and only if

$$\det \begin{pmatrix} 1 & x & bx \\ x & bx & bcx \\ bx & bcx & \alpha_3^2 bcx \end{pmatrix} \geq 0,$$

or equivalently, $x \leq a$.

Case 2 ($j \geq 1$): Without loss of generality we may assume that $j = 1$ and $a = 1$. Thus $\alpha_1 = \sqrt{x}$. Then by Theorem 2.1, T_x is subnormal if and only if $x = b$. On the other hand, by (2.12), T_x is 2-hyponormal if and only if

$$\det \begin{pmatrix} 1 & 1 & x \\ 1 & x & cx \\ x & cx & \alpha_3^2 cx \end{pmatrix} \geq 0 \quad \text{and} \quad \det \begin{pmatrix} 1 & x & cx \\ x & cx & \alpha_3^2 cx \\ cx & \alpha_3^2 cx & \alpha_3^2 \alpha_4^2 cx \end{pmatrix} \geq 0.$$

Thus a direct calculation with the specific forms of α_3, α_4 given in (3.4) shows that T_x is 2-hyponormal if and only if $(x-b)\left(x - \frac{b(c^2-2c+b)}{b-1}\right) \leq 0$ and $x \leq b$. Since $b \leq \frac{b(c^2-2c+b)}{b-1}$, it follows that T_x is 2-hyponormal if and only if $x = b$. This completes the proof. □

4. Proof of Theorem 2.2

With the notation in (2.6), we let

$$p_n := u_n\, v_{n+1} - w_n \qquad (n \geq 0).$$

We then have:

Lemma 4.1. *If* $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ *is a strictly increasing weight sequence then the following statements are equivalent:*

(i) W_α *is 2-hyponormal;*

(ii) $\alpha_{n+1}^2(u_{n+1}+u_{n+2})^2 \leq u_{n+1}v_{n+2} \qquad (n \geq 0)$*;*

(iii) $\frac{\alpha_n^2}{\alpha_{n+2}^2}\frac{u_{n+2}}{u_{n+3}} \leq \frac{u_{n+1}}{u_{n+2}} \qquad (n \geq 0)$*;*

(iv) $p_n \geq 0 \quad (n \geq 0)$*.*

Proof. This follows from a straightforward calculation. □

Proof of Theorem 2.2. If α is not strictly increasing then α is flat, by the argument of [7, Corollary 6], i.e., $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$. Then

$$D_n(s) = \begin{pmatrix} \alpha_0^2+|s|^2\alpha_0^4 & \bar{s}\alpha_0^3 \\ s\alpha_0^3 & |s|^2\alpha_0^4 \end{pmatrix} \oplus 0_\infty \tag{4.1}$$

(cf. (2.5)), so that (2.9) is evident. Thus we may assume that α is strictly increasing, so that $u_n > 0$, $v_n > 0$ and $w_n > 0$ for all $n \geq 0$. Recall that if we write $d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i$ then the $c(n,i)$'s satisfy the following recursive formulas (cf. (2.8)):

$$c(n+2,i) = u_{n+2}\, c(n+1,i) + v_{n+2}\, c(n+1,i-1) - w_{n+1}\, c(n,i-1) \quad (n \geq 0,\ 1 \leq i \leq n). \tag{4.2}$$

Also, $c(n,n+1) = v_0 \cdots v_n$ (again by (2.8)) and $p_n := u_n v_{n+1} - w_n \geq 0$ $(n \geq 0)$, by Lemma 4.1. A straightforward calculation shows that

$$\begin{aligned} d_0(t) &= u_0 + v_0\, t; \\ d_1(t) &= u_0u_1 + (v_0u_1 + p_0)\, t + v_0v_1\, t^2; \\ d_2(t) &= u_0u_1u_2 + (v_0u_1u_2 + u_0p_1 + u_2p_0)\, t + (v_0v_1u_2 + v_0p_1 + v_2p_0)\, t^2 + v_0v_1v_2\, t^3. \end{aligned} \tag{4.3}$$

Evidently,

$$c(n,\, i) \geq 0 \qquad (0 \leq n \leq 2,\ 0 \leq i \leq n+1). \tag{4.4}$$

Define

$$\beta(n,i) := c(n,i) - v_0 \cdots v_{i-1}u_i \cdots u_n \qquad (n \geq 1,\ 1 \leq i \leq n).$$

For every $n \geq 1$, we now have

$$c(n,i) = \begin{cases} u_0 \cdots u_n \geq 0 & (i = 0) \\ v_0 \cdots v_{i-1}u_i \cdots u_n \ + \ \beta(n,i) & (1 \leq i \leq n) \\ v_0 \cdots v_n \geq 0 & (i = n+1). \end{cases} \tag{4.5}$$

For notational convenience we let $\beta(n,0) := 0$ for every $n \geq 0$.

Claim 1. *For $n \geq 1$,*

$$c(n,\, n) \geq u_n\, c(n-1,\, n) \geq 0. \tag{4.6}$$

Proof of Claim 1. We use mathematical induction. For $n = 1$,

$$c(1,1) = v_0 u_1 + p_0 \;\; \geq \;\; u_1\, c(0,1) \geq 0,$$

and

$$\begin{aligned} c(n+1,n+1) &= u_{n+1}\, c(n,n+1) + v_{n+1}\, c(n,n) - w_n c(n-1,n) \\ &\geq u_{n+1}\, c(n,n+1) + v_{n+1}\, u_n c(n-1,n) - w_n\, c(n-1,n) \quad \text{(by inductive hypothesis)} \\ &= u_{n+1}\, c(n,n+1) + p_n\, c(n-1,n) \\ &\geq u_{n+1}\, c(n,n+1), \end{aligned}$$

which proves Claim 1.

Claim 2. *For $n \geq 2$,*

$$\beta(n,\, i) \geq u_n\, \beta(n-1,i) \geq 0 \qquad (0 \leq i \leq n-1). \tag{4.7}$$

Proof of Claim 2. We use mathematical induction. If $n = 2$ and $i = 0$, this is trivial. Also,

$$\beta(2,1) = u_0\, p_1 + u_2\, p_0 = u_0\, p_1 + u_2\, \beta(1,1) \geq u_2\, \beta(1,1) \geq 0.$$

Assume that (4.7) holds. We shall prove that

$$\beta(n+1,\, i) \geq u_{n+1}\, \beta(n,i) \geq 0 \qquad (0 \leq i \leq n).$$

For,

$$\begin{aligned} \beta(n+1,i) + v_0 \cdots v_{i-1} u_i \cdots u_{n+1} &= c(n+1,i) \qquad \text{(by (4.2))} \\ &= u_{n+1} c(n,i) + v_{n+1} c(n,i-1) - w_n c(n-1,i-1) \\ &= u_{n+1} \Big(\beta(n,i) + v_0 \cdots v_{i-1} u_i \cdots u_n \Big) \\ &\quad + v_{n+1} \Big(\beta(n,i-1) + v_0 \cdots v_{i-2} u_{i-1} \cdots u_n \Big) \\ &\quad - w_n \Big(\beta(n-1,i-1) + v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1} \Big), \end{aligned}$$

so that

$$\begin{aligned} \beta(n+1,i) &= u_{n+1}\beta(n,i) + v_{n+1}\beta(n,i-1) - w_n\beta(n-1,i-1) \\ &\quad + v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1}\, (u_n v_{n+1} - w_n) \\ &= u_{n+1}\beta(n,i) + v_{n+1}\beta(n,i-1) - w_n\beta(n-1,i-1) + (v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1})\, p_n \\ &\geq u_{n+1}\beta(n,i) + v_{n+1} u_n \beta(n-1,i-1) - w_n\beta(n-1,i-1) \\ &\qquad \text{(by the inductive hypothesis and Lemma 4.1;} \\ &\qquad \text{observe that } i-1 \leq n-1, \text{ so (4.7) applies)} \\ &= u_{n+1}\beta(n,i) + p_n\, \beta(n-1,i-1) \\ &\geq u_{n+1}\, \beta(n,i), \end{aligned}$$

which proves Claim 2.

By Claim 2 and (4.5), we can see that $c(n,i) \geq 0$ for all $n \geq 0$ and $1 \leq i \leq n-1$. Therefore (4.4), (4.5), Claim 1 and Claim 2 imply

$$c(n,i) \;\geq\; v_0 \cdots v_{i-1} u_i \cdots u_n \qquad (n \geq 0,\ 0 \leq i \leq n+1).$$

This completes the proof. □

5. Proof of Theorem 2.3

To prove Theorem 2.3 we need:

Lemma 5.1 ([15, Lemma 2.3]). *Let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ be a strictly increasing weight sequence. If W_α is 2-hyponormal then the sequence of quotients*

$$\Theta_n := \frac{u_{n+1}}{u_{n+2}} \qquad (n \geq 0) \tag{5.1}$$

is bounded away from 0 and from ∞. More precisely,

$$1 \leq \Theta_n \leq \frac{u_1}{u_2}\left(\frac{||W_\alpha||^2}{\alpha_0\alpha_1}\right)^2 \qquad \text{for sufficiently large } n. \tag{5.2}$$

In particular, $\{u_n\}_{n=0}^{\infty}$ is eventually decreasing.

Proof of Theorem 2.3. By Theorem 2.2, W_α is *strictly* positively quadratically hyponormal, in the sense that all coefficients of $d_n(t)$ are *positive* for all $n \geq 0$. Note that finite rank perturbations of α affect a finite number of values of $u_n,\ v_n$ and w_n. More concretely, if α' is a perturbation of α in the weights $\{\alpha_0, \cdots, \alpha_N\}$, then $u_n,\ v_n,\ w_n$ and p_n are invariant under α' for $n \geq N+3$. In particular, $p_n \geq 0$ for $n \geq N+3$.

Claim 1. *For $n \geq 3,\ 0 \leq i \leq n+1$,*

$$\begin{aligned} c(n,i) =& u_n\, c(n-1,i) + p_{n-1}\, c(n-2,i-1) + \sum_{k=4}^{n} p_{k-2}\left(\prod_{j=k}^{n} v_j\right) c(k-3,\, i-n+k-2) \\ &+ v_n \cdots v_3\, \rho_{i-n+1}, \end{aligned} \tag{5.3}$$

where

$$\rho_{i-n+1} = \begin{cases} 0 & (i < n-1) \\ u_0 p_1 & (i = n-1) \\ v_0 p_1 + v_2 p_0 & (i = n) \\ v_0 v_1 v_2 & (i = n+1) \end{cases}$$

(cf. [12, Proof of Theorem 4.3]).

Proof of Claim 1. We use induction. For $n = 3,\ 0 \leq i \leq 4$,

$$\begin{aligned} c(3,i) &= u_3\, c(2,\, i) + v_3\, c(2,\, i-1) - w_2\, c(1,\, i-1) \\ &= u_3\, c(2,i) + v_3\Big(u_2\, c(1,i-1) + v_2\, c(1,\, i-2) - w_1\, c(0,\, i-2)\Big) - w_2\, c(1,i-1) \\ &= u_3\, c(2,i) + p_2\, c(1,i-1) + v_3\Big(v_2\, c(1,i-2) - w_1\, c(0,i-2)\Big) \\ &= u_3\, c(2,i) + p_2\, c(1,i-1) + v_3\, \rho_{i-2}, \end{aligned}$$

where by (4.3),

$$\rho_{i-2}=\begin{cases} 0 & (i<2)\\ u_0p_1 & (i=2)\\ v_0p_1+v_2p_0 & (i=3)\\ v_0v_1v_2 & (i=4).\end{cases}$$

Now,

$$\begin{aligned} c(n+1,i) &= u_{n+1}c(n,i)+v_{n+1}c(n,i-1)-w_nc(n-1,i-1)\\ &= u_{n+1}c(n,i)+v_{n+1}\Bigg(u_nc(n-1,i-1)+p_{n-1}c(n-2,i-2)\\ &\quad+\sum_{k=4}^{n}p_{k-2}\left(\prod_{j=k}^{n}v_j\right)c(k-3,i-n+k-3)+v_n\cdots v_3\rho_{i-n}\Bigg)-w_n\,c(n-1,i-1)\\ &= u_{n+1}c(n,i)+p_nc(n-1,i-1)+v_{n+1}p_{n-1}c(n-2,i-2)\\ &\quad+v_{n+1}\sum_{k=4}^{n}p_{k-2}\left(\prod_{j=k}^{n}v_j\right)c(k-3,i-n+k-3)+v_{n+1}\cdots v_3\rho_{i-n}\\ &\qquad\text{(by inductive hypothesis)}\\ &= u_{n+1}c(n,i)+p_nc(n-1,i-1)+\sum_{k=4}^{n+1}p_{k-2}\left(\prod_{j=k}^{n+1}v_j\right)c(k-3,i-n+k-3)\\ &\quad+v_{n+1}\cdots v_3\rho_{i-n}, \end{aligned}$$

which proves Claim 1.

Write $u'_n,\ v'_n, w'_n, p'_n, \rho'_n$, and $c'(\cdot,\cdot)$ for the entities corresponding to α'. If $p_n>0$ for every $n=0,\cdots,N+2$, then in view of Claim 1, we can choose a small perturbation such that $p'_n>0$ $(0\le n\le N+2)$ and therefore $c'(n,i)>0$ for all $n\ge 0$ and $0\le i\le n+1$, which implies that $W_{\alpha'}$ is also positively quadratically hyponormal. If instead $p_n=0$ for some $n=0,\cdots,N+2$, careful inspection of (5.3) reveals that without loss of generality we may assume $p_0=\cdots=p_{N+2}=0$. By Theorem 2.2, we have that for a sufficiently small perturbation α' of α,

$$c'(n,i)>0\ \ (0\le n\le N+2,\ 0\le i\le n+1)\quad\text{and}\quad c'(n,n+1)>0\ \ (n\ge 0). \tag{5.4}$$

Write

$$k_n:=\frac{v_n}{u_n}\qquad (n=2,3,\cdots).$$

Claim 2. *$\{k_n\}_{n=2}^{\infty}$ is bounded.*

Proof of Claim 2. Observe that

$$\begin{aligned} k_n=\frac{v_n}{u_n} &= \frac{\alpha_n^2\alpha_{n+1}^2-\alpha_{n-1}^2\alpha_{n-2}^2}{\alpha_n^2-\alpha_{n-1}^2}\\ &= \alpha_n^2+\alpha_{n-1}^2+\alpha_n^2\frac{\alpha_{n+1}^2-\alpha_n^2}{\alpha_n^2-\alpha_{n-1}^2}+\alpha_{n-1}^2\frac{\alpha_{n-1}^2-\alpha_{n-2}^2}{\alpha_n^2-\alpha_{n-1}^2}. \end{aligned} \tag{5.5}$$

Therefore if W_α is 2-hyponormal then by Lemma 5.1, the sequences

$$\left\{\frac{\alpha_{n+1}^2-\alpha_n^2}{\alpha_n^2-\alpha_{n-1}^2}\right\}_{n=2}^{\infty} \quad \text{and} \quad \left\{\frac{\alpha_{n-1}^2-\alpha_{n-2}^2}{\alpha_n^2-\alpha_{n-1}^2}\right\}_{n=2}^{\infty}$$

are both bounded, so that $\{k_n\}_{n=2}^{\infty}$ is bounded. This proves Claim 2.

Write $k := \sup_n k_n$. Without loss of generality we assume $k<1$ (this is possible from the observation that $c\alpha$ induces $\{c^2k_n\}$). Choose a sufficiently small perturbation α' of α such that if we let

$$h := \sup_{\substack{0\le \ell\le N+2\\ 0\le m\le 1}} \left| \sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+3} v'_j\right) c'(k-3,\,\ell) + v'_{N+3}\cdots v'_3\,\rho'_m \right| \tag{5.6}$$

then

$$c'(N+3,\,i) - \frac{1}{1-k}h \;>\; 0 \qquad (0\le i\le N+3) \tag{5.7}$$

(this is always possible because by Theorem 2.2, we can choose a sufficiently small $|p'_i|$ such that

$$c'(N+3,i) > v_0\cdots v_{i-1}u_i\cdots u_{N+3}-\epsilon \quad \text{and} \quad |h| < (1-k)\big(v_0\cdots v_{i-1}u_i\cdots u_{N+3}-\epsilon\big)$$

for any small $\epsilon>0$).

Claim 3. *For $j\ge 4$ and $0\le i\le N+j$,*

$$c'(N+j,\,i) \;\ge\; u_{N+j}\cdots u_{N+4}\left(c'(N+3,\;i) - \sum_{n=1}^{j-3} k^n\,h\right). \tag{5.8}$$

Proof of Claim 3. We use induction. If $j=4$ then by Claim 1 and (5.6),

$$\begin{aligned} c'(N+4,\,i) &= u'_{N+4}c'(N+3,i) + p'_{N+3}c'(N+2,i-1)\\ &\quad + v'_{N+4}\sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+3} v'_j\right) c'(k-3,i-N+k-6) + v'_{N+4}\cdots v'_3\rho'_{i-(N+3)}\\ &\ge u'_{N+4}c'(N+3,i) + p'_{N+3}c'(N+2,i-1) - v'_{N+4}h\\ &\ge u_{N+4}\big(c'(N+3,i)-k_{N+4}h\big)\\ &\ge u_{N+4}\big(c'(N+3,i)-k\,h\big) \end{aligned}$$

because $u'_{N+4}=u_{N+4},\ v'_{N+4}=v_{N+4}$ and $p'_{N+3}=p_{N+3}\ge 0$. Now suppose (5.8) holds for some $j\ge 4$. By Claim 1, we have that for $j\ge 4$,

$$\begin{aligned} c'(N+j+1,i) &= u'_{N+j+1}c'(N+j,\,i) + p'_{N+j}c(N+j-1,\,i-1)\\ &\quad + \sum_{k=4}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3,i-N+k-j-3) + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)}\\ &= u'_{N+j+1}c'(N+j,\,i) + p'_{N+j}c(N+j-1,\,i-1)\\ &\quad + \sum_{k=N+5}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3,i-N+k-j-3)\\ &\quad + \sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3,i-N+k-j-3) + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)}. \end{aligned}$$

Since $p'_n = p_n > 0$ for $n \geq N+3$ and $c'(n,\ell) > 0$ for $0 \leq n \leq N+j$ by the inductive hypothesis, it follows that

$$p'_{N+j}c(N+j-1,\, i-1) + \sum_{k=N+5}^{N+j+1} p'_{k-2} \left(\prod_{j=k}^{N+j+1} v'_j \right) c'(k-3, i-N+k-j-3) \geq 0. \tag{5.9}$$

By inductive hypothesis and (5.9),

$$\begin{aligned}
&c'(N+j+1,i)\\
&\geq u'_{N+j+1}c'(N+j,\, i) + \sum_{k=4}^{N+4} p'_{k-2} \left(\prod_{j=k}^{N+j+1} v'_j \right) c'(k-3, i-N+k-j-3) + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)}\\
&\geq u_{N+j+1}u_{N+j}\cdots u_{N+4} \left(c'(N+3,i) - \sum_{n=1}^{j-3} k^n h \right)\\
&\quad + v_{N+j+1}v_{N+j}\cdots v_{N+4} \left(\sum_{k=4}^{N+4} p'_{k-2} \left(\prod_{j=k}^{N+3} v'_j \right) c'(k-3, i-N+k-j-3) + v'_{N+3}\cdots v'_3\rho'_{i-(N+j)} \right)\\
&\geq u_{N+j+1}u_{N+j}\cdots u_{N+4} \left(c'(N+3,i) - \sum_{n=1}^{j-3} k^n h \right) - v_{N+j+1}v_{N+j}\cdots v_{N+4}\, h\\
&= u_{N+j+1}u_{N+j}\cdots u_{N+4} \left(c'(N+3,i) - \sum_{n=1}^{j-3} k^n h - k_{N+j+1}k_{N+j}\cdots k_{N+4}\, h \right)\\
&\geq u_{N+j+1}u_{N+j}\cdots u_{N+4} \left(c'(N+3,i) - \sum_{n=1}^{j-2} k^n h \right),
\end{aligned}$$

which proves Claim 3.

Since $\sum_{n=1}^{j} k^n < \frac{1}{1-k}$ for every $j > 1$, it follows from Claim 3 and (5.7) that

$$c'(N+j,\, i) > 0 \quad \text{for } j \geq 4 \text{ and } 0 \leq i \leq N+j. \tag{5.10}$$

It thus follows from (5.4) and (5.10) that $c'(n,i) > 0$ for every $n \geq 0$ and $0 \leq i \leq n+1$. Therefore $W_{\alpha'}$ is also positively quadratically hyponormal. This completes the proof. □

Corollary 5.2. *Let W_α be a weighted shift such that $\alpha_{j-1} < \alpha_j$ for some $j \geq 1$, and let T_x be the weighted shift with weight sequence*

$$\alpha_x : \alpha_0, \cdots, \alpha_{j-1}, x, \alpha_{j+1}, \cdots .$$

Then $\{x : T_x$ is 2-hyponormal$\}$ is a proper closed subset of $\{x : \ T_x$ is quadratically hyponormal$\}$ whenever the latter set is non-empty.

Proof. Write

$$H_2 := \{x : T_x \text{ is 2-hyponormal}\}.$$

Without loss of generality, we can assume that H_2 is non-empty, and that $j = 1$. Recall that a 2-hyponormal weighted shift with two equal weights is of the form $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$ or $\alpha_0 < \alpha_1 = \alpha_2 = \alpha_3 = \cdots$. Let $x_m := \inf H_2$. By Proposition 6.7 below, T_{x_m} is hyponormal. Then $x_m > \alpha_0$. By assumption, $x_m < \alpha_2$. Thus $\alpha_0, x_m, \alpha_2, \alpha_3, \cdots$ is strictly increasing. Now we apply Theorem 2.3 to obtain x' such that $\alpha_0 < x' < x_m$ and $T_{x'}$ is quadratically hyponormal. However $T_{x'}$ is not 2-hyponormal by the definition of x_m. The proof is complete. □

The following question arises naturally:

Question 5.3. *Let α be a strictly increasing weight sequence and let $k \geq 3$. If W_α is a k-hyponormal weighted shift, does it follow that W_α is weakly k-hyponormal under a small perturbation of the weight sequence?*

6. OTHER RELATED RESULTS

§6.1 Subnormal Extensions

Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence, let $x_i > 0$ for $1 \leq i \leq n$, and let $(x_n, \cdots x_1)\alpha : x_n, \cdots, x_1, \alpha_0, \alpha_1, \cdots$ be the augmented weight sequence. We say that $W_{(x_n,\cdots,x_1)\alpha}$ is an *extension* (or *n-step extension*) of W_α. Observe that

$$W_{(x_n,\cdots,x_1)\alpha}|_{\bigvee\{e_n,e_{n+1},\cdots\}} \cong W_\alpha.$$

The hypothesis $F \neq c\,P_{\{e_0\}}$ in Theorem 2.1 is essential. Indeed, there exist infinitely many one-step subnormal extension of a subnormal weighted shift whenever one such extension exists. Recall ([7, Proposition 8]) that if W_α is a weighted shift whose restriction to $\bigvee\{e_1, e_2, \cdots\}$ is subnormal with associated measure μ, then W_α is subnormal if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;

(ii) $\alpha_0^2 \leq \left(||\frac{1}{t}||_{L^1(\mu)}\right)^{-1}$.

Also note that there may not exist any one-step subnormal extension of the subnormal weighted shift: for example, if W_α is the Bergman shift then the corresponding Berger measure is $\mu(t) = t$, and hence $\frac{1}{t}$ is not integrable with respect to μ; therefore W_α does not admit any subnormal extension. A similar situation arises when μ has an atom at $\{0\}$.

More generally we have:

Theorem 6.1 (Subnormal Extensions). *Let W_α be a subnormal weighted shift with weights $\alpha : \alpha_0, \alpha_1, \cdots$ and let μ be the corresponding Berger measure. Then $W_{(x_n,\cdots,x_1)\alpha}$ is subnormal if and only if*

(i) $\frac{1}{t^n} \in L^1(\mu)$;

(ii) $x_j = \left(\frac{||\frac{1}{t^{j-1}}||_{L^1(\mu)}}{||\frac{1}{t^j}||_{L^1(\mu)}}\right)^{\frac{1}{2}}$ *for $1 \leq j \leq n-1$;*

(iii) $x_n \leq \left(\frac{||\frac{1}{t^{n-1}}||_{L^1(\mu)}}{||\frac{1}{t^n}||_{L^1(\mu)}}\right)^{\frac{1}{2}}$.

In particular, if we put

$$S := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : W_{(x_n,\cdots,x_1)\alpha} \ \ \text{is subnormal}\}$$

then either $S = \emptyset$ or S is a line segment in $\mathbb{R}^n$.

Proof. Write $W_j := W_{(x_n,\cdots,x_1)\alpha}|_{\bigvee\{e_{n-j},e_{n-j+1},\cdots\}}$ $(1 \leq j \leq n)$ and hence $W_n = W_{(x_n,\cdots,x_1)\alpha}$. By the argument used to establish (3.2) we have that W_1 is subnormal with associated measure ν_1 if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;
(ii) $d\nu_1 = \frac{x_1^2}{t} d\mu$, or equivalently, $x_1^2 = \left(\int_0^{||W_\alpha||^2} \frac{1}{t}\, d\mu(t)\right)^{-1}$.

Inductively W_{n-1} is subnormal with associated measure ν_{n-1} if and only if

(i) W_{n-2} is subnormal;
(ii) $\frac{1}{t^{n-1}} \in L^1(\mu)$;
(iii) $d\nu_{n-1} = \frac{x_{n-1}^2}{t} d\nu_{n-2} = \cdots = \frac{x_{n-1}^2 \cdots x_1^2}{t^{n-1}} d\mu$, or equivalently, $x_{n-1}^2 = \frac{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-2}}\, d\mu(t)}{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-1}}\, d\mu(t)}$.

Therefore W_n is subnormal if and only if

(i) W_{n-1} is subnormal;
(ii) $\frac{1}{t^n} \in L^1(\mu)$;
(iii) $x_n^2 \leq \left(\int_0^{||W_\alpha||^2} \frac{1}{t} d\nu_{n-1}\right)^{-1} = \left(\int_0^{||W_\alpha||^2} \frac{x_{n-1}^2 \cdots x_1^2}{t^n} d\mu(t)\right)^{-1} = \frac{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-1}}\, d\mu(t)}{\int_0^{||W_\alpha||^2} \frac{1}{t^n}\, d\mu(t)}$.

□

Corollary 6.2. *If W_α is a subnormal weighted shift with associated measure μ, there exists an n-step subnormal extension of W_α if and only if $\frac{1}{t^n} \in L^1(\mu)$.*

For the next result we refer to the notation in (2.1) and (2.2).

Corollary 6.3. *A recursively generated subnormal shift with $\varphi_0 \neq 0$ admits an n-step subnormal extension for every $n \geq 1$.*

Proof. The assumption about φ_0 implies that the zeros of $g(t)$ are positive, so that $s_0 > 0$. Thus for every $n \geq 1$, $\frac{1}{t^n}$ is integrable with respect to the corresponding Berger measure $\mu = \rho_0\delta_{s_0} + \cdots + \rho_{r-1}\delta_{s_{r-1}}$. By Corollary 6.2, there exists an n-step subnormal extension. □

We need not expect that for arbitrary recursively generated shifts, 2-hyponormality and subnormality coincide as in Theorem 3.2. For example, if $\alpha : \sqrt{\frac{1}{2}}, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$ then by (2.12) and Theorem 6.1,

(i) T_x is 2-hyponormal $\Longleftrightarrow 4 - \sqrt{6} \leq x \leq 2$;
(ii) T_x is subnormal $\Longleftrightarrow x = 2$.

A straightforward calculation shows, however, that T_x is 3-*hyponormal* if and only if $x = 2$; for,

$$A(0;3) := \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2}x & \frac{3}{2}x \\ \frac{1}{2} & \frac{1}{2}x & \frac{3}{2}x & 5x \\ \frac{1}{2}x & \frac{3}{2}x & 5x & 17x \\ \frac{3}{2}x & 5x & 17x & 58x \end{pmatrix} \geq 0 \iff x = 2.$$

This behavior is typical of general recursively generated weighted shifts: we show in [13] that subnormality is equivalent to k-hyponormality for some $k \geq 2$.

§6-2 Convexity and Closedness

Next, we will show that canonical rank-one perturbations of k-hyponormal weighted shifts which preserve k-hyponormality form a convex set. To see this we need an auxiliary result.

Lemma 6.4. *Let $I = \{1, \cdots, n\} \times \{1, \cdots, n\}$ and let J be a symmetric subset of I. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and let $C = (c_{ij}) \in M_n(\mathbb{C})$ be given by*

$$c_{ij} = \begin{cases} c\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \qquad (c > 0).$$

If A and C are positive semidefinite then $B = (b_{ij}) \in M_n(\mathbb{C})$ defined by

$$b_{ij} = \begin{cases} b\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \qquad (b \in [1, c] \text{ or } [c, 1])$$

is also positive semidefinite.

Proof. Without loss of generality we may assume $c > 1$. If $b = 1$ or $b = c$ the assertion is trivial. Thus we assume $1 < b < c$. The result is now a consequence of the following observation. If $[D]_{(i,j)}$ denotes the (i,j)-entry of the matrix D then

$$\begin{aligned} \left[\frac{c-b}{c-1}\left(A + \frac{b-1}{c-b}C\right)\right]_{(i,j)} &= \begin{cases} \frac{c-b}{c-1}\left(1 + \frac{b-1}{c-b}c\right) a_{ij} & \text{if } (i,j) \in J \\ \frac{c-b}{c-1}\left(1 + \frac{b-1}{c-b}\right) a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \\ &= \begin{cases} b\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \\ &= [B]_{(i,j)}, \end{aligned}$$

which is positive semidefinite because positive semidefinite matrices in $M_n(\mathbb{C})$ form a cone. □

An immediate consequence of Lemma 6.4 is that positivity of a matrix forms a convex set with respect to a fixed diagonal location; i.e., if

$$A_x = \begin{pmatrix} * & * & * \\ * & x & * \\ * & * & * \end{pmatrix}$$

then $\{x : A_x \text{ is positive semidefinite}\}$ is convex.

We now have:

Theorem 6.5. *Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence, let $k \geq 1$, and let $j \geq 0$. Define $\alpha^{(j)}(x) : \alpha_0, \cdots, \alpha_{j-1}, x, \alpha_{j+1}, \cdots$. Assume W_α is k-hyponormal and define*

$$\Omega_\alpha^{k,j} := \{x : \ W_{\alpha^{(j)}(x)} \text{ is } k\text{-hyponormal}\}.$$

Then $\Omega_{\alpha}^{k,j}$ is a closed interval.

Proof. Suppose $x_1, x_2 \in \Omega_{\alpha}^{k,j}$ with $x_1 < x_2$. Then by (2.11), the $(k+1)\times(k+1)$ Hankel matrix

$$A_{x_i}(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \qquad (n \geq 0;\ i = 1, 2)$$

is positive, where A_{x_i} corresponds to $\alpha^{(j)}(x_i)$. We must show that $tx_1 + (1-t)x_2 \in \Omega_{\alpha}^{k,j}$ $(0 < t < 1)$, i.e.,

$$A_{tx_1+(1-t)x_2}(n;k) \ \geq\ 0 \quad (n \geq 0,\ 0 < t < 1).$$

Observe that it suffices to establish the positivity of the $2k$ Hankel matrices corresponding to $\alpha^{(j)}(tx_1 + (1-t)x_2)$ such that $tx_1 + (1-t)x_2$ appears as a factor in at least one entry but not in every entry. A moment's thought reveals that without loss of generality we may assume $j = 2k$. Observe that

$$A_{z_1}(n;k) - A_{z_2}(n;k) = (z_1^2 - z_2^2)\, H(n;k)$$

for some Hankel matrix $H(n;k)$. For notational convenience, we abbreviate $A_z(n;k)$ as A_z. Then

$$A_{tx_1+(1-t)x_2} = \begin{cases} t^2 A_{x_1} + (1-t)^2 A_{x_2} + 2t(1-t) A_{\sqrt{x_1 x_2}} & \text{for } 0 \leq n \leq 2k \\ \left(t + (1-t)\frac{x_2}{x_1}\right)^2 A_{x_1} & \text{for } n \geq 2k+1. \end{cases}$$

Since $A_{x_1} \geq 0,\ A_{x_2} \geq 0$ and $A_{\sqrt{x_1 x_2}}$ have the form described by Lemma 6.4 and since $x_1 < \sqrt{x_1 x_2} < x_2$ it follows from Lemma 6.4 that $A_{\sqrt{x_1 x_2}} \geq 0$. Thus evidently, $A_{tx_1+(1-t)x_2} \geq 0$, and therefore $tx_1 + (1-t)x_2 \in \Omega_{\alpha}^{k,j}$. This shows that $\Omega_{\alpha}^{k,j}$ is an interval. The closedness of the interval follows from Proposition 6.7 below. ☐

In [17] and [18], it was shown that there exists a non-subnormal polynomially hyponormal operator. Also in [22], it was shown that there exists a non-subnormal polynomially hyponormal operator if and only if there exists one which is also a weighted shift. However, no concrete weighted shift has yet been found. As a strategy for finding such a shift, we would like to suggest the following:

Question 6.6. *Does it follow that the polynomial hyponormality of the weighted shift is stable under small perturbations of the weight sequence?*

If the answer to Question 6.6 were affirmative then we would easily find a polynomially hyponormal non-subnormal (even non-2-hyponormal) weighted shift; for example, if

$$\alpha : 1, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^{\wedge}$$

and T_x is the weighted shift associated with α, then by Theorem 3.2, T_x is subnormal $\Leftrightarrow x = 2$, whereas T_x is polynomially hyponormal $\Leftrightarrow 2 - \delta_1 < x < 2 + \delta_2$ for some $\delta_1, \delta_2 > 0$ provided the answer to Question 6.6 is yes; therefore for sufficiently small $\epsilon > 0$,

$$\alpha_\epsilon : 1, \sqrt{2+\epsilon}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^{\wedge}$$

would induce a non-2-hyponormal polynomially hyponormal weighted shift.

The answer to Question 6.6 for weak k-hyponormality is negative. In fact we have:

Proposition 6.7.

(i) *The set of k-hyponormal operators is sot-closed.*

(ii) *The set of weakly k-hyponormal operators is sot-closed.*

Proof. Suppose $T_\eta \in \mathcal{L}(\mathcal{H})$ and $T_\eta \to T$ in *sot*. Then, by the Uniform Boundedness Principle, $\{||T_\eta||\}_\eta$ is bounded. Thus $T_\eta^{*i}T_\eta^j \to T^{*i}T^j$ in *sot* for every i, j, so that $M_k(T_\eta) \to M_k(T)$ in *sot* (where $M_k(T)$ is as in (1.2)). (i) In this case $M_k(T_\eta) \geq 0$ for all η, so $M_k(T) \geq 0$, i.e., T is k-hyponormal.

(ii) Here, $M_k(T_\eta)$ is weakly positive for all η. By (1.3), $M_k(T)$ is also weakly positive, i.e., T is weakly k-hyponormal. □

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