Integr. equ. oper. theory 47 (2003) 307–314
0378-620X/030307-8, DOI 10.1007/s00020-002-1164-1

Integral Equations and Operator Theory

Weyl's Theorem for Algebraically Paranormal Operators

Raúl E. Curto and Young Min Han

Abstract. Let T be an algebraically paranormal operator acting on Hilbert space. We prove : (i) Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$; (ii) a-Browder's theorem holds for $f(S)$ for every $S \prec T$ and $f \in H(\sigma(S))$; (iii) the spectral mapping theorem holds for the Weyl spectrum of T and for the essential approximate point spectrum of T.

Mathematics Subject Classification (2000). Primary 47A10, 47A53; Secondary 47B20.

Keywords. Weyl's theorem, Browder's theorem, a-Browder's theorem, algebraically paranormal operator, single valued extension property.

1. Introduction

Throughout this note let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ we shall write $N(T)$ and $R(T)$ for the null space and range of T, respectively. Also, let $\alpha(T) := dimN(T)$, $\beta(T) := dimN(T^*)$, and let $\sigma(T)$, $\sigma_a(T)$ and $\pi_0(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite co-dimension. The *index* of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

T is called *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm "of finite ascent and descent:" equivalently ([Har2, Theorem 7.9.3]) if T is Fredholm and $T-\lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in \mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by ([Har1],[Har2])

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

The research of the first author was partially supported by NSF grants DMS-9800931 and DMS-0099357.

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$. If we write iso $K = K \setminus \operatorname{acc} K$ then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty \},$$

and

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T).$$

We say that *Weyl's theorem holds for* T if

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T), \tag{1.1}$$

and that Browder's theorem holds for T if

$$\sigma(T) \setminus \omega(T) = p_{00}(T). \tag{1.2}$$

In this note we investigate the validity of Weyl's theorem and of Browder's theorem for algebraically paranormal operators.

We consider the sets

$$\Phi_+(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : R(T) \text{ is closed and } \alpha(T) < \infty\},$$

$$\Phi_-(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : R(T) \text{ is closed and } \beta(T) < \infty\},$$

and

$$\Phi_+^-(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T \in \Phi_+(\mathcal{H}) \text{ and } i(T) \leq 0\}.$$

By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in \mathcal{K}(\mathcal{H})\}$$

is the *essential approximate point spectrum*, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in \mathcal{K}(\mathcal{H})\}$$

is the *Browder essential approximate point spectrum.*

In [Rak1, Theorem 3.1], it was shown that $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(\mathcal{H})\}$.

Example 1.1. Let $U_+ \in \mathcal{B}(\ell_2)$ be the unilateral shift. Then $\sigma_{ea}(U_+^*)$ is the closed unit disk.

Proof. Since $i(U_+^* - \bar{\lambda}) = -i(U_+ - \lambda) = 1$ for all $|\lambda| < 1$, $U_+ - \lambda \notin \Phi_+^-(l_2)$ whenever $|\lambda| < 1$. Since $\sigma_{ea}(U_+^*) \subseteq \sigma(U_+^*)$ and $\sigma_{ea}(U_+^*)$ is closed, $\sigma_{ea}(U_+^*)$ must be the closed unit disk. □

We say that a-Browder's theorem holds for T if

$$\sigma_{ea}(T) = \sigma_{ab}(T). \tag{1.3}$$

It is known ([DjHa],[HarLe]) that if $T \in \mathcal{B}(\mathcal{H})$ then

$$\begin{array}{ccc} \text{Weyl's theorem} & & \\ & \searrow & \\ & & \text{Browder's theorem} \\ & \nearrow & \\ a\text{-Browder's theorem} & & \end{array}$$

In [Wey], H. Weyl proved that (1.1) holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators ([Cob]), and to several classes of operators including seminormal operators ([Ber1],[Ber2]). Recently, the second named author and W.Y. Lee [HanLe] showed that Weyl's theorem holds for algebraically hyponormal operators. In this note, we extend this result to algebraically paranormal operators.

2. Weyl's Theorem for Algebraically Paranormal Operators

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *paranormal* if

$$||Tx||^2 \le ||T^2x||||x|| \quad \text{for all } x \in \mathcal{H}.$$

We say that T is *algebraically paranormal* if there exists a nonconstant complex polynomial p such that $p(T)$ is paranormal.

In general,

$$\text{hyponormal} \Rightarrow p\text{-hyponormal} \Rightarrow \text{paranormal} \Rightarrow \text{algebraically paranormal}. \tag{2.1}$$

Algebraic paranormality is preserved under translation by scalars and under restriction to invariant subspaces. Moreover, if T is paranormal and invertible then T^{-1} is paranormal. Indeed, given $x \in \mathcal{H}$ let $y := T^{-1}x$ and $z := T^{-1}y$, so $Tz = y$ and $T^2z = x$. Then

$$\begin{aligned} ||T^{-1}x||^2 = ||y||^2 = ||Tz||^2 \le ||T^2z||||z|| = ||x||||T^{-2}x|| \\ = ||(T^{-1})^2x||||x||. \end{aligned}$$

Before we state our main theorem (Theorem 2.4), we need some notation and three preliminary results.

We write $r(T)$ and $W(T)$ for the spectral radius and numerical range of T, respectively. It is well known that $r(T) \le ||T||$ and that $W(T)$ is convex with convex hull $\operatorname{conv}\sigma(T) \subseteq \overline{W(T)}$. T is called *convexoid* if $\operatorname{conv}\sigma(T) = \overline{W(T)}$, and *normaloid* if $r(T) = ||T||$.

Lemma 2.1. *Let T be a paranormal operator, $\lambda \in \mathbb{C}$, and assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Proof. We consider two cases:
Case I ($\lambda = 0$): Since T is paranormal, T is normaloid. Therefore $T = 0$.
Case II ($\lambda \neq 0$): Here T is invertible, and since T is paranormal, we see that T^{-1} is also paranormal. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $||T||||T^{-1}|| = |\lambda| \left|\frac{1}{\lambda}\right| = 1$. It follows from [Mla, Lemma 3] that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda$. □

In [DuDj], B.P. Duggal and S.V. Djordjević proved that quasinilpotent algebraically p-hyponormal operators are nilpotent, using the so-called Berberian extension. We now establish a similar result for algebraically paranormal operators; our proof uses different tools.

Lemma 2.2. *Let T be a quasinilpotent algebraically paranormal operator. Then T is nilpotent.*

Proof. Suppose that $p(T)$ is paranormal for some nonconstant polynomial p . Since $\sigma(p(T)) = p(\sigma(T))$, the operator $p(T) - p(0)$ is quasinilpotent. It follows from Lemma 2.1 that $c\, T^m(T-\lambda_1)\cdots(T-\lambda_n) \equiv p(T) - p(0) = 0$ (where $m \geq 1$). Since $T - \lambda_i$ is invertible for every $\lambda_i \neq 0$, we must have $T^m = 0$. □

It is well known that every paranormal operator is *isoloid* (cf. [ChRa]), that is, every isolated point in $\sigma(T)$ is an eigenvalue. We now extend this result to algebraically paranormal operators.

Lemma 2.3. *Let T be an algebraically paranormal operator. Then T is isoloid.*

Proof. Let $\lambda \in \text{iso}\, \sigma(T)$ and let $P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1} d\mu$ be the associated Riesz idempotent, where D is a closed disk centered at λ which contains no other points of $\sigma(T)$. We can then represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T is algebraically paranormal, $p(T)$ is paranormal for some nonconstant polynomial p. Since $\sigma(T_1) = \{\lambda\}$, we must have $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$. Therefore $p(T_1) - p(\lambda)$ is quasinilpotent. Since $p(T_1)$ is paranormal, it follows from Lemma 2.1 that $p(T_1) - p(\lambda) = 0$. Put $q(z) := p(z) - p(\lambda)$. Then $q(T_1) = 0$, and hence T_1 is algebraically paranormal. Since $T_1 - \lambda$ is quasinilpotent and algebraically paranormal, it follows from Lemma 2.2 that $T_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(T_1)$, and hence $\lambda \in \pi_0(T)$. This shows that T is isoloid. □

In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$. Also, we say that $T \in \mathcal{B}(\mathcal{H})$ has the single valued extension property (SVEP) if for every open set $U \subseteq \mathbb{C}$ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$.

Theorem 2.4. *Let T be an algebraically paranormal operator. Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. We first show that Weyl's theorem holds for T. Suppose that $\lambda \in \sigma(T) \setminus \omega(T)$. Then $T-\lambda$ is Weyl and not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume to the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ such that $\dim N(T-\mu) > 0$ for all $\mu \in U$. It follows from [Fin, Theorem 10] that T does not have SVEP. On the other hand, since $p(T)$ is paranormal for some nonconstant polynomial p, it follows from [ChRa, Corollary 2.10] that $p(T)$ has SVEP. Hence by [LaNe, Theorem 3.3.9], T has SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(T) \setminus \omega(T)$, and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(T)$.
Conversely, suppose that $\lambda \in \pi_{00}(T)$, with associated Riesz idempotent $P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk centered at λ which contains no other points of $\sigma(T)$. As before, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

We consider two cases:
Case I ($\lambda = 0$): Here T_1 is algebraically paranormal and quasinilpotent, so from Lemma 2.2 it follows that T_1 is nilpotent. We claim that $\dim R(P) < \infty$. For, if $N(T_1)$ were infinite dimensional, then $0 \notin \pi_{00}(T)$, a contradiction. Therefore T_1 is a finite dimensional operator, therefore Weyl. But since T_2 is invertible, we can conclude that T is Weyl. Thus $0 \in \sigma(T) \setminus \omega(T)$.
Case II ($\lambda \neq 0$): By the proof of Lemma 2.3, $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is a finite dimensional operator, so $T_1 - \lambda$ is Weyl. Since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl.
Thus Weyl's theorem holds for T. Next we claim that $f(\omega(T)) = \omega(f(T))$ for all $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since $\omega(f(T)) \subseteq f(\omega(T))$ with no other restriction on T, it suffices to show that $f(\omega(T)) \subseteq \omega(f(T))$. Suppose $\lambda \notin \omega(f(T))$. Then $f(T) - \lambda$ is Weyl and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)\cdots(T - \alpha_n)g(T), \tag{2.2}$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in C$ and $g(T)$ is invertible. Since the operators on the right-hand side of (2.2) commute, every $T-\alpha_i$ is Fredholm. Since T is algebraically paranormal, T has SVEP [ChRa, Corollary 2.10]. It follows from [AiMo, Theorem 2.6] that $i(T - \alpha_i) \leq 0$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda \notin f(\omega(T))$, and hence $f(\omega(T)) = \omega(f(T))$.
Now recall ([LeLe, Lemma]) that if T is isoloid then

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)) \quad \text{for every } f \in H(\sigma(T)).$$

Since T is isoloid (by Lemma 2.3) and Weyl's theorem holds for T,

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\omega(T)) = \omega(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. This completes the proof. □

From the proof of Theorem 2.4, we obtain the following useful consequence.

Corollary 2.5. *Let T be algebraically paranormal. Then*

$$\omega(f(T)) = f(\omega(T)) \text{ for every } f \in H(\sigma(T)).$$

3. a-Browder's Theorem for Algebraically Paranormal Operators

In general, we cannot expect that Weyl's theorem holds for operators having only SVEP. Consider the following example: let $T \in \mathcal{B}(l_2)$ be defined by

$$T(x_1, x_2, x_3, \cdots) := (\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots).$$

Then T is quasinilpotent, and so T has SVEP (in fact, T and T^* are decomposable). But $\sigma(T) = \omega(T) = \{0\}$ and $\pi_{00}(T) = \{0\}$, hence Weyl's theorem does not hold for T. However, a-Browder's theorem holds for T, as Theorem 3.3 below shows. We first need the following auxiliary result, essentially due to C.K. Fong [Fon]; for completeness, we include a proof. Recall that $X \in \mathcal{B}(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. $S \in \mathcal{B}(\mathcal{H})$ is said to be a *quasiaffine transform* of T (notation: $S \prec T$) if there is a quasiaffinity X such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are *quasisimilar*.

Lemma 3.1. *Suppose T has SVEP and $S \prec T$. Then S has SVEP.*

Proof. Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \longrightarrow \mathcal{H}$ be an analytic function such that $(S-\lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $S \prec T$, there exists a quasiaffinity X such that $XS = TX$. So $X(S-\lambda) = (T-\lambda)X$ for all $\lambda \in U$. Since $(S-\lambda)f(\lambda) = 0$ for all $\lambda \in U$, $0 = X(S-\lambda)f(\lambda) = (T-\lambda)Xf(\lambda)$ for all $\lambda \in U$. But T has SVEP, so $Xf(\lambda) = 0$ for all $\lambda \in U$. Since X is one-to-one, $f(\lambda) = 0$ for all $\lambda \in U$. Therefore S has SVEP. □

For $T \in \mathcal{B}(\mathcal{H})$, it is known that the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in H(\sigma(T))$ with no restrictions on T ([Rak2, Theorem 3.3]). The next theorem shows that for algebraically paranormal operators the spectral mapping theorem holds for the essential approximate point spectrum.

Theorem 3.2. *Assume that T or T^* is algebraically paranormal. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$.*

Proof. Let $f \in H(\sigma(T))$. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+^-(\mathcal{H})$ and

$$f(T) - \lambda = c(T-\alpha_1)(T-\alpha_2)\cdots(T-\alpha_n)g(T), \tag{3.1}$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$, and $g(T)$ is invertible. If T is algebraically paranormal, it follows from [AiMo, Theorem 2.6] that $i(T-\alpha_i) \le 0$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

Suppose now that T^* is algebraically paranormal. Then T^* has SVEP, and so by [AiMo, Theorem 2.8] $i(T-\alpha_i) \geq 0$ for each $i = 1, 2, \cdots, n$. Since

$$0 \leq \sum_{i=1}^{n} i(T-\alpha_i) = i(f(T)-\lambda) \leq 0,$$

$T-\alpha_i$ is Weyl for each $i = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_{ea}(T))$, and so $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. This completes the proof. □

Theorem 3.3. *Assume T has SVEP and let $S \prec T$. Then a-Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(S))$.*

Proof. We first show that a-Browder's theorem holds for S. It is well known that $\sigma_{ea}(S) \subseteq \sigma_{ab}(S)$. Conversely, suppose that $\lambda \in \sigma_a(S) \setminus \sigma_{ea}(S)$. Then $S-\lambda \in \Phi_+^-(\mathcal{H})$ and $S-\lambda$ is not bounded below. Since S has SVEP (by Lemma 3.1) and $S-\lambda \in \Phi_+^-(\mathcal{H})$, it follows from [AiMo, Theorem 2.6] that $S-\lambda$ has finite ascent. Therefore, by [Rak2, Theorem 2.1], $\lambda \in \sigma_a(S) \setminus \sigma_{ab}(S)$. Thus, a-Browder's theorem holds for S.
It now follows from Theorem 3.2 that

$$\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S))$$

(all $f \in H(\sigma(S))$), and so a-Browder's theorem holds for $f(S)$. □

Corollary 3.4. *Let T be an algebraically paranormal operator and let $S \prec T$. Then a-Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(S))$.*

Proof. Straightforward from Theorem 3.3 and the fact that algebraically paranormal operators have SVEP. □

References

[AiMo] P. Aiena and O. Monsalve, Operators which do not have the single valued extension property, J. Math. Anal. Appl. **250** (2000), 435–448.

[Ber1] S.K. Berberian, An extension of Weyl's theorem to a class of not necessarily normal operators, Michigan Math. J. **16** (1969), 273–279.

[Ber2] S.K. Berberian, The Weyl spectrum of an operator, Indiana Univ. Math. J. **20** (1970), 529–544.

[ChRa] N.N. Chourasia and P.B. Ramanujan, Paranormal operators on Banach spaces, Bull. Austral. Math. Soc. **21** (1980), 161–168.

[Cob] L.A. Coburn, Weyl's theorem for nonnormal operators, Michigan Math. J. **13** (1966), 285–288.

[CuDa] R.E. Curto and A.T. Dash, Browder spectral systems, Proc. Amer. Math. Soc. **103** (1988), 407–413.

[DjHa] S.V. Djordjević and Y.M. Han, Browder's theorems and spectral continuity, Glasgow Math. J. **42** (2000), 479-486.

[DuDj] B.P. Duggal and S.V. Djordjević, Weyl's theorem in the class of algebraically p-hyponormal operators, Comment. Math. Prace Mat. **40** (2000), 49–56.

[Fin] J.K. Finch, The single valued extension property on a Banach space, Pacific J. Math. **58** (1975), 61–69.

[Fon] C.K. Fong, Quasi-affine transforms of subnormal operators, Pacific J. Math. **70** (1977), 361–368.

[HanLe] Y.M. Han and W.Y. Lee, Weyl's theorem holds for algebraically hyponormal operators, Proc. Amer. Math. Soc. **128** (2000), 2291–2296.

[Har1] R.E. Harte, Fredholm, Weyl and Browder theory, Proc. Royal Irish Acad. **85A** (1985), 151–176.

[Har2] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Dekker, New York, 1988.

[HarLe] R.E. Harte and W.Y. Lee, Another note on Weyl's theorem, Trans. Amer. Math. Soc. **349** (1997), 2115–2124.

[Lau] K.B. Laursen, Operators with finite ascent, Pacific J. Math. **152** (1992), 323–336.

[LaNe] K.B. Laursen and M.M. Neumann, An Introduction to Local Spectral Theory, London Mathematical Society Monographs New Series 20, Clarendon Press, Oxford, 2000.

[LeLe] S.H. Lee and W.Y. Lee, A spectral mapping theorem for the Weyl spectrum, Glasgow Math. J. **38** (1996), 61–64.

[Mla] W. Mlak, Hyponormal contractions, Colloq. Math. **18** (1967), 137– 141.

[Rak1] V. Rakočević, On the essential approximate point spectrum II, Mat. Vesnik. **36** (1984), 89–97.

[Rak2] V. Rakočević, Approximate point spectrum and commuting compact perturbations, Glasgow Math. J. **28** (1986), 193–198.

[Wey] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollsteig ist, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

Raúl E. Curto
Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242-1419
USA
E-mail: curto@math.uiowa.edu

Young Min Han
Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242-1419
USA
E-mail: yhan@math.uiowa.edu

Submitted: November 7, 2002