Operators Cauchy Dual to 2-hyperexpansive Operators: The Multivariable Case

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Abstract. As a natural outgrowth of the work done in [18] and [21], we introduce an abstract framework to study generating m-tuples, and use it to analyze hypercontractivity and hyperexpansivity in several variables. These two notions encompass (joint) hyponormality and subnormality, as well as toral and spherical isometric-ness; for instance, the Drury-Arveson 2-shift is a spherical complete hyperexpansion. Our approach produces a unified theory that simultaneously covers toral and spherical hypercontractions (and hyperexpansions). As a byproduct, we arrive at a dilation theory for completely hypercontractive and completely hyperexpansive generating tuples. We can then analyze in detail the Cauchy duals of toral and spherical 2-hyperexpansive tuples. We also discuss various applications to the theory of hypercontractive and hyperexpansive tuples.

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Contents

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1. Introduction

The present paper is a sequel to the work initiated in [18] and continued in [21] on the operators Cauchy dual to 2-hyperexpansive operators. In this part, we carry out our investigations in the multivariable setup. In [18], the rich spectral theory of the Cauchy dual operators has been exploited to establish a Berger-Shaw theory for 2-hyperexpansive operators. On the other hand, in [21] it is shown that finer analysis of the Cauchy dual operators leads to Cowen-Douglas decompositions for certain unbounded 2-hyperexpansions. Over the last decade, the Cauchy dual operators have been developed as a powerful tool not just in the model theory of left-invertible operators but also in the function theory on the unit disc (see, for instance, [58], [59], [44], [18], [19] and [21]). Thus, it seems natural to consider and analyze the multivariable analogs of the notion of Cauchy dual, particularly in those cases when the underlying domains are the unit polydisc and the unit ball.

The paper is organized as follows. We first introduce the language of generating tuples, well-suited for handling, in a unified manner, operator tuples satisfying certain "toral" and "spherical" inequalities. The main examples of generating tuples include toral and spherical ones (see Example 2.2 below). Section 3 is devoted to the basic theory of the so-called completely hypercontractive generating tuples. These tuples are reminiscent of the subnormal tuples with constrained joint spectra. We mainly discuss the dilation theory for such tuples. In particular, we show that these generating tuples admit unique multiplicative dilations (Theorem 3.9); in this context, multiplicative generating tuples are the proper analogues of normal tuples. Among various applications, we characterize completely hypercontractive generating tuples by the contractivity of their minimal multiplicative dilations (Corollary 3.11). Although the results of Section 3 are not required in the remaining sections, we believe that hypercontractive generating tuples bear the same relation to hyperexpansive generating tuples as the subnormal contractive tuples to the completely hyperexpansive tuples. Thus, we consider Section 3 as a revisit to the theory of subnormal tuples (an antithesis of the subject under investigation), via the theory of generating tuples.

Section 4 is devoted to completely hyperexpansive generating tuples, which is a tractable subclass of 2-hyperexpansive generating tuples. The theory of completely hyperexpansive generating tuples may be regarded as an antithesis of that of completely hypercontractive generating tuples. In fact, this is one of the central themes in the recent investigations concerning the rich interplay between hypercontractive and hyperexpansive operators (see, for instance, [11], [12], [14], [58], [18], and [19]). As an interesting example, we note that there is a completely hyperexpansive spherical generating tuple associated with the Drury-Arveson 2-shift (refer to [31] and [4] for the basic theory of Drury-Arveson m-shifts; refer also to [5] and [6]).

We then proceed to obtain several elementary properties of such tuples. In particular, one may associate with a completely hyperexpansive generating tuple a canonical 1-variable weighted shift (see the discussion following Corollary 4.2). It turns out that, via this association, the spherical generating 1-tuple associated with the Drury-Arveson 2-shift leads to the Dirichlet shift! We obtain a dilation theorem for a subclass of completely hyperexpansive generating tuples which includes, in particular, 2-isometric ones (Theorem 4.6). Our proof of Theorem 4.6 is based on the Stinespring's Dilation Theorem and the Lévy-Khinchin representation for completely hyperexpansive generating tuples. This is the best possible result in the sense that a completely hyperexpansive generating tuple admits a multiplicative dilation if and only if it is isometric. We also mention that Theorem 4.6 is new even in the single variable case.

Essentially all the results in Sections 5 and 6 rely heavily on the properties of generating tuples derived in Section 4. In Sections 5 and 6 we introduce and discuss toral and spherical Cauchy dual tuples, respectively. Toral and spherical Cauchy dual tuples act naturally on the classes of toral and spherical 2-hyperexpansive operator tuples, respectively. We present several examples, and we also discuss toral and spherical counterparts of some of the results obtained in [18] and [21]. In particular, we show that the toral Cauchy dual to a toral 2-hyperexpansive tuple consists of *hyponormal* contractions. We use the properties of toral Cauchy dual tuples to obtain the *hypercyclicity* of certain scalar multiples, and a structure theorem for the C^*-algebra of toral 2-hyperexpansive multivariable weighted shifts. While we have obtained new structural results, the structure theory of spherical Cauchy dual tuples is rather mysterious and not well understood; we conclude the paper with a couple of open questions.

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2. Generating m-tuples: Abstract Framework

If $\mathbb{N}$ denotes the set of non-negative integers, we let $\mathbb{N}^m$ denote the cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times). Let $p \equiv (p_1, \cdots, p_m)$ and $n \equiv (n_1, \cdots, n_m)$ be in $\mathbb{N}^m$. We write $|p| := \sum_{i=1}^m p_i$ and $p \leq n$ if $p_i \leq n_i$ for $i = 1, \cdots, m$. For $p \leq n$, we let $\binom{n}{p} := \prod_{i=1}^m \binom{n_i}{p_i}$.

Given a Banach space $\mathcal{X}$, a tuple $T \equiv (T_1, \cdots, T_m)$ of bounded linear operators acting on $\mathcal{X}$, and $p \in \mathcal{N}^m$, we let $T^p := (T_1^{p_1}, \cdots, T_m^{p_m})$, where $T_i^{p_i}$ denotes the product of T_i times itself p_i times. Two specific instances of $\mathcal{X}$ are of central importance to us: $\mathcal{X} = \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, and $\mathcal{X} = B(\mathcal{H})$, the C^*-algebra of bounded linear operators on $\mathcal{H}$. When $T \equiv (T_1, \cdots, T_m)$ is a tuple of commuting bounded linear operators acting on $\mathcal{H}$, we let $T^* := (T_1^*, \cdots, T_m^*)$.

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space, let $B(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$, and let $Q \equiv (Q_1, \cdots, Q_m)$ be a commuting m-tuple of positive, bounded, linear operators acting on $B(\mathcal{H})$. We refer to Q as a generating m-tuple on $\mathcal{H}$.*

Example 2.2. *We are interested in the following choices of Q associated with a commuting m-tuple T of bounded linear operators $T_1, \cdots, T_m$ acting on $\mathcal{H}$:*

Toral Generating m-tuple: $Q_t := (Q_1, \cdots, Q_m)$, where $Q_i(X)$ $:= T_i^* X T_i$ $(X \in B(\mathcal{H}))$.

Spherical Generating 1-tuple: $Q_s(X) := \sum_{i=1}^m T_i^* X T_i$ $(X \in B(\mathcal{H}))$.
Further, we are interested in the multi-sequence $\{Q^n(I)\}_{n \in \mathbb{N}^m}$ associated with Q. Notice that the multi-sequences associated with Q_t and Q_s are $\{T^{*n} T^n\}_{n \in \mathbb{N}^m}$ and $\left\{\sum_{p \in \mathbb{N}^m, |p|=n} \frac{n!}{p!} T^{*p} T^p\right\}_{n \in \mathbb{N}}$, respectively. The reader will detect an immediate connection with J. Agler's hereditary functional calculus ([1], [8]*); in fact, the multi-sequences associated with the generating tuples Q_t and Q_s can be written as $\{(y_1 x_1, \cdots, y_m x_m)^n(T)\}_{n \in \mathbb{N}^m}$ and $\{(\sum_{i=1}^m y_i x_i)^n(T)\}_{n \in \mathbb{N}}$, respectively. On the other hand, and in view of the way Q_s and Q_t are defined, the theory of elementary operators is undoubtedly relevant here; we mention the pioneering work of L. Fialkow (*[32], [33], [34], [35]*), B. Magajna (*[45], [46], [47], [48]*) and M. Mathieu (*[49], [50], [51], [52], [53]*), and the subsequent contributions made in* [27] *and* [57]. □

One may raise the obvious question: Why generating m-tuples? Firstly, the setup of generating tuples makes possible a unified treatment of operator tuples satisfying certain "toral" and "spherical" inequalities. Moreover, this framework allows one to look at the operator tuples "related" to different domains in $\mathbb{C}^m$, for example, operator 2-tuples (T_1, T_2) related to

$$\left\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\right\},$$

where the generating 1-tuple associated with (T_1, T_2) is given by

$$Q(X) := T_1^* X T_1 + T_2^{*2} X T_2^2 \ (X \in B(\mathcal{H})).$$

To support the last assertion, let us discuss one more example of a generating tuple. For a 4-tuple (T_1, T_2, T_3, T_4), consider the generating 2-tuple (Q_1, Q_2) on $\mathcal{H}$ given by

$$Q_1(X) := T_1^* X T_1 + T_2^* X T_2, \ Q_2(X) := T_3^* X T_3 + T_4^* X T_4 \ (X \in B(\mathcal{H})).$$

Notice that (Q_1, Q_2) is related to the bi-ball

$$\left\{(z_1, z_2, w_1, w_2) \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 < 1, |w_1|^2 + |w_2|^2 < 1\right\}.$$

We believe that the theory of generating tuples provides, to a certain extent, a "coordinate-free" approach to multivariable operator theory. We regard it as a unified theory for operator tuples related to various domains. Although we are mainly interested in the operator tuples related to the unit polydisc $\mathbb{D}^m$ and the unit ball $\mathbb{B}_1^m$, the results of Sections 3 and 4 are quite general. We encourage the reader to specialize the results of these sections to the generating tuples discussed above.

Another motivation comes from the recent investigation [36] (which itself is motivated by the considerations of [4]) of operator tuples satisfying certain "spherical" equalities; for instance, spherical generating tuples Q_s satisfying $B_n(Q_s) = 0$ (see (3.1) below). We hope that the abstract framework of generating tuples may prove handy in dealing with such operator tuples as well, as it makes contact with the theory of completely positive maps.

For future reference, let us mention that generating tuples admit a *polynomial functional calculus.* For a finite set $\{c_\alpha\} \subseteq \mathbb{C}$ and a generating m-tuple R, let

$$p(R) := \sum_\alpha c_\alpha R^\alpha,$$

where $p(z) \equiv \sum_\alpha c_\alpha z^\alpha$. The polynomial functional calculus for R is given by $\Phi : p \longmapsto p(R)(I)$. We will show later that R actually admits a *continuous functional calculus*, under some additional positivity hypotheses. Further, in this case $[0,1]^m$ is a *spectral set* for R in the sense that the norm of Φ is less than or equal to 1 (Corollary 3.11).

3. Hypercontractive Generating Tuples

Let Q be a generating m-tuple on $\mathcal{H}$. For $n \in \mathbb{N}^m$, set

$$B_n(Q) := \sum_{p \in \mathbb{N}^m, 0 \le p \le n} (-1)^{|p|} \binom{n}{p} Q^p(I), \tag{3.1}$$

where $Q^0(I) := I$. We say that Q is *contractive* if $B_n(Q) \geq 0$ for all $n \in \mathbb{N}^m$ with $|n| = 1$; Q is said to be *completely hypercontractive* if

$$B_n(Q) \geq 0 \text{ for all } n \in \mathbb{N}^m.$$

Remark 3.1. *By* [16, Chapter 4, Proposition 6.11]*, a generating m-tuple Q on $\mathcal{H}$ is completely hypercontractive if and only if for every $h \in \mathcal{H}$ there is a positive Radon measure μ_h on $[0,1]^m$ such that*

$$\langle Q^n(I)h,\, h\rangle = \int_{[0,1]^m} t^n d\mu_h(t) \ (n \in \mathbb{N}^m). \tag{3.2}$$

It is clear from (3.2) *that μ_h is a finite Borel measure.*

Recall that a *semispectral measure* is a positive operator-valued measure defined on a Borel σ-algebra, which is countably additive in the weak operator topology.

Lemma 3.2. *A generating m-tuple Q on $\mathcal{H}$ is completely hypercontractive if and only if there exists a unique semispectral measure E on $[0,1]^m$ such that*

$$Q^n(I) = \int_{[0,1]^m} t^n dE(t) \ (\textit{for all } n \in \mathbb{N}^m). \tag{3.3}$$

In this case, there exist a Hilbert space $\mathcal{K}$, an isometry $R \in B(\mathcal{H}, \mathcal{K})$, and a commuting m-tuple A of positive contractions on $\mathcal{K}$ such that

$$Q^n(I) = R^* A^n R \ (\textit{for all } n \in \mathbb{N}^m). \tag{3.4}$$

Proof. The derivation of (3.3) is merely an adaptation of the proof of [39, Theorem 4.2] to the multivariable situation, via an appropriate version of the Polarization Identity. For the verification of (3.4), if F denotes the spectral dilation of E on $\mathcal{K}$ as guaranteed by the Naimark Dilation Theorem, then the commuting tuple $A \equiv (A_1, \cdots, A_m)$ of positive contractions given by $A_i := \int_{[0,1]^m} t_i dF(t)$ $(i = 1, \cdots, m)$ does the job. For details, the reader is referred to the proof of [39, Theorem 4.4]. Finally, the identity in (3.4) may also be deduced from Theorem 3.9 below. $\square$

An m-tuple $S \equiv (S_1, \cdots, S_m)$ of commuting operators in $\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an m-tuple $N \equiv (N_1, \cdots, N_m)$ of commuting normals in $\mathcal{B}(\mathcal{K})$ such that

$$N_i h = S_i h \text{ for every } h \in \mathcal{H} \text{ and } 1 \leq i \leq m.$$

For the definitions and the basic theory of various joint spectra (including the Taylor spectrum), the reader is referred to [25]. For an m-tuple $T \equiv (T_1, \cdots, T_m)$ of bounded linear operators on $\mathcal{H}$, we reserve the symbols $\sigma(T)$, $\sigma_p(T)$ and $\sigma_\ell(T)$ for the Taylor spectrum, point spectrum and left spectrum of T, respectively. It is well known that $\sigma_p(T) \subseteq \sigma_\ell(T) \subseteq \sigma(T)$, and that the spectral mapping theorem for polynomial mappings holds for both the Taylor and the left spectra.

The following two propositions yield some important examples of completely hypercontractive generating tuples.

Proposition 3.3. *(Toral Case) Let S denote a subnormal m-tuple on $\mathcal{H}$ with a normal extension N. Let Q_t, P_t denote the toral generating m-tuples associated with S and N, respectively (see Example* 2.2*). Then the following statements hold.*

(i) For all $n \in \mathbb{N}$, we have

$$Q_t^n(I)h = P_{\mathcal{H}} P_t^n(I)h \ (h \in \mathcal{H}). \tag{3.5}$$

(ii) If the Taylor joint spectrum $\sigma(N)$ of N is contained in the closed unit polydisc $\overline{\mathbb{D}}^m$ in $\mathbb{C}^m$, then Q_t is completely hypercontractive.

Proof. For a Hilbert space $\mathcal{H}$, we will denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, the inner-product and the norm on $\mathcal{H}$, respectively.

(i): This has been already noted in [10]. For completeness, we include the details here. For $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{N}^m$, notice that

$$\begin{aligned} \langle P_{\mathcal{H}} N^{*\alpha} N^\alpha x, \, y \rangle_{\mathcal{H}} &= \langle N^{*\alpha} N^\alpha x, \, y \rangle_{\mathcal{K}} = \langle N^\alpha x, \, N^\alpha y \rangle_{\mathcal{K}} \\ &= \langle S^\alpha x, \, S^\alpha y \rangle_{\mathcal{H}} = \langle S^{*\alpha} S^\alpha x, \, y \rangle_{\mathcal{H}} . \end{aligned}$$

The desired conclusion is now immediate.

(ii): Let B_n be as in (3.1). Note that $P_t(I) := (P_1(I), \cdots, P_m(I))$ is a commuting m-tuple since N is doubly commuting (by Fuglede's Theorem).

For $h \in \mathcal{H}$, notice that, in view of (3.5),

$$\begin{aligned}\langle B_n(Q_t)h,\, h\rangle &= \sum_{p\in\mathbb{N}^m, 0\le p\le n} (-1)^{|p|}\binom{n}{p}\langle P_{\mathcal{H}}P_t^p(I)h,\, h\rangle_{\mathcal{H}} \\ &= \sum_{p\in\mathbb{N}^m, 0\le p\le n} (-1)^{|p|}\binom{n}{p}\langle P_t(I)^p h,\, h\rangle_{\mathcal{K}} \\ &= \left\langle \prod_{j=1}^m (I-P_j(I))^{n_j}h,\, h\right\rangle_{\mathcal{K}}. \qquad (3.6)\end{aligned}$$

The expression in (3.6) is positive in view of the Spectral Theorem provided $\sigma(N) \subseteq \overline{\mathbb{D}}^m$. □

Proposition 3.4. *(Spherical Case) Let S denote a subnormal m-tuple on $\mathcal{H}$ with a normal extension N. Let Q_s, P_s denote the spherical generating 1-tuples associated with S and N respectively (see Example 2.2). Then the following statements hold:*
(i) For all $n \in \mathbb{N}$, we have

$$Q_s^n(I)h = P_{\mathcal{H}}P_s^n(I)h \ (h \in \mathcal{H}). \qquad (3.7)$$

(ii) If the Taylor joint spectrum $\sigma(N)$ of N is contained in the closed unit ball $\overline{\mathbb{B}_1^m}$ in $\mathbb{C}^m$, then Q_s is completely hypercontractive.

Proof. This is straightforward from the proof of Proposition 3.3, properly adapted to spherical generating tuples. □

Recall that a commuting m-tuple T of bounded linear operators on $\mathcal{H}$ is a *toral (resp. spherical) complete hypercontraction* if the toral generating m-tuple Q_t (resp. the spherical generating 1-tuple Q_s) associated with T as given in Example 2.2 is a complete hypercontraction. Similarly, one may define torally and spherically contractive tuples. Also, an m-tuple T acting on $\mathcal{H}$ is a *toral (resp. spherical) isometry* if $T_i^*T_i = I$ for all $i = 1, \cdots, m$ (resp. $\sum_{i=1}^m T_i^*T_i = I$).

In the toral (resp. spherical) case, complete hypercontractivity characterizes the torally (resp. spherically) contractive subnormal tuples. This is the content of [8, Theorem 4.1] (resp. [9, Theorem 5.2]). We see below that (3.7) also characterizes such tuples. Indeed, keeping aside the subnormality, we are able to characterize arbitrary completely hypercontractive generating tuples by (3.7) (see Definition 3.6 below). Our proof utilizes Stinespring's Dilation Theorem in a crucial manner; for the sake of completeness, we now state this result in the form that we will use it. (For related results, the reader is referred to [54] and [7].)

Let $\mathcal{A}$ denote a unital C^*-algebra. By an *operator system* we mean a self-adjoint subspace of $\mathcal{A}$ containing the unit. Let $M_n(\mathcal{A})$ denote the C^*-algebra of all $n \times n$ matrices with entries from $\mathcal{A}$. A mapping ϕ from $\mathcal{A}$ into another C^*-algebra $\mathcal{B}$ is said to be *positive* if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. Let $\mathcal{S} \subseteq \mathcal{A}$ denote an operator system. If $\phi : \mathcal{S} \to \mathcal{B}$ is a linear map, then we define $\phi_n : M_n(\mathcal{S}) \to M_n(\mathcal{B})$ by $\phi_n([a_{i,j}]) := [\phi(a_{i,j})]$, where $[a_{i,j}] \in M_n(\mathcal{S})$. We say that ϕ is *completely positive* if ϕ_n is positive for all $n \geq 1$.

The following fundamental theorem, due to Stinespring, characterizes the completely positive maps from a C^*-algebra into $B(\mathcal{H})$.

Stinespring's Dilation Theorem *(*[54, Theorem 4.1]*) Let $\mathcal{A}$ be a unital C^*-algebra. For every completely positive map $\phi : \mathcal{A} \to B(\mathcal{H})$, there exists a Hilbert space $\mathcal{K}$ and a unital *-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$ such that*

$$\phi(a) = V^* \pi(a) V \ (a \in \mathcal{A}),$$

where $V : \mathcal{H} \to \mathcal{K}$ is a bounded operator. Furthermore, we have $\|\phi(1)\| = \|V\|^2$. In particular, if ϕ is unital then V is an isometry.

Remark 3.5. *We may choose π, V and $\mathcal{K}$ in Stinespring's Dilation Theorem such that $\mathcal{K} = \pi(\mathcal{A})V\mathcal{H}$. When this happens, we refer to such a triple $(\pi, V, \mathcal{K})$ as the minimal Stinespring representation of ϕ.*

Definition 3.6. *A generating m-tuple $Q \equiv (Q_1, \cdots, Q_m)$ on $\mathcal{H}$ is said to be multiplicative if $Q^n(I) = Q(I)^n$ for every $n \in \mathbb{N}^m$, where*

$$Q(I) := (Q_1(I), \cdots, Q_m(I)).$$

Given two generating tuples P and Q on $\mathcal{K}$ and $\mathcal{H}$ respectively, we say that P is a multiplicative dilation of Q if P is multiplicative, $\mathcal{H} \subseteq \mathcal{K}$, and

$$Q^n(I)h = P_{\mathcal{H}} P^n(I)h \ (n \in \mathbb{N}^m, h \in \mathcal{H}).$$

A multiplicative dilation P of Q is said to be minimal if

$$\mathcal{K} = \bigvee \{P^n(I)h : n \in \mathbb{N}^m, h \in \mathcal{H}\}.$$

Example 3.7. *(a) The toral generating m-tuple (resp. spherical generating 1-tuple) associated with a toral isometry (resp. spherical isometry) is multiplicative.*

(b) Recall that an operator T acting on a Hilbert space $\mathcal{H}$ is said to be quasinormal if T commutes with T^*T, i.e., $T^*TT = TT^*T$. A toral generating 1-tuple associated with a quasinormal operator is multiplicative. However, we do not know if the converse is true; that is, if a toral generating 1-tuple associated with an operator T is multiplicative, must T be necessarily quasinormal? When $T^*T - TT^* \geq 0$, then T is indeed quasinormal. Moreover, if a toral generating 1-tuple associated with T is multiplicative, then the norm and spectral radius of T are equal, using the spectral radius formula. □

Remark 3.8. *Let Q be a multiplicative generating m-tuple on $\mathcal{H}$. Note that $Q(I)$ is a commuting m-tuple. If E denotes the spectral measure of $Q(I)$ then by the Spectral Theorem for commuting normal operators, Q satisfies*

$$Q^n(I) = \int_{[0,\|Q_1(I)\|]\times\cdots\times[0,\|Q_m(I)\|]} t^n dE(t) \ (n \in \mathbb{N}^m), \tag{3.8}$$

where E is a spectral measure. Conversely, any Q determined by a spectral measure E as in (3.8) is necessarily multiplicative. If, in addition, Q is contractive then it is easy to see that Q is completely hypercontractive. Needless to say, in the latter case, the semispectral measure E of Q, as given in (3.3), coincides with the spectral measure of $Q(I)$ by the uniqueness part of Lemma 3.2.

Theorem 3.9. *Every completely hypercontractive generating m-tuple Q on $\mathcal{H}$ admits a minimal multiplicative dilation P on $\mathcal{K}$. Moreover, the semispectral measure E occurring in the representation (3.3) of Q is related to the spectral measure F of $P(I) := (P_1(I), \cdots, P_m(I))$ by*

$$E(\sigma) = P_{\mathcal{H}} F(\sigma)|_{\mathcal{H}} \ (\text{for } \sigma \text{ a Borel subset of } [0,1]^m).$$

Furthermore, if P and S acting on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, are two minimal multiplicative dilations of Q then there exists a Hilbert space isomorphism $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $Uh = h$ $(h \in \mathcal{H})$ and

$$UP^n(I) = S^n(I)U \ (n \in \mathbb{N}^m).$$

Proof. The idea of the proof of the existence part is entirely simple and appears to be known among the specialists. Indeed, in a similar way, we may characterize the Hausdorff moment sequences (refer to [54]). For the sake of completeness, we include a detailed proof. Let $\mathcal{S}$ denote the operator system given by

$$\mathcal{S} := \{p \in C([0,1]^m) : p \in \mathbb{C}[t_1, \cdots, t_m]\},$$

where $\mathbb{C}[t_1, \cdots, t_m]$ denotes the space of complex polynomials in m indeterminates, and $C([0,1]^m)$ denotes the C^*-algebra of complex-valued continuous functions on $[0,1]^m$ endowed with the sup norm $\|\cdot\|_\infty$. Given $p \in \mathcal{S}$, write $p(t) \equiv \sum_{k\in\mathbb{N}^m, |k|\le n} \alpha_k t^k$ and consider the mapping $\phi : \mathcal{S} \to B(\mathcal{H})$

$$\phi(p) \equiv \phi\left(\sum_{k\in\mathbb{N}^m, |k|\le n} \alpha_k t^k\right) := \sum_{k\in\mathbb{N}^m, |k|\le n} \alpha_k Q^k(I) \ (n \in \mathbb{N}, \alpha_k \in \mathbb{C}).$$

By (3.2) in Remark 3.1, we have

$$\langle \phi(p)h,\, h\rangle = \int_{[0,1]^m} p(t) d\mu_h(t)$$

for every $p \in \mathcal{S}$ and every $h \in \mathcal{H}$, where μ_h is a positive Borel measure given by

$$\mu_h(\sigma) = \langle E(\sigma)h,\, h\rangle \ (\sigma \text{ is a Borel subset of } [0,1]^m) \tag{3.9}$$

with E the semispectral measure appearing in (3.3). Thus, ϕ is positive on $\mathbb{C}[t_1, \cdots, t_m]$. By the Stone-Weierstrass Theorem [23], we may extend ϕ to a positive map $\tilde{\phi}$ on the whole of $C([0,1]^m)$. A result of Stinespring says that any positive map from $C([0,1]^m)$ into a C^*-algebra is always completely positive [54, Theorem 3.11], and therefore Stinespring's Dilation Theorem is applicable. Thus there exists a minimal Stinespring representation $(\pi, V, \mathcal{K})$ of $\tilde{\phi}$. Since $\phi(1) = I$, V is isometric. Hence $V\mathcal{H} \subseteq \mathcal{K}$ may be identified with $\mathcal{H}$. Moreover, for the commuting m-tuple $N := (\pi(t_1), \cdots, \pi(t_m))$ of positive operators in $B(\mathcal{K})$, we have

$$Q_i^{n_i}(I) = \phi(t_i^{n_i}) = P_{\mathcal{H}}\pi(t_i^{n_i})|_{\mathcal{H}} = P_{\mathcal{H}} N_i^{n_i} \ (i = 1, \cdots, m).$$

If we define the generating m-tuple P by setting $P^n(X) := N^n$ $(n \in \mathbb{N}^m, X \in B(\mathcal{K}))$ then clearly P is a multiplicative dilation of Q, which is minimal since so is the triplet $(\pi, V, \mathcal{K})$. This establishes the first part of the theorem.

For the second part, we first notice that for any polynomial p, $p(Q)(I) = P_{\mathcal{H}} p(P(I))|_{\mathcal{H}}$. Let μ_h be as in (3.9). Let f be an μ_h-integrable function for every $h \in \mathcal{H}$. By the density of polynomials in $C([0,1]^m)$, we may choose a sequence $\{p_n\}$ of polynomials (depending of course on h) converging to f in $L^1(\mu_h)$ [56, Theorem 3.14], where $L^1(\mu_h)$ denotes the Banach space of μ_h-integrable functions. By a routine application of the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \langle p_n(Q)(I)h,\, h\rangle_{\mathcal{H}} = \lim_{n\to\infty} \int_{[0,1]^m} p_n(t) d\mu_h = \int_{[0,1]^m} f(t) d\mu_h.$$

In case f is a characteristic function χ_σ (for a Borel set σ), by the Spectral Theorem we obtain

$$\begin{aligned} \langle P_{\mathcal{H}} F(\sigma)h,\, h\rangle_{\mathcal{H}} &= \lim_{n\to\infty} \langle p_n(P(I))h,\, h\rangle_{\mathcal{K}} = \lim_{n\to\infty} \langle P_{\mathcal{H}} p_n(P(I))h,\, h\rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty} \langle p_n(Q)(I)h,\, h\rangle_{\mathcal{H}} = \mu_h(\sigma) = \langle E(\sigma)h,\, h\rangle_{\mathcal{H}}, \end{aligned}$$

which completes the proof of the second part.

To see the uniqueness part, define U by

$$U\left(\sum_{i=1}^{k} \alpha_i P^{n(i)}(I)h_i\right) := \sum_{i=1}^{k} \alpha_i S^{n(i)}(I)h_i$$

for finitely many non-zero $\alpha_i \in \mathbb{C}$, $n(i) \in \mathbb{N}^m$, and $h_i \in \mathcal{H}$. Observe that

$$\begin{aligned} \left\| \sum_{i=1}^{k} \alpha_i S^{n(i)}(I)h_i \right\|^2_{\mathcal{K}_2} &= \sum_{i,j} \alpha_i \overline{\alpha_j} \langle S^{n(i)+n(j)}(I)h_i,\, h_j\rangle_{\mathcal{K}_2} \\ &= \sum_{i,j} \alpha_i \overline{\alpha_j} \langle P_{\mathcal{H}} S^{n(i)+n(j)}(I)h_i,\, h_j\rangle_{\mathcal{H}} \\ &= \sum_{i,j} \alpha_i \overline{\alpha_j} \langle Q^{n(i)+n(j)}(I)h_i,\, h_j\rangle_{\mathcal{H}}. \end{aligned}$$

Similarly, we can see that

$$\left\| \sum_{i=1}^{k} \alpha_i P^{n(i)}(I)h_i \right\|^2_{\mathcal{K}_1} = \sum_{i,j} \alpha_i \overline{\alpha_j} \langle Q^{n(i)+n(j)}(I)h_i,\, h_j\rangle_{\mathcal{H}}. \tag{3.10}$$

We can now extend U linearly and isometrically from $\mathcal{K}_1$ onto $\mathcal{K}_2$, which gives the desired Hilbert space isomorphism. □

Remark 3.10. *The proof of the existence part of Theorem* 3.9 *may also be deduced from* [7, Scholium A].

The completely hypercontractive generating tuples arise naturally out of multiplicative generating tuples, as we now prove.

Corollary 3.11. *Let Q be a generating m-tuple on $\mathcal{H}$. Then the following statements are equivalent.*

(i) Q is completely hypercontractive.

(ii) Q admits a minimal multiplicative and contractive dilation P.

In this case, we have

$$\|f(Q)(I)\| \leq \sup\{|f(t)| : t \in [0,1]^m\} \ (\textit{for every } f \in C([0,1]^m)).$$

Proof. (i) implies (ii): By the preceding theorem, Q admits a minimal multiplicative dilation P. Since P is minimal, to establish contractivity it suffices to check that $\|P_k(I)x\|_{\mathcal{K}} \leq \|x\|_{\mathcal{K}}$ for every k and every x in

$$\text{linspan}\{P^n(I)h : n \in \mathbb{N}^m, h \in \mathcal{H}\}.$$

We may conclude from (3.10) and the second half of Lemma 3.2 that

$$\begin{aligned}
\left\| P_k(I) \sum_{i=1}^{k} \alpha_i P^{n(i)}(I) h_i \right\|_{\mathcal{K}}^2 &= \left\| \sum_{i=1}^{k} \alpha_i P^{n(i)+\epsilon(k)}(I) h_i \right\|_{\mathcal{K}}^2 \\
&= \sum_{i,j} \alpha_i \overline{\alpha_j} \langle Q^{n(i)+n(j)+2\epsilon(k)} h_i,\, h_j \rangle_{\mathcal{H}} \\
&= \sum_{i,j} \alpha_i \overline{\alpha_j} \langle R^* A^{n(i)+n(j)+2\epsilon(k)} R h_i,\, h_j \rangle_{\mathcal{H}} \\
&= \left\| A_k \sum_{i=1}^{k} \alpha_i A^{n(i)} R h_i \right\|_{\mathcal{H}}^2,
\end{aligned}$$

where $\epsilon(k)$ denotes the m-tuple with 1 in the k-th entry and zeros elsewhere. Since each A_k is contractive,

$$\left\| A_k \sum_{i=1}^{k} \alpha_i A^{n(i)} R h_i \right\|_{\mathcal{H}}^2 \leq \left\| \sum_{i=1}^{k} \alpha_i A^{n(i)} R h_i \right\|_{\mathcal{H}}^2 = \left\| \sum_{i=1}^{k} \alpha_i P^{n(i)}(I) h_i \right\|_{\mathcal{K}}^2$$

in view of the preceding calculation.

(ii) implies (i): The proof of this part is similar to that of Proposition 3.3(ii), and hence omitted. Finally, the verification of the last part is similar to that of the second part of Theorem 3.9. □

Corollary 3.12. *A completely hypercontractive generating m-tuple Q on $\mathcal{H}$ satisfies*

$$Q_i^2(I) \geq Q_i(I)^2 \ (i = 1, \cdots, m).$$

Proof. Let P denote the multiplicative dilation of Q as guaranteed by Theorem 3.9, and let $P_{\mathcal{H}}$ denote the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Observe that for any $i = 1, \cdots, m$,

$$\begin{aligned}
Q_i^2(I) - Q_i(I)^2 &= P_{\mathcal{H}} N_i(I)^2|_{\mathcal{H}} - (P_{\mathcal{H}} N_i(I)|_{\mathcal{H}})^2 \\
&= P_{\mathcal{H}} N_i(I)(I - P_{\mathcal{H}})(P_{\mathcal{H}} N_i(I))^*|_{\mathcal{H}},
\end{aligned}$$

where the operator on the right-hand-side is clearly positive. □

In the proof of the next corollary, we need the following result of V. Müller and A. Soltysiak (see also [22]).

Lemma 3.13. *([42, Theorem 1]) Let T be a commuting m-tuple of bounded linear operators on a Hilbert space. Then*

$$\sup_{(z_1,\cdots,z_n)\in\sigma(T)} \left(|z_1|^2+\cdots+|z_n|^2\right)^{\frac{1}{2}} = \lim_{n\to\infty} \|Q_s^n(I)\|^{\frac{1}{2n}},$$

where Q_s is the spherical generating 1*-tuple associated with T.*

Corollary 3.14. *Let S be a spherical complete hypercontractive m-tuple of commuting operators on $\mathcal{H}$. Then the Taylor spectrum $\sigma(S)$ is contained in the closed ball $\overline{\mathbb{B}^m_r}$ of radius $r := \sqrt{\|P(I)\|} \in [0,1]$ centered at* 0*, where P is the minimal multiplicative dilation of the spherical generating* 1*-tuple Q_s associated with S.*

Proof. By Corollary 3.11, we must have $0 \leq \|P(I)\| \leq 1$. Also,

$$\|Q^n(I)\|^{\frac{1}{2n}} \leq \sqrt{\|P(I)\|} \text{ (for any } n \in \mathbb{N}).$$

The desired conclusion now follows from Lemma 3.13. □

We remark that a variant of Corollary 3.14 may also be obtained from [9, Theorem 5.2], and Proposition 3.4.

Let ϕ be a real-valued map on $\mathbb{N}^m$. For $1 \leq i \leq m$, define the difference operators ∇_i by $(\nabla_i\phi)(n) := \phi(n) - \phi(n+\epsilon(i))$ $(n \in \mathbb{N}^m)$, where $\epsilon(i)$ is the m-tuple with 1 in the i-th entry and zeros elsewhere. The operator ∇^k is inductively defined for every $k \in \mathbb{N}^m$ through the relations $\nabla^0\phi := \phi$, $\nabla^{k+\epsilon(i)}\phi := \nabla_i\nabla^k\phi$. A real-valued map ϕ on $\mathbb{N}^m$ is said to be *completely monotone* if $(\nabla^k\phi)(n) \geq 0$ for all $n, k \in \mathbb{N}^m$.

In the literature there are several equivalent formulations of subnormality of operator tuples. These include mainly the Itô characterization [38] (which is a direct extension of the Bram-Halmos criterion), the Embry-Lubin conditions, and the multi-dimensional Halmos-Szymanski boundedness conditions (see [61, Theorem 2.1]; refer also to [24] for the single variable case). The aforementioned conditions are related to the complete monotonicity of the real-valued sequence $\{a_k\}_{k\in\mathbb{N}^m}$, where

$$a_k := \sum_{0\leq i,j\leq n} \langle Q_t^{i+j+k}(I)f_j,\, f_i\rangle.$$

Indeed, $\nabla^0 a_k \geq 0$ are precisely the Embry-Lubin conditions. On the other hand, $\nabla_i^1 a_k \geq 0$ $(1 \leq i \leq m)$ are precisely the boundedness conditions. Our next application of Theorem 3.9 asserts that the complete monotonicity of $\{a_k\}_{k\in\mathbb{N}^m}$ characterizes completely hypercontractive generating tuples Q.

Before we do that, we would like to point out here an interesting connection between the Embry-Lubin conditions, the boundedness conditions and the complete hypercontractivity. In view of the Vandermonde's convolution formula, it is easy to see that, for a rather special choice of f_i, namely,

$$f_i := (-1)^i \binom{n}{i} h \ (h \in \mathcal{H}, \ 0 \leq i \leq n)$$

in $\nabla^0 a_k \geq 0$, we have $B_{2n}(Q) \geq 0$ for every $n \in \mathbb{N}^m$ (see (3.1)). Further, the same choice of f_i's in $\nabla_i^1 a_k \geq 0$ yields $B_{2n+\epsilon(i)}(Q) \geq 0$ ($n \in \mathbb{N}^m, i = 1, \cdots, m$) in view of

$$B_{n+\epsilon(i)}(Q) = B_n(Q) - Q_i(B_n(Q)). \tag{3.11}$$

Corollary 3.15. *Let Q be a generating m-tuple on $\mathcal{H}$. For $n \in \mathbb{N}^m$ and for a finite set of vectors $\{f_i\}_{0 \leq i \leq n}$ in $\mathcal{H}$, let*

$$a_k := \sum_{0 \leq i,j \leq n} \langle Q^{i+j+k}(I) f_j, \, f_i \rangle \ (k \in \mathbb{N}^m).$$

Then the following statements are equivalent.

(i) Q is completely hypercontractive.

(ii) The real-valued multi-sequence $\{a_k\}_{k \in \mathbb{N}^m}$ is completely monotone for any finite set of vectors $\{f_i\}_{0 \leq i \leq n}$ in $\mathcal{H}$.

Proof. (i) implies (ii): Let P denote the contractive multiplicative dilation of Q as guaranteed by Theorem 3.9 and Corollary 3.11. Then

$$\begin{aligned}
&\textstyle\sum_{0 \leq p \leq q} (-1)^p \binom{n}{p} a_{k+p} \\
&= \sum_{0 \leq p \leq q} (-1)^p \binom{n}{p} \sum_{0 \leq i,j \leq n} \left\langle Q^{i+j+k+p}(I) f_j, f_i \right\rangle_{\mathcal{H}} \\
&= \sum_{0 \leq p \leq q} (-1)^p \binom{n}{p} \sum_{0 \leq i,j \leq n} \left\langle P^{i+j+k+p}(I) f_j, f_i \right\rangle_{\mathcal{K}} \\
&= \sum_{0 \leq i,j \leq n} \left\langle \Pi_{s=1}^m (I - P_s(I))^{q_s} P^{j+k/2}(I) f_j, P^{i+k/2}(I) f_i \right\rangle_{\mathcal{K}} \\
&= \left\langle \Pi_{s=1}^m (I - P_s(I))^{q_s} \sum_{0 \leq j \leq n} P^{j+k/2}(I) f_j, \sum_{0 \leq i \leq n} P^{i+k/2}(I) f_i \right\rangle_{\mathcal{K}},
\end{aligned}$$

which is non-negative since P is contractive. The conclusion of (ii) now follows from [16, Chapter 4, Proposition 6.11].

The remaining implication is immediate from the discussion prior to the statement of Corollary 3.15. $\square$

For a generating m-tuple Q on $\mathcal{H}$, let B_n be as in (3.1). If p is a positive integer then we say that Q is *p-isometric* if

$$B_n(Q) = 0 \text{ for all } n \in \mathbb{N}^m \text{ with } |n| = p.$$

If $p = 1$ then we refer to the p-isometric generating tuple simply as the *isometric* generating tuple. It is easy to see from (3.11) that any isometric generating tuple is p-isometric (for all $p \geq 1$) as well as completely hypercontractive. One may naturally ask, Are these the only p-isometric, completely hypercontractive generating tuples? The answer to this question, even in the special cases of toral and spherical generating tuples, is rather subtle. This is a consequence of a result of Athavale and Pedersen [13], which was obtained by moment-theoretic methods. We include here a direct verification. Notably, our method can be used to obtain an alternative proof of [13, Proposition 8] as well.

Lemma 3.16. *Let Q be a generating m-tuple on $\mathcal{H}$ with the minimal multiplicative dilation P on $\mathcal{K}$. Let p be an analytic polynomial in the complex variables $z_1, \cdots, z_m$. Then the following statements are equivalent.*
(i) $p(Q)(I) = 0$.
(ii) $p(P)(I) = 0$.

Proof. For any $x, y \in \mathcal{H}$, notice that

$$\langle p(P)(I)x,\, y\rangle_{\mathcal{K}} = \langle p(Q)(I)x,\, y\rangle_{\mathcal{H}} \ \ (x, y \in \mathcal{H}). \tag{3.12}$$

It is now clear from (3.12) that $p(Q)(I) = 0$ if $p(P)(I) = 0$. To see the implication $(i) \Rightarrow (ii)$, suppose $p(Q) = 0$ and pick $h \in \mathcal{K}$. Since P is the minimal multiplicative dilation of Q,

$$\mathcal{K} = \bigvee \{P^k(I)h : k \in \mathbb{N}^m,\ h \in \mathcal{H}\}.$$

Thus we can choose a sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that h_n converges to h in $\mathcal{K}$, where

$$\mathcal{F} := \text{linear span of } \left\{P^k(I)h : k \in \mathbb{N}^m,\ h \in \mathcal{H}\right\}.$$

Since $\langle p(P)h_n,\, h_n\rangle_{\mathcal{K}}$ converges to $\langle p(P)h,\, h\rangle_{\mathcal{K}}$ as n tends to infinity, it suffices to check that

$$\left\langle p(P)(I)P^k(I)x,\, P^l(I)y\right\rangle_{\mathcal{K}} = 0 \text{ for any } k, l \in \mathbb{N}^m \text{ and } x, y \in \mathcal{H}.$$

Since P is multiplicative, in view of (3.12) above, we have

$$\begin{aligned} \left\langle p(P)(I)P^k(I)x,\, P^l(I)y\right\rangle_{\mathcal{K}} &= \left\langle P^{k+l} \circ p(P)(I)x,\, y\right\rangle_{\mathcal{K}} \\ &= \left\langle Q^{k+l} \circ p(Q)(I)x,\, y\right\rangle_{\mathcal{H}}. \end{aligned}$$

Since $p(Q)(I) = 0$, the proof is complete. □

Proposition 3.17. *Let Q be a completely hypercontractive generating m-tuple on $\mathcal{H}$. Assume that for a positive scalar k, kQ is p-isometric. Then kQ is isometric.*

Proof. By Theorem 3.9, Q admits a minimal multiplicative dilation, say, P on $\mathcal{K}$. Suppose that for some $p \geq 1$, kQ is a p-isometry. Then, by Lemma 3.16 (applied to the minimal multiplicative dilation P of Q), kP is also p-isometric. It follows that

$$\prod_{i=1}^{m} (I - kP_i(I))^{n_i} = 0 \text{ (for all } n \in \mathbb{N}^m \text{ such that } |n| = p).$$

It is now easy to see that $kP_i(I) = I$ (for all $i = 1, \cdots, m$) . Thus $kQ_i(I) = I$ (for all $i = 1, \cdots, m$), that is, kQ is isometric. □

Remark 3.18. *A special case of Proposition* 3.17*, and the above argument, appear in* [60]*.*

Fix a positive integer $m \geq 1$. Suppose that $\{e_n\}_{n\in\mathbb{N}^m}$ is an orthonormal basis for $\mathcal{H}$ and that $\left\{w_n^{(i)} : 1 \leq i \leq m, n \in \mathbb{N}^m\right\}$ is a bounded subset of the positive real line. An m-variable weighted shift $T = (T_1, \cdots, T_m)$ is defined through the relations

$$T_i e_n := w_n^{(i)} e_{n+\epsilon(i)} \ (1 \leq i \leq m),$$

where $\epsilon(i)$ is the m-tuple with 1 in the i-th entry and zeros elsewhere. We denote by $T : \{w_n^{(i)}\}$ the multivariable weighted shift T with weight sequence $\left\{w_n^{(i)} : 1 \leq i \leq m, n \in \mathbb{N}^m\right\}$.

Example 3.19. *For positive integers m and a such that $m \geq a$, let $\mathcal{H}_{a,m}$ denote the reproducing kernel Hilbert space of analytic functions on the unit ball $\mathbb{B}_1^m$ associated with the positive definite kernel*

$$\frac{1}{(1 - \langle z, w\rangle)^a},$$

where $\langle z, w\rangle = \sum_{i=1}^m z_i\overline{w}_i$ $(z, w \in \mathbb{B}_1^m)$. Then the operator $M_{z,a}$ of multiplication by the coordinate function $z = (z_1, \cdots, z_m)$ in $\mathcal{H}_{a,m}$ is a spherical $(m - a + 1)$-isometry [36, Theorem 4.2]*. $M_{z,a}$ can also be realized as the multivariable weighted shift with weights*

$$\left\{\sqrt{\frac{n_i + 1}{|n| + a}} : 1 \leq i \leq m, \ n \in \mathbb{N}^m\right\}.$$

Assume further that $a \neq m$. It is then easy to see that $M_{z,a}$ is not a spherical isometry. Apply Proposition 3.17 *with $k := \|Q_s(I)\|$ and $Q := \frac{1}{k}Q_s$, where Q_s is the spherical generating* 1*-tuple associated with $M_{z,a}$. Then $\frac{1}{\sqrt{k}}M_{z,a}$ is not completely hypercontractive. Hence, by* [9, Theorem 5.2]*, $M_{z,a}$ is not subnormal.* □

Remark 3.20. *By the preceding example, the Drury-Arveson m-shift $M_{z,1}$ is not subnormal. This provides a new verification of* [4, Section 3, Corollary 1]*. One can also use the Six-Point Test in* [26] *to prove that the Drury-Arveson* 2*-shift is not even jointly hyponormal.*

4. Hyperexpansive Generating Tuples

We say that a generating tuple Q is
completely hyperexpansive if

$$B_n(Q) \leq 0 \text{ for all } n \in \mathbb{N}^m \setminus \{0\}.$$

p-hyperexpansive if

$$B_n(Q) \leq 0 \text{ for all } n \in \mathbb{N}^m \setminus \{0\} \text{ with } |n| \leq p.$$

p-expansive if

$$B_n(Q) \leq 0 \text{ for all } n \in \mathbb{N}^m \setminus \{0\} \text{ with } |n| = p.$$

A commuting tuple T of m bounded linear operators on $\mathcal{H}$ is a *toral (resp. spherical) complete hyperexpansion* if the toral generating m-tuple (resp. the spherical generating 1-tuple) associated with T as given in Example 2.2 is completely hyperexpansive. Similarly, one may introduce toral and spherical p-expansions, p-hyperexpansions, p-isometries. The class of toral complete hyperexpansions has been studied in [14] while the class of toral (resp. spherical) p-isometries has been studied in [28] and [29] (resp. [36] and [43]).

It follows from [36, Theorem 4.2] that the Drury-Arveson 2-shift $M_{z,1}$ in the Hilbert space $\mathcal{H}_{1,2}$ is spherically 2-isometric. Hence, by the next proposition, $M_{z,1}$ is a spherical complete hyperexpansion.

Most of the observations of the next proposition are known in the one-variable case (refer to [55], [14], [40]). We ask the reader to compare the conclusion of the second half of Proposition 4.1(iv) with that of Corollary 3.12.

Proposition 4.1. *Let Q be a 2-expansive generating m-tuple on $\mathcal{H}$. The following statements hold.*

(i) Q is 2-hyperexpansive.

(ii) For each i and a non-zero $h \in \mathcal{H}$, the sequence

$$\left\{ \frac{\left\| \sqrt{Q_i^{n+1}(I)}h \right\|}{\left\| \sqrt{Q_i^n(I)}h \right\|} \right\}_{n \in \mathbb{N}} \tag{4.1}$$

monotonically decreases to 1, where $\sqrt{P}$ denotes the positive square-root of the positive operator P.

(iii) Q is 2-isometric if and only if Q satisfies

$$Q^n(I) = I + \sum_{i=1}^{m} n_i(Q_i(I) - I) \ (n \in \mathbb{N}^m). \tag{4.2}$$

In this case, Q is completely hyperexpansive.

(iv) For non-zero reals α, β such that $\alpha^2 + \beta^2 = 1$, Q satisfies the inequality

$$4Q_i \circ Q_j(I) \leq \frac{Q_i(I)^2}{\alpha^2} + \frac{Q_j(I)^2}{\beta^2} \ (1 \leq i, j \leq m).$$

In particular, Q satisfies

$$Q_i^2(I) \leq Q_i(I)^2 \ (1 \leq i \leq m).$$

Proof. Observe that the 2-expansivity of Q_i implies

$$\langle Q_i^{n-1}(I)h,\, h\rangle - 2\langle Q_i^n(I)h,\, h\rangle + \langle Q_i^{n+1}(I)h,\, h\rangle \leq 0 \ (n \in \mathbb{N}_+). \tag{4.3}$$

(i): It suffices to check that $Q_i(I) \geq I$ for all i. The argument is similar to [55, Lemma 1]. By (4.3), $\{Q_i^{k+1}(I) - Q_i^k(I)\}_{k \geq 0}$ is a non-increasing sequence of self-adjoint operators. It follows that for any positive integer n,

$$\begin{aligned} 0 \leq Q_i^n(I) &= \sum_{k=0}^{n-1} \left(Q_i^{k+1}(I) - Q_i^k(I)\right) + I \\ &\leq n\left(Q_i(I) - I\right) + I. \end{aligned}$$

This leads to $Q_i(I) \geq \frac{n-1}{n} I$ for all positive integers n, and hence $Q_i(I) \geq I$.

(ii): It follows from (4.3) that

$$\begin{aligned}\left\|\sqrt{Q_i^{n-1}(I)}h\right\|\left\|\sqrt{Q_i^{n+1}(I)}h\right\| &\leq \frac{\left\|\sqrt{Q_i^{n-1}(I)}h\right\|^2+\left\|\sqrt{Q_i^{n+1}(I)}h\right\|^2}{2}\\ &= \frac{\langle Q_i^{n-1}(I)h,\,h\rangle+\langle Q_i^{n+1}(I)h,\,h\rangle}{2}\\ &\leq \langle Q_i^{n}(I)h,\,h\rangle=\left\|\sqrt{Q_i^{n}(I)}h\right\|^2.\end{aligned}$$

Therefore,

$$\frac{\left\|\sqrt{Q_i^{n+1}(I)}h\right\|}{\left\|\sqrt{Q_i^{n}(I)}h\right\|} \leq \frac{\left\|\sqrt{Q_i^{n}(I)}h\right\|}{\left\|\sqrt{Q_i^{n-1}(I)}h\right\|},$$

so the sequence (4.1) is monotonically decreasing. To calculate its limit, observe that $\langle Q_i^{n-1}(I)h,\,h\rangle \neq 0$ if $h\neq 0$ in view of (i), divide by $\langle Q_i^{n-1}(I)h,\,h\rangle$ in (4.3) to get

$$1-2\frac{\langle Q_i^{n}(I)h,\,h\rangle}{\langle Q_i^{n-1}(I)h,\,h\rangle}+\frac{\langle Q_i^{n+1}(I)h,\,h\rangle}{\langle Q_i^{n}(I)h,\,h\rangle}\frac{\langle Q_i^{n}(I)h,\,h\rangle}{\langle Q_i^{n-1}(I)h,\,h\rangle}\leq 0,$$

and let n tend to infinity.

(iv): Notice that the inequality

$$I-Q_i(I)-Q_j(I)+Q_i\circ Q_j(I)\leq 0 \tag{4.4}$$

can be rewritten as

$$\left(\alpha I-\frac{Q_i(I)}{2\alpha}\right)^2+\left(\beta I-\frac{Q_j(I)}{2\beta}\right)^2+\left(Q_i\circ Q_j(I)-\frac{Q_i(I)^2}{4\alpha^2}-\frac{Q_j(I)^2}{4\beta^2}\right)\leq 0,$$

which yields the first inequality. To see the remaining part, let $\alpha=\beta$ and $i=j$.

(iii): We prove by induction on m that if Q is 2-isometric then it satisfies (4.2). Suppose $m=1$. Then, as in the verification of (i),

$$Q^n(I)=I+\sum_{k=0}^{n-1}\left(Q^{k+1}(I)-Q^k(I)\right)=I+n(Q(I)-I).$$

Now suppose that (4.2) holds true for $m-1$ variables. Notice that the equality in (4.4) is equivalent to (4.2) for $n\equiv\epsilon(i)+\epsilon(j)$ $(1\leq i,j\leq m)$. Then, by the induction hypothesis,

$$\begin{aligned} Q^n(I) &= Q_m^{n_m}\left(I + \sum_{i=1}^{m-1} n_i(Q_i(I) - I)\right) \\ &= Q_m^{n_m}(I) + \sum_{i=1}^{m-1} n_i\left(Q_m^{n_m} \circ Q_i(I) - Q_m^{n_m}(I)\right) \\ &= Q_m^{n_m}(I) + \sum_{i=1}^{m-1} n_i(Q_i(I) - I) = I + \sum_{i=1}^{m} n_i(Q_i(I) - I). \end{aligned}$$

The converse is easy to see. The second part follows from (3.11) and (i) above. □

Corollary 4.2. *Let Q be a 2-hyperexpansive generating 1-tuple. Then*

$$\lim_{n\to\infty} \|Q^n(I)\|^{\frac{1}{n}} = 1. \tag{4.5}$$

In particular, the Taylor spectrum of any spherical 2-hyperexpansion is contained in the closed unit ball.

Proof. This is a global analogue of [40, Lemma 3.2(v)], and can be proved along the same lines. From the proof of Proposition 4.1(i), it can be seen that any 2-hyperexpansive generating 1-tuple Q satisfies

$$Q^n(I) \le n(Q(I) - I) + I \ (n \in \mathbb{N}).$$

Thus $\|Q^n(I)\| \le \|n(Q(I) - I) + I\|$, and hence $\limsup_{n\to\infty} \|Q^n(I)\|^{\frac{1}{n}} \le 1$. Since Q is expansive, we must have $\liminf_{n\to\infty} \|Q^n(I)\|^{\frac{1}{n}} \ge 1$, which establishes (4.5). The remaining part follows from Lemma 3.13. □

Let $\{e_n\}_{\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Given a generating 1-tuple Q on $\mathcal{H}$ and a vector $h \in \mathcal{H}$, we consider the one-variable weighted shift $W_{Q,h}$ given by

$$W_{Q,h}\, e_n := \frac{\left\|\sqrt{Q^{n+1}(I)}h\right\|}{\left\|\sqrt{Q^n(I)}h\right\|}\, e_{n+1} \ (n \in \mathbb{N}).$$

It is easy to see that Q is completely hyperexpansive (resp. 2-hyperexpansive, resp. 2-isometric) if and only if $W_{Q,h}$ is completely hyperexpansive (resp. 2-hyperexpansive, resp. 2-isometric) for each $h \in \mathcal{H}$ (cf. [40, Proposition 6.7]). This elementary observation has some interesting consequences. Firstly, Proposition 4.1(ii) follows from the fact that the weight-sequence of any 2-hyperexpansive weighted shift decreases to 1 [40]. This also shows that $W_{Q,h}$ is a compact perturbation of the unilateral shift. Secondly, recall that the self-commutator of any 2-hyperexpansive one-variable weighted shift is trace-class [40]; it follows that if Q_s is a 2-hyperexpansive spherical 1-tuple then for any $h \in \mathcal{H}$, we have

$$\sum_{k=1}^{\infty} \left| \frac{\left\|\sqrt{Q_s^{k+1}(I)}h\right\|^2}{\left\|\sqrt{Q_s^{k}(I)}h\right\|^2} - \frac{\left\|\sqrt{Q_s^{k}(I)}h\right\|^2}{\left\|\sqrt{Q_s^{k-1}(I)}h\right\|^2} \right| < \infty.$$

In view of the above discussion, it seems desirable to examine the weight sequence of $W_{Q,h}$. For 2-isometric multi-variable weighted shifts, this weight sequence can be completely determined (see Corollary 4.3). Interestingly, if Q_s is a spherical generating 1-tuple associated with the Drury-Arveson 2-shift with orthonormal basis $\{f_k\}_{k\in\mathbb{N}_+^m}$, then the weighted shift W_{Q_s,f_0} is nothing but the classical *Dirichlet* shift! Recall that the classical Dirichlet shift is the weighted shift operator with weight-sequence $\left\{\sqrt{\frac{n+2}{n+1}}\right\}$. In this regard, we note that the norm $\|\cdot\|_{\mathcal{H}_{1,2}}$ is induced by the norm $\|\cdot\|_D$ in the following sense:

$$\|f\|_{\mathcal{H}_{1,2}} = \int_{\partial\mathbb{B}_1^2} \|f_z\|_D^2 d\sigma(z),$$

where $f_z(w) \equiv f(zw)$ and $d\sigma$ is the normalized Lebesgue measure on $\partial\mathbb{B}_1^2$ [36]. (Here $\mathcal{H}_{1,2}$ and D are the Drury-Arveson and Dirichlet spaces, respectively.)

Corollary 4.3. *Let $T : \{w_k^{(i)}\}$ be a spherical 2-isometric m-variable weighted shift with respect to the orthonormal basis $\{e_k : k \in \mathbb{N}^m\}$, and let Q_s be the spherical generating 1-tuple associated with T. Then*

$$\left\|\sqrt{Q_s^n(I)}e_k\right\| = \sqrt{1 + n\left(\sum_{i=1}^{m}\left(w_k^{(i)}\right)^2 - 1\right)}$$

for every $n, k \in \mathbb{N}^m$.

Proof. This is immediate from Proposition 4.1(iii). □

By [16, Chapter 4, Proposition 6.12], Q is completely hyperexpansive if and only if for every $h \in \mathcal{H}$ there exists a positive Radon measure μ_h on $[0,1]^m \setminus \{\mathbf{1}\}$ such that

$$\langle Q^n(I)h,\, h\rangle = \|h\|^2 + \sum_{i=1}^{m} n_i\alpha_i + \int_{[0,\ 1]^m\setminus\{\mathbf{1}\}} (1-t^n)d\mu_h(t)\ (n \in \mathbb{N}^m),$$

where $\mathbf{1}$ denotes the m-tuple $(1, \cdots, 1)$ and $(\alpha_1, \cdots, \alpha_m) \in \mathbb{C}^m$. The following result may be obtained by imitating the ingenious proof of [12, Theorem 1], where the non-traditional idea of deriving the Lévy-Khinchin representation from the solution of the multi-dimensional Hausdorff Moment Problem is central.

Lemma 4.4. *A generating m-tuple Q on $\mathcal{H}$ is completely hyperexpansive if and only if there exist semispectral measures $E_1, \cdots, E_m$ on $[0,1]^m$ such that*

$$Q^n(I) = I + \sum_{i=1}^{m} n_i E_i(\{\mathbf{1}\}) + \int_{[0,\ 1]^m \setminus \{\mathbf{1}\}} (1 - t^n) \frac{d(E_1 + \cdots + E_m)(t)}{m - t_1 - \cdots - t_m} \ (n \in \mathbb{N}^m). \tag{4.6}$$

Remark 4.5. *We refer to the representation (4.6) in Lemma 4.4 as the Lévy-Khinchin representation for completely hyperexpansive generating tuples. In case of a 2-isometric generating m-tuple Q, note that the semispectral measure E_i $(1 \leq i \leq m)$ given by*

$$\begin{aligned} E_i(\sigma) &= Q_i(I) - I \text{ if } \mathbf{1} \text{ belongs to the Borel subset } \sigma \text{ of } [0,1]^m \\ &= 0 \text{ otherwise} \end{aligned}$$

satisfies (4.6) (see Proposition 4.1(iii)).

One may anticipate the complete-hyperexpansivity analogue of Theorem 3.9. However, the situation is much more complicated.

Theorem 4.6. *Let Q be a completely hyperexpansive generating m-tuple on $\mathcal{H}$. Suppose further that the measure F given by*

$$F(\sigma) := \int_\sigma \frac{d(E_1 + \cdots + E_m)(t)}{m - t_1 - \cdots - t_m} \ (\sigma \text{ a Borel subset of } [0,1]^m \setminus \{\mathbf{1}\})$$

is a $B(\mathcal{H})$-valued Borel measure, where $E_1, \cdots, E_m$ are the semispectral measures appearing in the Lévy-Khinchin representation of Q (see (4.6)). Then there exist a multiplicative generating tuple P on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ such that

$$p(Q)(I) = -V^*(p(P)(I))V \text{ for any } p \in \mathcal{S},$$

where $\mathcal{S}$ is the self-adjoint subspace given by
$\mathcal{S} := \{p \in C([0,1]^m) : p \text{ polynomial in } \mathbf{t} \equiv (t_1, \cdots, t_m),\ p(\mathbf{1}) = 0,\ \nabla p(\mathbf{1}) = 0\}$,
with $\nabla \equiv (\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m})$. Moreover, P is minimal in the sense that

$$\mathcal{K} = \bigvee \{P^n(I)h : \ n \in \mathbb{N}^m, h \in \mathcal{H}\}.$$

Proof. Consider the mapping $\phi : \mathcal{S} \to B(\mathcal{H})$ given by

$$\phi\,(p) \equiv \phi\left(\sum_{k\in\mathbb{N}^m, |k|\leq n} \alpha_k t^k\right) := -\sum_{k\in\mathbb{N}^m, |k|\leq n} \alpha_k Q^k(I)\ (n \in \mathbb{N}, \alpha_k \in \mathbb{C}).$$

It is easy to see from Lemma 4.4 that

$$\phi(p) = \int_{[0,\ 1]^m\setminus\{\mathbf{1}\}} p(t)\ dF(t)$$

for every $p \in \mathcal{S}$. Thus ϕ is positive. We may extend ϕ linearly to the linear span of $\mathcal{S}$ and $\mathbb{C}$ by setting

$$\tilde{\phi}(p+\alpha) := \phi(p) + \alpha A\ (p \in \mathcal{S}, \alpha \in \mathbb{C}),$$

where A is the total mass $F([0,1]^m \setminus \{\mathbf{1}\})$ of F. Then $\tilde{\phi}$ is also positive since

$$\tilde{\phi}(p+\alpha) = \int_{[0,1]^m\setminus\{\mathbf{1}\}} (p(t)+\alpha) dF(t)$$

for any $p \in \mathcal{S}$ and any $\alpha \in \mathbb{C}$. By the Stone-Weierstrass Theorem, we may extend $\tilde{\phi}$ to a positive map (still denoted by $\tilde{\phi}$) on the whole of $C([0,1]^m)$. A result of Stinespring says that any positive map from $C([0,1]^m)$ into a C^*-algebra is always completely positive [54, Theorem 3.11], so $\tilde{\phi}$ is completely positive. We can now apply Stinespring's Dilation Theorem to conclude that there exists a minimal Stinespring representation $(\pi, V, \mathcal{K})$ of $\tilde{\phi}$ with $\|V\| = \sqrt{\|A\|}$. For the commuting m-tuple $N := (\pi(t_1), \cdots, \pi(t_m))$ of positive operators in $B(\mathcal{K})$,

$$p(Q)(I) = -V^*(p(N)))V \text{ for any } p \in \mathcal{S}.$$

If we define the generating m-tuple P by setting $P^n(X) := N^n$ $(n \in \mathbb{N}^m, X \in B(\mathcal{K}))$ then clearly P is a multiplicative generating m-tuple, which is minimal since so is the triple $(\pi, V, \mathcal{K})$. □

Remark 4.7. *(i) Any 2-isometric generating tuple satisfies the hypotheses of Theorem* 4.6.
(ii) In Theorem 4.6*, observe that* $\tilde{\phi}(1) = A$ *and* $\tilde{\phi}(1) = V^*V$*, so it follows that* $V^*V = A$*.*
(iii) In general, we cannot expect multiplicative dilations for completely hyperexpansive generating tuples. Indeed, any 2*-hyperexpansive generating tuple with a multiplicative dilation is necessarily isometric.*

One may of course wish to relate the spectral measure of $N(I)$ to the semispectral measures $E_1, \cdots, E_m$ appearing in the Lévy-Khinchin representation of Q. Also, one would like to know whether the multiplicative generating tuple of Theorem 4.6 is necessarily contractive on the range of V. At present, these questions seem to be intractable.

The following special case of Theorem 4.6 is worth-mentioning.

Corollary 4.8. *Let Q and $\mathcal{H}$ be as in Theorem 4.6. Then there exists a commuting m-tuple $P = (P_1, \cdots, P_m)$ of positive linear operators on $\mathcal{K} \supseteq \mathcal{H}$ and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ such that*

$$B_n(Q) = -V^* \left(\prod_{i=1}^{m} (I - P_i)^{n_i} \right) V$$

for $n \in \mathbb{N}^m \setminus \{0\}$ with either $n_i = 0$ or $n_i \geq 2$, where B_n is as in (3.1). Moreover, $\mathcal{K} = \bigvee_{n \in \mathbb{N}^m} P^n \mathcal{H}$.

The toral and spherical complete hypercontractions arise naturally out of subnormal tuples (see the second paragraph following Proposition 3.4). On the other hand, the toral and spherical 2-hyperexpansions are almost never subnormal (Proposition 3.3(i) and 3.4(i), and Remark 4.7(iii)). Unlike the toral case (see the discussion prior to [18, Lemma 2.19]), spherical 2-isometric m-tuples ($m > 1$) may contain non-isometric hyponormal contractions [4, Corollary to Proposition 5.3]. (Recall that $T \in B(\mathcal{H})$ is *hyponormal* if $T^*T - TT^* \geq 0$.)

The next result is the hyperexpansive analogue of Proposition 3.17 (refer to [14, Proposition 4] for a special case).

Proposition 4.9. *Let Q be a 2-hyperexpansive generating m-tuple on $\mathcal{H}$, and assume that Q is p-isometric for some $p \geq 2$. Then Q is 2-isometric.*

Proof. The conditions $I - Q_i(I) \leq 0$ and $I - 2Q_i(I) + Q_i^2(I) \leq 0$ guarantee that the positive operator-valued sequence $\{Q_i^{n+1}(I) - Q_i^n(I)\}_{n \geq 0}$ is monotonically non-increasing and bounded, so that it converges in the strong operator topology in $B(\mathcal{H})$ (refer to [41]). Thus there exists a positive operator A in $B(\mathcal{H})$ such that

$$Q_i^{n+1}(I) - Q_i^n(I) \to A \text{ (sot)} \quad \text{as } n \to \infty,$$

where sot stands for the strong operator topology.

Suppose $B_{n+\epsilon(i)}(Q) \leq 0$ for some $n \in \mathbb{N}^m$ with $n_i \geq 2$. We claim that $B_n(Q) \leq 0$. Since $B_{n+\epsilon(i)}(Q) = B_n(Q) - Q_i(B_n(Q))$, we have $B_n(Q) \leq Q_i(B_n(Q))$. An induction argument shows that $B_n(Q) \leq Q_i^m(B_n(Q))$ ($m \geq 1$). Thus it suffices to check that

$$Q_i^m(B_n(Q)) \to 0 \text{ (sot) as } m \to \infty.$$

Note that

$$\begin{aligned} B_n(Q) &= B_{n-\epsilon(i)}(Q) - Q_i(B_{n-\epsilon(i)}(Q)) \\ &= \sum_{0 \le p \le n-\epsilon(i)} (-1)^p \binom{n-\epsilon(i)}{p} (Q^p(I) - Q^{p+\epsilon(i)}(I)), \end{aligned}$$

so that

$$Q_i^m(B_n(Q)) = \sum_{0 \le p \le n-\epsilon(i)} (-1)^p \binom{n-\epsilon(i)}{p} (Q^{p+m\epsilon(i)}(I) - Q^{p+(m+1)\epsilon(i)}(I))$$

$(m \geq 1)$. Letting $m \to \infty$ in the preceding equality leads to

$$Q^m(B_n(Q)) \to \sum_{0 \le p \le n-\epsilon(i)} (-1)^p \binom{n-\epsilon(i)}{p} Q_1^{p_1} \cdots Q_{i-1}^{p_{i-1}} Q_{i+1}^{p_{i+1}} \cdots Q_m^{p_m}(-A).$$

Since $n_i \geq 2$, $Q^m(B_n(Q)) \to 0$ (sot) as $m \to \infty$. This completes the proof of the claim. Similarly, we may check that if $B_{n+\epsilon(i)}(Q) \geq 0$ for some $n \in \mathbb{N}^m$ with $n_i \geq 2$ then $B_n(Q) \geq 0$.

Now suppose Q is p-isometric for $p \geq 3$. In particular,

$$B_{\epsilon(j)+(p-1)\epsilon(i)}(Q) = 0 \ (1 \leq i, j \leq m).$$

By the discussion in the second paragraph and finite induction, we must have

$$B_{\epsilon(j)+2\epsilon(i)}(Q) = 0 \ (1 \leq i, j \leq m). \tag{4.7}$$

To see that Q is 2-isometric, it suffices to check that

$$B_{\epsilon(j)+\epsilon(i)}(Q) = 0 \ (1 \leq i, j \leq m).$$

Since $I - Q_i(I) - Q_j(I) + Q_i \circ Q_j(I) \leq 0$, as in the first paragraph, we may check that there exists a positive operator B in $B(\mathcal{H})$ such that

$$Q_i^n(Q_j(I) - I) \to B \text{ (sot) as } n \to \infty.$$

By (3.11), (4.7), and a finite induction argument, for any $n \geq 1$,

$$\begin{aligned} B_{\epsilon(j)+\epsilon(i)}(Q) &= Q_i^n(B_{\epsilon(j)+\epsilon(i)}(Q)) \\ &= Q_i^n(I - Q_i(I) - Q_j(I) + Q_i \circ Q_j(I)) \\ &= Q_i^{n+1}(Q_j(I) - I) - Q_i^n(Q_j(I) - I), \end{aligned}$$

which converges to 0 in sot. Hence, $B_{\epsilon(j)+\epsilon(i)}(Q) = 0$. □

We now establish the hyperexpansivity analogue of Corollary 3.15. Since the difficult part of its proof (based on Lemma 4.4) follows essentially along the lines of the proof of [12, Theorem 2], we omit it. The easier half can be obtained as in the proof of Corollary 3.15.

Recall now that a real-valued map ϕ on $\mathbb{N}^m$ is said to be *completely alternating* if $(\nabla^k \phi)(n) \leq 0$ for all $n \in \mathbb{N}^m, k \in \mathbb{N}^m \setminus \{0\}$.

Theorem 4.10. *Let Q be a generating m-tuple on $\mathcal{H}$. For $n \in \mathbb{N}^m$ and for a finite set of vectors $\{f_i\}_{0 \leq i \leq n}$ in $\mathcal{H}$, let*

$$a_k := \sum_{0 \leq i,j \leq n} \langle Q^{i+j+k}(I) f_j,\, f_i \rangle \ (k \in \mathbb{N}^m).$$

Then the following are equivalent.
(i) Q is completely hyperexpansive.
(ii) The real-valued multi-sequence $\{a_k\}_{k \in \mathbb{N}^m}$ is completely alternating for any finite set of vectors $\{f_i\}_{0 \leq i \leq n}$ in $\mathcal{H}$ such that $\sum_{0 \leq i \leq n} f_i = 0$.

5. Toral Cauchy Dual Tuples

Recall that $T \in B(\mathcal{H})$ is left invertible if and only if T^*T is invertible.

Definition 5.1. *Let T be an m-tuple of left-invertible bounded linear operators T_i on $\mathcal{H}$. Let $Q_t \equiv (Q_1, \cdots, Q_m)$ denote the toral generating m-tuple associated with T. We refer to the m-tuple $T^{\mathfrak{t}} := (T_1^{\mathfrak{t}}, \cdots, T_m^{\mathfrak{t}})$ as the operator tuple torally Cauchy dual to T, where $T_i^{\mathfrak{t}} := T_i(Q_i(I))^{-1}$ $(i = 1, \cdots, m)$.*

Remark 5.2. *Clearly, $T^{\mathfrak{t}}$ is an m-tuple of left-invertible bounded linear operators such that $(T^{\mathfrak{t}})^{\mathfrak{t}} = T$. Moreover, for every $i = 1, \cdots, m$ we have $(T_i^{\mathfrak{t}})^* T_i^{\mathfrak{t}} = (T_i^* T_i)^{-1}$. Thus, if $T \equiv (T_1, \cdots, T_m)$ is torally expansive (that is, $T_i^* T_i \geq I$ for all i) then $T^{\mathfrak{t}}$ is torally contractive. Also, if U is a toral isometry then $U^{\mathfrak{t}} = U$.*

Example 5.3. *Recall the definition of multivariable weighted shift given right before Example* 3.19*. Observe that if T is a multivariable weighted shift with weight sequence $\{w_n^{(i)}\}$, then T_i commutes with T_j if and only if $w_n^{(i)} w_{n+\epsilon(i)}^{(j)} = w_n^{(j)} w_{n+\epsilon(j)}^{(i)}$ for all $n \in \mathbb{N}^m$. Also, T_i is left-invertible if and only if $\inf_{n \in \mathbb{N}^m} w_n^{(i)} > 0$. It was noted in* [14, Example 1] *that the multivariable weighted shift $D_\lambda : \left\{ \sqrt{\frac{\lambda + |n|}{1+|n|}} \right\}$ is a toral complete hyperexpansion if and only if $1 \leq \lambda \leq 2$. Also, if $1 < \lambda < 2$ then D_λ is not a toral 2-isometry.*

Assume now that the commuting m-variable weighted shift T satisfies $w_n^{(i)} w_{n+\epsilon(i)}^{(j)} = w_n^{(j)} w_{n+\epsilon(j)}^{(i)}$ $(n \in \mathbb{N}^m,\ 1 \leq i, j \leq m)$, and that

$\min_{i=1}^m \left\{\inf_{n\in\mathbb{N}^m} w_n^{(i)}\right\} > 0$. *Then T consists of left-invertible operators. Moreover, the operator tuple $T^{\mathfrak{t}}$ torally Cauchy dual to T given by*

$$T_i^{\mathfrak{t}} e_n := \frac{1}{w_n^{(i)}} e_{n+\epsilon(i)} \ (1 \le i \le m)$$

is also a commuting m-variable weighted shift with weight sequence

$$\left\{\frac{1}{w_n^{(i)}} : 1 \le i \le m, n \in \mathbb{N}^m\right\}.$$

In particular, the operator tuple $D_\lambda^{\mathfrak{t}}$ torally Cauchy dual to D_λ is the multivariable weighted shift $B_\lambda : \left\{\sqrt{\frac{1+|n|}{\lambda+|n|}}\right\}$.

One may conclude from [14, Proposition 6] *that for a toral complete hyperexpansion $T : \{w_n^{(i)}\}$, the toral Cauchy dual tuple $T^{\mathfrak{t}}$ is a torally contractive subnormal tuple. Thus $B_\lambda : \left\{\sqrt{\frac{1+|n|}{\lambda+|n|}}\right\}$ is a torally contractive subnormal tuple ($1 \le \lambda \le 2$).* □

For $\mu \in \mathbb{C}^m$ and an m-tuple S of bounded linear operators $S_1, \cdots, S_m$, let $\ker(S - \mu I) := \bigcap_{i=1}^m \ker(S_i - \mu_i I)$ and let

$$\text{nullity}(S - \mu I) := \dim \ker(S - \mu I).$$

For a commuting m-tuple S of bounded linear operators in $B(\mathcal{H})$, we let $e_{\lambda,S,x}$ denote the formal $\mathcal{H}$-valued power series in λ given by

$$e_{\lambda,S,x} := \sum_{n\in\mathbb{N}^m} \lambda^n S^n x \ (\lambda \in \mathbb{C}^m, x \in \mathcal{H}). \tag{5.1}$$

Theorem 5.4. *Let T be an m-tuple of commuting left-invertible, bounded linear operators $T_1, \cdots, T_m$ on $\mathcal{H}$. Assume that the toral Cauchy dual operator tuple $T^{\mathfrak{t}}$ is commuting. Suppose nullity(T^*) ≥ 1 and let $r := (\|T_1^{\mathfrak{t}}\|^{-1}, \cdots, \|T_m^{\mathfrak{t}}\|^{-1})$. Then for any $s \in \mathbb{R}_+^m$ with $s \le r$, we have*

$$\bigvee_{n\in\mathbb{N}^m} (T^{\mathfrak{t}})^n (ker(T^*)) = \bigvee_{\mu\in s\mathbb{D}^m} \{e_{\mu,T^{\mathfrak{t}},h} : h \in ker(T^*)\}. \tag{5.2}$$

Moreover, $e_{\lambda,T^{\mathfrak{t}},h}$ belongs to ker($T^* - \lambda I$) for any $\lambda \in r\mathbb{D}^m$ provided

$$T_i^*(T^{\mathfrak{t}})^{k-k_i\epsilon(i)}|_{ker(T^*)} = 0 \ (k \in \mathbb{N}^m). \tag{5.3}$$

In this case, we have nullity($T^* - \lambda$) $\ge$ nullity(T^*) for every $\lambda \in r\mathbb{D}^m$.

Proof. Suppose $\ker(T^*) \ne \{0\}$. Pick up a nonzero $h \in \ker(T^*)$. The idea of the proof is same as that of [21, Lemma 2.15]. Notice that the $\mathcal{H}$-valued power series $e_{\lambda,T^{\mathfrak{t}},h}$ in the variable λ is absolutely convergent on the open polydisc $r\mathbb{D}^m$. Since $T^{\mathfrak{t}}$ is commuting, it follows from the operator identity $T_i^* T_i^{\mathfrak{t}} = I$ and (5.3) that

$$T_i^* e_{\lambda, T^{\mathfrak{t}}, h} = \lambda_i e_{\lambda, T^{\mathfrak{t}}, h} \ (i = 1, \cdots, m).$$

Since $e_{\lambda, T^{\mathfrak{t}}, h} = 0$ if and only if $h = 0$, observe that $\{e_{\lambda, T^{\mathfrak{t}}, h_i}\}_{i=1}^n$ is linearly independent if so is $\{h_i\}_{i=1}^n$.

In view of the preceding discussion, it suffices to check the equality in (5.2). Observe first that

$$\bigvee_{\mu \in r\mathbb{D}^m} \{e_{\mu, T^{\mathfrak{t}}, h} : h \in \ker(T^*)\} \subseteq \bigvee_{n \in \mathbb{N}^m} \{\left(T^{\mathfrak{t}}\right)^n h : h \in \ker(T^*)\}.$$

For $x \in \mathcal{H}$ and $h \in \ker(T^*)$, define $f_{x,h} : s\mathbb{D}^m \to \mathbb{C}$ by

$$f_{x,h}(\mu) := \sum_{n \in \mathbb{N}^m} \overline{\langle x, \, (T^{\mathfrak{t}})^n h \rangle} \mu^n \ (\mu \in s\mathbb{D}^m).$$

Since $e_{\mu, T, h}$ is absolutely convergent, $f_{x,h}$ is a well-defined function analytic in $s\mathbb{D}^m$. In fact, $f_{x,h}(\mu) = \langle e_{\mu, T^{\mathfrak{t}}, h}, \, x \rangle$ (all $\mu \in s\mathbb{D}^m$). Let

$$x \in \left(\bigvee \{e_{\mu, T^{\mathfrak{t}}, h} : \mu \in s\mathbb{D}^m, h \in \ker(T^*)\} \right)^{\perp}.$$

Thus $\langle x, \, e_{\mu, T^{\mathfrak{t}}, h} \rangle = 0$ for every $\mu \in s\mathbb{D}^m$ and every $h \in \ker(T^*)$. It follows that

$$\sum_{n \in \mathbb{N}^m} \langle x, \, (T^{\mathfrak{t}})^n h \rangle \overline{\mu}^n = 0 \text{ for every } \mu \in s\mathbb{D}^m,$$

where $\overline{z} \in \mathbb{C}^m$ denotes the component-wise complex conjugate of $z \in \mathbb{C}^m$. Thus the analytic function $f_{x,h}$ is identically zero in $s\mathbb{D}^m$. Hence $\langle x, \, (T^{\mathfrak{t}})^n h \rangle = 0$ for every $n \in \mathbb{N}^m$ and every $h \in \ker(T^*)$. It follows that $x \in (\bigvee_{n \in \mathbb{N}^m} \{\left(T^{\mathfrak{t}}\right)^n h : h \in \ker(T^*)\})^{\perp}$. This completes the verification of (5.2). ☐

An operator S in $B(\mathcal{H})$ is said to be *hypercyclic* if its orbit $\{S^n h : n \geq 0\}$ is dense in $\mathcal{H}$ for some vector $h \in \mathcal{H}$. If U is a unilateral shift, then αU^* is hypercyclic for every complex number α of modulus bigger than 1 [15, Example 1.9]. This result of Rolewicz was extended to unilateral weighted shifts by Salas [15, Example 1.15]. The following result may be regarded as the multivariable analogue of the results of Rolewicz and Salas. However, notice that the following cannot be deduced directly from the corresponding one-variable result [20, Theorem 1].

Corollary 5.5. *Let $T : \{w_n^{(i)}\}$ be a commuting m-variable weighted shift such that $\min_{i=1}^m \left\{\inf_n w_n^{(i)}\right\} > 0$. Then λT_i^* is hypercyclic for any $\lambda \in \mathbb{C}$ such* *that* $|\lambda| > \left(\inf_n w_n^{(i)}\right)^{-1}$ $(i = 1, \cdots, m)$*. If, in addition, T is torally 2-hyperexpansive then λT_i^* is hypercyclic for any $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$* $(i = 1, \cdots, m)$.

Proof. Choose $s \in \mathbb{R}_+$ sufficiently small such that $|\lambda| s < 1$. Let

$$U := \text{linspan}\{e_{\mu, T^{\mathfrak{t}}, h} : h \in \ker(T^*), \mu \in s\mathbb{D}^m\},$$
$$S_{ij} := \frac{(T_i^{\mathfrak{t}})^j}{\lambda^j} \ (j \in \mathbb{N}_+, 1 \leq i \leq m).$$

By Example 5.3, T satisfies the hypotheses of the preceding theorem. Since the left-hand-side of (5.2) equals $\mathcal{H}$ for any multivariable weighted shift T, U is a dense subset of $\mathcal{H}$. Also, it is easy to see that (5.3) holds true. It follows from Theorem 5.4 that $e_{\mu, T^{\mathfrak{t}}, h}$ belongs to $\ker(T^* - \mu)$. Moreover, since $\|T_i^{\mathfrak{t}}\| = \left(\inf_n w_n^{(i)}\right)^{-1}$ $(i = 1, \cdots, m)$, T and S_{ij} satisfy

$$\|\lambda^n T_i^{*n} x\| \to 0, \ \|S_{in} x\| \to 0 \text{ as } n \to \infty \ (x \in U). \tag{5.4}$$

Hence, by the Hypercyclicity Criterion [15, Theorem 1.6], λT_i^* is hypercyclic.

To see the remaining part, simply note that $\inf_n w_n^{(i)} = 1$ $(i = 1, \cdots, m)$ by Proposition 4.1(ii). □

Corollary 5.6. *Let T be an m-tuple of commuting bounded linear operators on $\mathcal{H}$, and suppose T is torally* 2*-hyperexpansive and satisfies* (5.3)*. If $T^{\mathfrak{t}}$ is commuting and nullity$(T^*) \geq 1$, then $\sigma(T) = \overline{\mathbb{D}^m}$.*

Proof. Notice that none of the T_i's can be invertible since nullity$(T_i^*) \neq \{0\}$. Hence, we may conclude from [18, Lemma 2.14 and Theorem 2.9] that $\|T_i^{\mathfrak{t}}\| = 1$ for all i. Thus we may choose r in Theorem 5.4 to be $\mathbf{1}$. It follows that $\mathbb{D}^m \subseteq \sigma_p(T^*)$. We now argue as in the proof of [14, Proposition 5(i)] to conclude that $\sigma(T) = \overline{\mathbb{D}^m}$. □

Let us discuss another application of Theorem 5.4. Let T be an m-tuple of commuting left-invertible operators on $\mathcal{H}$, and assume that $T^{\mathfrak{t}}$ is commuting. Suppose further that nullity$(T^*) = 1$ and fix a non-zero $h \in \ker(T^*)$ with $\|h\|_{\mathcal{H}} = 1$. Define $\kappa : r\mathbb{D}^m \times r\mathbb{D}^m \to \mathbb{C}$ by

$$\kappa(\lambda, \mu) := \langle e_{\lambda, T^{\mathfrak{t}}, h}, \, e_{\mu, T^{\mathfrak{t}}, h} \rangle_{\mathcal{H}} \ (\lambda, \mu \in r\mathbb{D}^m),$$

where r and $e_{\lambda, S, x}$ are as in the statement of Theorem 5.4. Since κ is a positive definite kernel on $r\mathbb{D}^m$, we can associate with κ a reproducing kernel Hilbert space $\mathscr{H}$ as described in [3]. Thus

$$\langle g, \, \kappa(\lambda, \cdot) \rangle_{\mathscr{H}} = g(\lambda) \ (\lambda \in r\mathbb{D}^m, g \in \mathscr{H}).$$

Set $U\kappa(\lambda, \cdot) = e_{\lambda,T^{\mathfrak{t}},h}$ ($\lambda \in r\mathbb{D}^m$). Observe that we may extend U linearly and isometrically from $\mathscr{H}$ onto $\bigvee\{e_{\lambda,T^{\mathfrak{t}},h} : \lambda \in r\mathbb{D}^m\} \subseteq \mathcal{H}$. By (5.2), U is surjective if and only if $T^{\mathfrak{t}}$ is cyclic with the cyclic vector h. If T satisfies (5.3) then it can be seen that

$$UM_{z_i}^* = T_i^* U \ (i = 1, \cdots, m),$$

where $(M_{z_1}, \cdots, M_{z_m})$ on $\mathscr{H}$ is the m-tuple of operators of multiplication by the coordinate functions $z_1, \cdots, z_m$.

Remark 5.7. *We note that not all toral* 2*-hyperexpansions satisfy condition (*5.3*) of Theorem* 5.4*. Indeed, if U_+ denotes the unilateral shift on a separable Hilbert space then (*5.3*) does not hold true for the commuting pair (U_+, U_+).*

The following is the toral analogue of [18, Theorem 2.9]. (We mention in passing that the important half of [18, Theorem 2.9] had already been observed in [59, Section 5].)

Theorem 5.8. *Let T be a toral 2-hyperexpansive m-tuple of commuting bounded linear operators on $\mathcal{H}$ and let $T^{\mathfrak{t}}$ denote the operator tuple torally Cauchy dual to T. Then for $1 \le i, j \le m$, we have*

$$2(T_i^{\mathfrak{t}})^*((T_j^{\mathfrak{t}})^*T_j^{\mathfrak{t}})^{-1}T_i^{\mathfrak{t}} \le I + (T_i^{\mathfrak{t}})^*T_i^{\mathfrak{t}}((T_j^{\mathfrak{t}})^*T_j^{\mathfrak{t}})^{-2}(T_i^{\mathfrak{t}})^*T_i^{\mathfrak{t}}.$$

In particular, the operator tuple $T^{\mathfrak{t}}$ torally Cauchy dual to T consists of hyponormal contractions.

Proof. By definition, T is torally 2-hyperexpansive if and only if the toral generating tuple Q_t associated with T is 2-hyperexpansive (see Example 2.2). Applying Proposition 4.1(iv) to Q_t with $\alpha = \frac{1}{\sqrt{2}} = \beta$, and rephrasing the same in terms of T, we may conclude that for $1 \le i, j \le m$,

$$2T_i^*T_j^*T_jT_i \le (T_i^*T_i)^2 + (T_j^*T_j)^2. \tag{5.5}$$

Since $T_k^*T_k = ((T_k^{\mathfrak{t}})^*T_k^{\mathfrak{t}})^{-1}$, we have

$$2(T_i^{\mathfrak{t}})^*((T_j^{\mathfrak{t}})^*T_j^{\mathfrak{t}})^{-1}T_i^{\mathfrak{t}} \le I + (T_i^{\mathfrak{t}})^*T_i^{\mathfrak{t}}((T_j^{\mathfrak{t}})^*T_j^{\mathfrak{t}})^{-2}(T_i^{\mathfrak{t}})^*T_i^{\mathfrak{t}},$$

as desired. The choice $i = j$ in the last inequality yields $T_i^{\mathfrak{t}*}(T_i^{\mathfrak{t}})^*T_i^{\mathfrak{t}})^{-1}T_i^{\mathfrak{t}} \le I$. Since $(T_k^{\mathfrak{t}})^*T_k^{\mathfrak{t}})^{-1} = T_k^*T_k$, we have $(T_iT_i^{\mathfrak{t}})^*T_iT_i^{\mathfrak{t}} \le I$. Thus $C_i := T_iT_i^{\mathfrak{t}}$ is a contraction such that $C_i^*T_i^{\mathfrak{t}} = (T_i^{\mathfrak{t}})^*$. Hence $T_i^{\mathfrak{t}}$ is a hyponormal contraction. $\square$

Corollary 5.9. *Let $T : \{w^{(i)}_n\}$ be a commuting torally 2-hyperexpansive m-variable weighted shift. Then, for all $1 \le i, j \le m$,*

$$2\left(w^{(i)}_n\right)^2\left(w^{(j)}_{n+\epsilon(i)}\right)^2 \le \left(w^{(i)}_n\right)^4 + \left(w^{(j)}_n\right)^4 \quad (n \in \mathbb{N}^m).$$

Proof. The desired conclusion follows from (5.5) specialized to the multi-variable weighted shift $T : \{w^{(i)}_n\}$ (see the Proof of Theorem 5.8). □

Let $\mathbb{T}^m$ denote the m-dimensional unit torus in $\mathbb{C}^m$.

Corollary 5.10. *Let T be a torally 2-hyperexpansive commuting m-variable weighted shift on $\mathcal{H}$. Let $C^*(T)$ denote the C^*-algebra generated by T and let J_T denote the commutator ideal of $C^*(T)$. Then the following statements hold.*
*(i) For every $\mu \in \mathbb{T}^m$ there exists a non-zero *-homomorphism ϕ_μ on $C^*(T)$ such that $\phi_\mu(T_i) = \mu_i$ $(i = 1, \cdots, m)$.*
(ii) There exists an isometric $*$-isomorphism $\phi : C^*(T)/J_T \to C(\mathbb{T}^m)$ such that $\phi(T_i + J_T) = z_i$, where z_i is given by $z_i(w) = w_i$ $(w \in \mathbb{T}^m)$.

Proof. Since $C^*(T_i) = C^*(T_i^{\mathfrak{t}})$ [18, Lemma 2.13], it follows that $C^*(T) = C^*(T^{\mathfrak{t}})$. Hence, in view of the preceding theorem, the desired conclusions follow from [17, Propositions 3 and 4] if we can show that the left spectrum $\sigma_\ell(T^{\mathfrak{t}})$ of $T^{\mathfrak{t}}$ is the m-dimensional unit torus $\mathbb{T}^m$.

Since T_i is non-invertible, by [18, Lemma 2.14], $\sigma_\ell(T_i^{\mathfrak{t}}) = \mathbb{T}$ for $i = 1, \cdots, m$. By the projection property for the left spectrum [25], we have $\sigma_\ell(T^{\mathfrak{t}}) \subseteq \prod_{i=1}^m \sigma_\ell(T_i^{\mathfrak{t}}) \subseteq \mathbb{T}^m$. Thus $\sigma_\ell(T^{\mathfrak{t}})$ is a nonempty compact subset of the m-dimensional unit torus. Since any multi-variable weighted shift possesses multi-circular spectral symmetry [30], it follows that $\sigma_\ell(T^{\mathfrak{t}})$ must be the entire torus $\mathbb{T}^m$. This completes the proof. □

6. Spherical Cauchy Dual Tuples

Recall that a commuting m-tuple T of bounded linear operators on $\mathcal{H}$ is said to be *jointly left-invertible* if there exists a positive number k such that

$$T_1^*T_1 + \cdots + T_m^*T_m \ge kI.$$

Definition 6.1. *Let T be a jointly left-invertible m-tuple of bounded linear operators on $\mathcal{H}$. Let Q_s be the spherical generating tuple associated with T (see Example 2.2). We refer to the m-tuple $T^{\mathfrak{s}} := (T_1^{\mathfrak{s}}, \cdots, T_m^{\mathfrak{s}})$ as the* *operator tuple spherically Cauchy dual to T, where $T_i^{\mathfrak{s}} := T_i(Q_s(I))^{-1}$ $(i = 1, \cdots, m)$.*

Remark 6.2. *Let P_s be the spherical generating tuple associated with $T^{\mathfrak{s}}$. Then, since $P_s(I) = (Q_s(I))^{-1}$, $T^{\mathfrak{s}}$ is jointly left-invertible and $(T^{\mathfrak{s}})^{\mathfrak{s}} = T$. If T is spherically expansive (that is, $Q_T(I) \geq I$) then $T^{\mathfrak{s}}$ is spherically contractive. Also, if S is a spherical isometry then $S^{\mathfrak{s}} = S$.*

Example 6.3. *Let T, e_n, w_n^i be as in Example 5.3. Set*

$$\beta_n := \sqrt{\sum_{i=1}^{m} \left(w_n^{(i)}\right)^2} \ (n \in \mathbb{N}^m).$$

Note that T is jointly left-invertible if and only if $w := \inf_{n \in \mathbb{N}^m} \beta_n > 0$. Let $M_{z,a}$ be the spherical $(m-a+1)$-isometric weighted shift as discussed in Example 3.19. By Proposition 4.9 and Proposition 4.1(iii), $M_{z,a}$ is a spherical complete hyperexpansion if and only if a is either $m-1$ or m.

Assume that $w_n^{(i)} w_{n+\epsilon(i)}^{(j)} = w_n^{(j)} w_{n+\epsilon(j)}^{(i)}$ for all $n \in \mathbb{N}^m$ $(1 \leq i, j \leq m)$ and that $w > 0$. Then the commuting m-variable weighted shift T is jointly left-invertible. Also, the operator tuple spherically Cauchy dual to T given by

$$T_i^{\mathfrak{s}} e_n := \frac{w_n^{(i)}}{\beta_n^2} e_{n+\epsilon(i)} \ (1 \leq i \leq m)$$

is an m-variable weighted shift with the weight sequence

$$\left\{ \frac{w_n^{(i)}}{\beta_n^2} : 1 \leq i \leq m, n \in \mathbb{N}^m \right\}.$$

A routine calculation shows that $T^{\mathfrak{s}}$ is commuting if and only if $\beta_{n+\epsilon(i)} = \beta_{n+\epsilon(j)}$ for all $1 \leq i, j \leq m$.

If T is a spherical complete hyperexpansive m-variable weighted shift then the one-variable weighted shift with weight-sequence

$$\frac{\left\| \sqrt{Q_s^n(I)} e_0 \right\|}{\left\| \sqrt{Q_s^{n+1}(I)} e_0 \right\|} \ (n \in \mathbb{N}),$$

is a subnormal contraction, where Q_s denotes the spherical generating 1-tuple associated with T. This is a consequence of [11, Proposition 4] *(see the discussion following Corollary 4.2). In the case of a spherical 2-isometric m-variable weighted shift, the last assertion can be made more precise in view of Corollary 4.3.* □

The classical Dirichlet shift is a rather special example of the completely hyperexpansive operator [2]. It is thus natural to ask whether the Dirichlet m-shift of the unit ball is a spherical complete hyperexpansion. Incidentally, in dimension $m > 1$, the Dirichlet m-shift is not even a spherical 2-hyperexpansion.

Recall that the Dirichlet space $\mathscr{D}$ on the unit ball B_1^m is the reproducing kernel Hilbert space associated with the reproducing kernel

$$\kappa(z, w) \equiv -\frac{\log(1 - \langle z,\, w\rangle)}{\langle z,\, w\rangle} \ (z, w \in B_1^m).$$

The tuple M_z of multiplication by $z_1, \cdots, z_m$ on $\mathscr{D}$ can be realized as an m-variable weighted shift, called the *Dirichlet m-shift T*, and given by

$$T_i e_\alpha := \frac{\sqrt{(\alpha_i + 1)(|\alpha| + 2)}}{|\alpha| + 1} e_{\alpha+\epsilon(i)} \ (i = 1, \cdots, m),$$

where $e_\alpha \equiv \frac{z^\alpha}{\|z^\alpha\|}$ $(\alpha \in \mathbb{N}^m)$ is an orthonormal basis for $\mathscr{D}$ [37, Example 1]. Notice that for any $\alpha \in \mathbb{N}^m$,

$$\langle Q_s(I)e_\alpha,\, e_\alpha\rangle = \sum_{j=1}^{m} \frac{(\alpha_j + 1)(|\alpha| + 2)}{(|\alpha| + 1)^2} = \frac{(|\alpha| + m)(|\alpha| + 2)}{(|\alpha| + 1)^2}.$$

In particular, T is spherically expansive. Let us calculate $\langle Q_s^2(I)e_\alpha,\, e_\alpha\rangle$:

$$\begin{aligned}\langle Q_s^2(I)e_\alpha,\, e_\alpha\rangle &= \sum_{i,j=1}^{m} \frac{(\alpha_j + 1)(|\alpha| + 2)}{(|\alpha| + 1)^2}\frac{(\alpha_i + 2)(|\alpha| + 3)}{(|\alpha| + 2)^2} \\ &= \frac{(|\alpha| + m)(|\alpha| + 2m)(|\alpha| + 3)}{(|\alpha| + 1)^2(|\alpha| + 2)}.\end{aligned}$$

A routine calculation shows that

$$\begin{aligned}\langle B_n(Q_s)e_\alpha,\, e_\alpha\rangle &= 1 - 2\langle Q_s(I)e_\alpha,\, e_\alpha\rangle + \langle Q_s^2(I)e_\alpha,\, e_\alpha\rangle \\ &= (m - 1)|\alpha|^2 + (2m^2 + m - 3)|\alpha| + 6m^2 - 8m + 2.\end{aligned}$$

In particular, T is a spherical 2-hyperexpansion if and only if $m = 1$.

Observe that if $T \in B(\mathcal{H})$ is a non-2-isometric complete hyperexpansion then the pair $(T, 0)$ is a spherical complete hyperexpansion but not a spherical 2-isometry. Let us ensure the existence of nontrivial spherical complete hyperexpansive tuples other than spherical 2-isometries.

Example 6.4. *Fix a one-variable non-2-isometric, completely hyperexpansive weighted shift with weight-sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *(see Example* 5.3*). We construct inductively a commuting* 2*-variable weighted shift which is spherically* *complete hyperexpansive but not 2-isometric. Choose positive weights* $w_0^{(1)}$ *and* $w_0^{(2)}$ *so that*

$$\left(w_0^{(1)}\right)^2 + \left(w_0^{(2)}\right)^2 = \alpha_0^2.$$

Next, choose positive weights $w_{\epsilon(1)}^{(1)}, w_{\epsilon(2)}^{(1)}, w_{\epsilon(2)}^{(2)}$*, and* $w_{\epsilon(1)}^{(2)}$ *such that*

$$\begin{aligned}\left(w_0^{(1)}\right)^2\left(w_{\epsilon(1)}^{(1)}\right)^2 + 2\left(w_0^{(1)}\right)^2\left(w_{\epsilon(1)}^{(2)}\right)^2 + \left(w_0^{(2)}\right)^2\left(w_{\epsilon(2)}^{(2)}\right)^2 &= \alpha_0^2\alpha_1^2,\\ w_0^{(1)}w_{\epsilon(1)}^{(2)} &= w_0^{(2)}w_{\epsilon(2)}^{(1)}.\end{aligned}$$

This is possible since there are 2 *equations in* 4 *variables. By induction, at the* n*th step, we arrive at the positive weights* $w_k^{(1)}$ *and* $w_k^{(2)}$ $(k \in \mathbb{N}^2, |k| = n-1)$ *such that*

$$\begin{aligned}&\sum_{|\beta|=n}\frac{n!}{\beta!}\left(w_0^{(1)}\right)^2\cdots\left(w_{(\beta_1-1)\epsilon(1)}^{(1)}\right)^2\cdot\\ &\left(w_{\beta_1\epsilon(1)}^{(2)}\right)^2\cdots\left(w_{\beta_1\epsilon(1)+(\beta_2-1)\epsilon(2)}^{(2)}\right)^2 = \alpha_0^2\cdots\alpha_{n-1}^2,\\ &w_k^{(1)}w_{k+\epsilon(1)}^{(2)} = w_k^{(2)}w_{k+\epsilon(2)}^{(1)}\ (k\in\mathbb{N}^2, |k| = n-2).\end{aligned}$$

Thus we obtain a commuting 2*-variable weighted shift* T *with weight-sequence* $\left\{w_k^{(i)} : 1 \le i \le 2, k \in \mathbb{N}^2\right\}$ *satisfying*

$$\frac{\left\|\sqrt{Q_s^{n+1}(I)}e_0\right\|}{\left\|\sqrt{Q_s^n(I)}e_0\right\|} = \alpha_n\ (n \in \mathbb{N}),$$

where Q_s *denotes the spherical generating* 1*-tuple associated with* T*. By our choice of the weight-sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *and the discussion following Corollary* 4.2*,* T *has the desired properties.* □

Remark 6.5. *A spherical complete hypercontraction is always a toral complete hypercontraction* [9]*. In view of this, it is natural to ask whether a spherical complete hyperexpansive tuple is necessarily a toral complete hyperexpansion? To see this, first notice that there is class of toral complete hyperexpansions with spectrum the closed unit polydisc (see Corollary* 5.6*). Secondly, by Corollary* 4.2*, the spectrum of any spherical* 2*-hyperexpansion is contained in the closed unit ball. Hence, the answer is No at least for operator tuples* T *satisfying (*5.3*), with* T^{t} *commuting, and nullity*$(T^*) \ge 1$*. The general case remains unanswered.*

One may regard the following result as another multivariable generalization of [18, Theorem 2.9] (see Corollary 6.8 below for a variant).

Theorem 6.6. *Let T be a spherical 2-hyperexpansive m-tuple of m commuting bounded linear operators on $\mathcal{H}$ and let $T^{\mathfrak{s}}$ denote the operator tuple spherically Cauchy dual to T. Then*

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\|T_j(T_i^{\mathfrak{s}})x\|^2 \leq \|x\|^2 \ (x \in \mathcal{H}).$$

In particular, for any vectors $x_1, \cdots, x_m \in \mathcal{H}$,

$$\left\|\sum_{i=1}^{m}(T_i^{\mathfrak{s}})^* x_i\right\|^2 \leq \sum_{i=1}^{m}\|T_i^{\mathfrak{s}} x_i\|^2 + \sum_{1\leq i\neq j\leq m}\|T_i^{\mathfrak{s}} x_j\|^2.$$

Proof. Applying Proposition 4.1(iv) to the spherical generating 1-tuple Q_s associated with T yields $Q_s^2(I) \leq (Q_s(I))^2$. Thus for any $x \in \mathcal{H}$

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{m}\|T_j(T_i^{\mathfrak{s}})x\|^2 &= \sum_{i=1}^{m}\sum_{j=1}^{m}\langle(T_j^*T_j)(T_i^{\mathfrak{s}})x,\, (T_i^{\mathfrak{s}})x\rangle \\
&= \sum_{i=1}^{m}\langle(\sum_{j=1}^{m}T_j^*T_j)(T_i^{\mathfrak{s}})x,\, (T_i^{\mathfrak{s}})x\rangle \\
&= \sum_{i=1}^{m}\langle Q_s(I)T_i(Q_s(I))^{-1}x,\, T_i(Q_s(I))^{-1}x\rangle \\
&= \langle\sum_{i=1}^{m}T_i^*Q_s(I)T_i(Q_s(I))^{-1}x,\, (Q_s(I))^{-1}x\rangle \\
&= \langle Q_s^2(I)(Q_s(I))^{-1}x,\, (Q_s(I))^{-1}x\rangle \\
&\leq \langle(Q_s(I))^2(Q_s(I))^{-1}x,\, (Q_s(I))^{-1}x\rangle = \|x\|^2.
\end{aligned}$$

This completes the proof of the first part.

To conclude the proof of the remaining part, first note that $(T_i^{\mathfrak{s}})^* = \sum_{j=1}^{m} C_{ij}(T_j^{\mathfrak{s}})$, where $C_{ij} := \left(T_j\,(T_i^{\mathfrak{s}})\right)^*\big)$ is spherically contractive. It follows that

$$\begin{aligned}\left\|\sum_{i=1}^{m}(T_i^{\mathfrak{s}})^* x_i\right\| &= \left\|\sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij}(T_j^{\mathfrak{s}})x_i\right\| \\ &\leq \sum_{i=1}^{m}\sum_{j=1}^{m}\|C_{ij}\|\|T_j^{\mathfrak{s}}x_i\| \\ &\leq \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\|C_{ij}\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\|T_j^{\mathfrak{s}}x_i\|^2\right)^{\frac{1}{2}} \\ &\leq 1\cdot\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\|T_j^{\mathfrak{s}}x_i\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence $\left\|\sum_{i=1}^{m}(T_i^{\mathfrak{s}})^* x_i\right\|^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{m}\|T_j^{\mathfrak{s}}x_i\|^2$ as desired. □

Let us specialize Theorem 6.6 to the multivariable weighted shifts.

Corollary 6.7. *Let $T : \{w_n^{(i)}\}$ be a commuting spherical 2-hyperexpansive m-variable weighted shift. Let β_n be as introduced in Example* 6.3. *Then*

$$\sum_{i=1}^{m}(w_n^{(i)})^2\beta_{n+\epsilon(i)}^2 \leq \beta_n^4 \ (n \in \mathbb{N}^m).$$

Moreover, for any $\alpha(i) \in \mathbb{N}_+^m$ such that $\alpha(i)-\epsilon(i) \neq \alpha(j)-\epsilon(j)$ $(1 \leq i \neq j \leq m)$, we have

$$\sum_{i=1}^{m}\left(\frac{w_{\alpha(i)-\epsilon(i)}^{(i)}}{\beta_{\alpha(i)-\epsilon(i)}^2}\right)^2 \leq \sum_{j=1}^{m}\frac{1}{\beta_{\alpha(j)}^2}.$$

We present a variant of Theorem 6.6.

Corollary 6.8. *Under the hypotheses of Theorem* 6.6*, we have*

$$\sum_{i=1}^{m}\|(T_i^{\mathfrak{s}})^* x\|^2 \ \leq \ m\sum_{i=1}^{m}\|T_i^{\mathfrak{s}}x\|^2 \ \ (x \in \mathcal{H}).$$

Proof. We may write $(T_i^{\mathfrak{s}})^* = \sum_{j=1}^{m} C_{ij}(T_j^{\mathfrak{s}})$ for spherically contractive m^2-tuple (C_{ij}) by Theorem 6.6. Since $\|C_{ij}\| \leq 1$, we have

$$\sum_{i=1}^{m}\|(T_i^{\mathfrak{s}})^* x\|^2 \leq \sum_{i}\left(\sum_{j=1}^{m}\|c_{ij}\|\|T_j^{\mathfrak{s}}x\|\right)^2 \leq m\sum_{i=1}^{m}\|T_i^{\mathfrak{s}}x\|^2$$

for any $x \in \mathcal{H}$. □

Corollary 6.8 suggests the following question:

Question 6.9. *Let $m > 1$ and let T be a spherical 2-hyperexpansive commuting m-tuple. Must the spherically Cauchy dual tuple $T^{\mathfrak{s}}$ necessarily satisfy the inequality*

$$\sum_{i=1}^{m} \left((T_i^{\mathfrak{s}})^* \, T_i^{\mathfrak{s}} - T_i^{\mathfrak{s}} \, (T_i^{\mathfrak{s}})^*\right) \geq 0 \ ? \tag{6.1}$$

Note that (6.1) is equivalent to the spherical contractivity of the m-tuple

$$W \equiv \left((T_1^{\mathfrak{s}})^* \sqrt{Q_s(I)}, \cdots, (T_m^{\mathfrak{s}})^* \sqrt{Q_s(I)}\right).$$

Also, observe that W^* is spherically contractive if T is spherically 2-hyperexpansive (Theorem 6.6). One would like to know under what conditions the spherical contractivity of W^* implies the same for W? Needless to say, this is the case if $m = 1$.

7. Concluding Remarks

In Section 2, we discussed two examples of generating tuples other than toral and spherical generating tuples. It is natural to consider as well the Cauchy dual tuples related to these generating tuples. Let us illustrate this for the case of a generating tuple related to the bi-ball $\mathbb{B}_1^2 \times \mathbb{B}_1^2$ in $\mathbb{C}^4$. If $T \equiv (T_1, T_2, T_3, T_4)$ is a commuting tuple of bounded linear operators on $\mathcal{H}$ such that (T_1, T_2) and (T_3, T_4) are jointly left invertible, then one may introduce the Cauchy dual tuple $(T^{\mathfrak{t}})^{\mathfrak{s}}$ of T by

$$\left(T^{\mathfrak{t}}\right)^{\mathfrak{s}} := (T_1(Q_s(I))^{-1}, T_2(Q_s(I))^{-1}, T_3(P_s(I))^{-1}, T_4(P_s(I))^{-1}),$$

where Q_s and P_s are the spherical generating tuples associated with (T_1, T_2) and (T_3, T_4), respectively. One would obviously like to have a version of [18, Theorem 2.9] for $(T^{\mathfrak{t}})^{\mathfrak{s}}$. One would further like to know whether it is possible to have a unified theory of Cauchy dual tuples. Although a Cauchy dual tuple is a complete unitary invariant, at present it is not clear to what extent the theory of Cauchy dual tuples can be exploited to unravel the multivariable function theory and operator theory. We believe these topics warrant substantial additional attention.

References

[1] J. Agler, *An abstract approach to model theory*, in *Surveys of some recent results in operator theory*, Vol. II, Pitman Res. Notes Math. Ser. **192**, Longman Sci. Tech., Harlow, 1988; pp. 1–23.

[2] A. Aleman, *The multiplication operators on Hilbert spaces of analytic functions*, Habilitationsschrift, FernUniversität Hagen, 1993.

[3] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[4] W. Arveson, *Subalgebras of C^*-algebras. III. Multivariable operator theory*, Acta Math. **181** (1998), 159–228.

[5] W. Arveson, *The curvature invariant of a Hilbert module over* $C[z_1, \cdots, z_d]$, J. Reine Angew. Math. **522** (2000), 173–236.

[6] W. Arveson, *The Dirac operator of a commuting d-tuple*, J. Funct. Anal. **189** (2002), 53–79.

[7] W. Arveson, *Dilation theory yesterday and today*, in A glimpse at Hilbert space operators, Oper. Theory Adv. Appl. **207**(2010), 99–123.

[8] A. Athavale, *Holomorphic kernels and commuting operators*, Trans. Amer. Math. Soc. **304** (1987), 101–110.

[9] A. Athavale, *Model theory on the unit ball in* $\mathbb{C}^n$, J. Operator Theory **27** (1992), 347–358.

[10] A. Athavale, *On a minimal normal dilation problem*, Proc. Amer. Math. Soc. **121** (1994), 843–850.

[11] A. Athavale, *On completely hyperexpansive operators*, Proc. Amer. Math. Soc. **124** (1996), 3745–3752.

[12] A. Athavale, *The complete hyperexpansivity analog of the Embry conditions*, Studia Math. **154** (2003), 233–242.

[13] A. Athavale and S. Pedersen, *Moment problems and subnormality*, J. Math. Anal. Appl. **146** (1990), 434–441.

[14] A. Athavale and V. Sholapurkar, *Completely hyperexpansive operator tuples*, Positivity **3** (1999), 245–257.

[15] F. Bayart and É. Matheron, *Dynamics of Linear Operators*, Cambridge University Press, Cambridge 2009.

[16] C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups*, Springer-Verlag, Berlin 1984.

[17] J. Bunce, *The joint spectrum of commuting nonnormal operators*, Proc. Amer. Math. Soc. **29** (1971) 499–505.

[18] S. Chavan, *On operators Cauchy dual to 2-hyperexpansive operators*, Proc. Edin. Math. Soc. **50** (2007), 637–652.

[19] S. Chavan, *On operators close to isometries*, Studia Math. **186** (2008), 275–293.

[20] S. Chavan, *Co-analytic, right-invertible operators are supercyclic*, Colloq. Math. **119** (2010), 137–142.

[21] S. Chavan, *On operators Cauchy dual to 2-hyperexpansive operators: the unbounded case*, Studia Math. **203** (2011), 129–162.

[22] M. Chō and W. Żelazko, *On geometric spectral radius of commuting n-tuples of operators*, Hokkaido Math. J. 21(1992), 251–258.

[23] J. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York 1997.

[24] J. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monographs, vol 36, Amer. Math. Soc. Providence, RI 1991.

[25] R. Curto, *Applications of several complex variables to multiparameter spectral theory*, in *Surveys of some recent results in operator theory*, Vol. II, Pitman Res. Notes Math. Ser. **192**, Longman Sci. Tech., Harlow, 1988; pp. 25–90.

[26] R. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, Proc. Symposia Pure Math. **51**(1990), 69–91.

[27] R. Curto, *Spectral theory of elementary operators*, in *Elementary operators and applications (Blaubeuren, 1991)*, World Sci. Publ., River Edge, NJ, 1992; pp. 3–52.

[28] R. Curto and F. Vasilescu, *Standard operator models in the polydisc-I*, Indiana Univ. Math. J. **42** (1993), 979–989.

[29] R. Curto and F. Vasilescu, *Standard operator models in the polydisc-II*, Indiana Univ. Math. J. **44** (1995), 727–746.

[30] R. Curto and K. Yan, *The spectral picture of Reinhardt measures*, J. Funct. Anal. **131**(1995), 279–301.

[31] S. Drury, *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc. **68** (1978), 300–304.

[32] L.A. Fialkow, *Essential spectra of elementary operators*, Trans. Amer. Math. Soc. **267**(1981), 157–174.

[33] L.A. Fialkow, *Spectral properties of elementary operators*, Acta Sci. Math. (Szeged) **46**(1983), 269–282.

[34] L.A. Fialkow, *Spectral properties of elementary operators, II*, Trans. Amer. Math. Soc. **290**(1985), 415–429.

[35] L.A. Fialkow, *Structural properties of elementary operators*, in *Elementary operators and applications (Blaubeuren, 1991)*, World Sci. Publ., River Edge, NJ, 1992; pp. 55–113.

[36] J. Gleason and S. Richter, *m-Isometric commuting tuples of operators on a Hilbert space*, Integral Equations Operator Theory **56** (2006), 181–196.

[37] K. Guo, J. Hu, and X. Xu, *Toeplitz algebras, subnormal tuples and rigidity on reproducing* $\mathbf{C}[z_1, \ldots, z_d]$*-modules*, J. Funct. Anal. **210** (2004), 214–247.

[38] T. Itô, *On the commutative family of subnormal operators*, J. Fac. Sci. Hokkaido Univ. Ser. I **14**(1958), 1–15.

[39] Z. Jabłoński, *Complete hyperexpansivity, subnormality and inverted boundedness conditions*, Integral Equations Operator Theory **44** (2002), 316–336.

[40] Z. Jabłoński and J. Stochel, *Unbounded 2-hyperexpansive operators*, Proc. Edinb. Math. Soc. **44** (2001), 613–629.

[41] C. Kubrusly, *An Introduction to Models and Decompositions in Operator Theory*, Birkhäuser, Inc. Boston 1997.

[42] V. Müller and A. Soltysiak, *Spectral radius formula for commuting Hilbert space operators*, Studia Math. **103** (1992), 329–333.

[43] V. Müller and F.H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. **117** (1993), 979–989.

[44] A. Olofsson, *Wandering subspace theorems*, Integral Equations Operator Theory **51**(2005), 395–409.

[45] B. Magajna, *A transitivity theorem for algebras of elementary operators*, Proc. Amer. Math. Soc. **118**(1993), 119–127.

[46] B. Magajna, *Interpolation by elementary operators*, Studia Math. **105**(1993), 77–92.

[47] B. Magajna, *The norm of a symmetric elementary operator*, Proc. Amer. Math. Soc. **132**(2004), 1747–1754.

[48] B. Magajna, *Approximation of maps on C^*-algebras by completely contractive elementary operators*, in *Elementary operators and their applications*, Oper. Theory Adv. Appl. **212**(2011), 25–39.

[49] M. Mathieu, Elementary operators on prime C^*-algebras, II, *Glasgow Math. J.* **30**(1988), 275–284.

[50] M. Mathieu, Elementary operators on prime C^*-algebras, I, *Math. Ann.* **284**(1989), 223–244.

[51] M. Mathieu, *How to use primeness to describe properties of elementary operators*, in *Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988)*, Proc. Sympos. Pure Math. **51**, Part 2, Amer. Math. Soc., Providence, RI, 1990; pp. 195–199.

[52] M. Mathieu, *Operator equations with elementary operators*, in *Multivariable operator theory (Seattle, WA, 1993)*, Contemporary Math. **185**(1995), 259–272.

[53] M. Mathieu, *The norm problem for elementary operators*, in *Recent progress in functional analysis (Valencia, 2000)*, North Holland Math. Stud. **189**, North-Holland, Amsterdam, 2001; pp. 363–368.

[54] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, Cambridge 2003.

[55] S. Richter, *Invariant subspaces of the Dirichlet shift*, J. Reine Angew. Math. **386**(1988), 205–220.

[56] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Co., New York 1987.

[57] E. Saksman and H.-O. Tylli, *Multiplications and elementary operators in the Banach space setting*, in Methods in Banach space theory, London Math. Soc. Lecture Note Ser. **337**, Cambridge Univ. Press, Cambridge, 2006; pp. 253–292.

[58] S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. **531** (2001), 147–189.

[59] S. Shimorin, *Complete Nevanlinna-Pick property of Dirichlet-type spaces*, J. Funct. Anal. **191**(2002), 276–296.

[60] V. Sholapurkar and A. Athavale, *Completely and alternatingly hyperexpansive operators*, J. Operator Theory 43(2000), 43–68.

[61] J. Stochel, *Characterizations of subnormal operators*, Studia Math. **97**(1991), 227–238.

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