# Toral and spherical Aluthge transforms of 2-variable weighted shifts

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**Abstract**

We introduce two natural notions of Aluthge transforms (toral and spherical) for 2-variable weighted shifts and study their basic properties. Next, we study the class of spherically quasinormal 2-variable weighted shifts, which are the fixed points for the spherical Aluthge transform. Finally, we briefly discuss the relation between spherically quasinormal and spherically isometric 2-variable weighted shifts. *To cite this article: R. Curto, J. Yoon, C. R. Acad. Sci. Paris, Ser. I (2016).*

**Résumé**

Nous introduisons deux notions naturelles des transformations d'Aluthge (toral et sphérique) pour les shifts pondérés à deux variables et nous étudions leurs propriétés. Ensuite, nous étudions la classe de shifts pondérés à deux variables sphériques et quasinormaux, qui sont les points fixes pour la transformation d'Aluthge sphérique. Enfin, nous discutons brièvement de la relation entre les shifts pondérés à deux variables qui sont sphériquement quasinormaux, et ceux qui sont sphériquement isomètriques. *Pour citer cet article : R. Curto, J. Yoon, C. R. Acad. Sci. Paris, Ser. I (2016).*

## Version française abrégée

Pour $T \in \mathcal{B}(\mathcal{H})$, la décomposition polaire de $T$ est $T \equiv U|T|$, oú $U$ est une isométrie partielle et $|T| := \sqrt{T^*T}$. La *transformation d'Aluthge* de $T$ est l'opérateur $\widetilde{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Cette transformation a été considereé pour la première fois dans [1] afin d'étudier des opérateurs $p$-hyponormal et log-hyponormal. Bref, l'idée derrière la transformation d'Aluthge est de convertir un opérateur à un autre qui partage avec

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le premier quelques propriétés spectrales, mais qui reste plus proche d'être un opérateur normal. Ces dernières années, la transformation d'Aluthge a reçu de l'attention considérable.

Dans cet article, nous présentons d'abord la décomposition polaire des opérateurs bornés et nous étudions deux transformations d'Aluthge (toral et sphérique) de shifts à deux variables $W_{(\alpha,\beta)} \equiv (T_1, T_2)$. Puisque a priori il y a plusieurs notions possibles, nous discutons de deux définitions plausibles et de propriétés fondamentales dans la deuxième partie. Notre recherche nous permettra de comparer les deux définitions quant à la façon dont elles généralisent la notion d'une variable. Après avoir discuté de quelques propriétés fondamentales de chaque transformation d'Aluthge, nous procédons à étudier les deux transformations dans le cas des shifts pondérés de deux variables. Nous considérons des sujets tels que la préservation de la hyponormalité conjointe, la continuité de norme, la quasinormalité sphérique et l'isométrie sphérique.

## 1. Introduction

For $T \in \mathcal{B}(\mathcal{H})$, the *polar decomposition* of $T$ is $T \equiv U|T|$, where $U$ is a partial isometry and $|T| := \sqrt{T^*T}$. The *Aluthge transform* of $T$ is the operator $\widetilde{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transform was first considered in [1] in order to study $p$-hyponormal and log-hyponormal operators. Roughly speaking, the idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties, but it is closer to being a normal operator. In recent years, the Aluthge transform has received substantial attention. Jung, Ko and Pearcy proved in [17] that $T$ has a nontrivial invariant subspace if and only if $\widetilde{T}$ does. (Since every normal operator has nontrivial invariant subspaces, the Aluthge transform has a natural connection with the invariant subspace problem.)

For a weighted shift $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \cdots)$, the Aluthge transform $\widetilde{W}_\alpha$ of $W_\alpha$ is also a unilateral weighted shift, given by $\widetilde{W}_\alpha \equiv \text{shift}(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots)$ (see [18]). We recall that $W_\alpha$ is hyponormal if and only if $\alpha_0 \le \alpha_1 \le \cdots$. Thus, if $W_\alpha$ is hyponormal, then $\widetilde{W}_\alpha$ is also hyponormal. However, the converse is not true in general. For example, if $W_\alpha \equiv \text{shift}\left(\frac{1}{2}, 2, \frac{1}{2}, 2, \cdots\right)$, then $W_\alpha$ is clearly not hyponormal but $\widetilde{W}_\alpha = U_+$ is subnormal. (Here and in what follows, $U_+$ denotes the (unweighted) unilateral shift.) In [18], S.H. Lee, W.Y. Lee, and the second-named author showed that for $k \ge 2$, the Aluthge transform, when acting on weighted shifts, need not preserve $k$-hyponormality. Finally, G. Exner proved in [14] that the Aluthge transform of a subnormal weighted shift need not be subnormal.

In this article, we first introduce the polar decompositions of a commuting pair of bounded operators and study two Aluthge transforms (toral and spherical) of 2-variable weighted shifts $W_{(\alpha,\beta)} \equiv (T_1, T_2)$. Since a priori there are several possible notions, we discuss two plausible definitions and their basic properties in Section 2. Our research will allow us to compare both definitions in terms of how well they generalize the 1-variable notion. After discussing some basic properties of each Aluthge transform, we proceed to study both transforms in the case of 2-variable weighted shifts. We consider such topics as preservation of joint hyponormality, norm continuity, spherical quasinormality, and spherical isometries.

We first recall that a commuting pair is subnormal if and only if it is $k$-hyponormal for every $k \ge 1$ (see [7, Theorem 2.3]). For $i = 1, 2$, we consider the polar decomposition $T_i = U_i\,|T_i|$ of $T_i$. Then, for a 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$, we define the *toral* Aluthge transform of $W_{(\alpha,\beta)}$ as follows: $\widetilde{W}_{(\alpha,\beta)} = (\widetilde{T}_1, \widetilde{T}_2) := \left(|T_1|^{\frac{1}{2}}U_1|T_1|^{\frac{1}{2}}, |T_2|^{\frac{1}{2}}U_2|T_2|^{\frac{1}{2}}\right)$. As we will see in Proposition 2.1, the commutativity of $\widetilde{W}_{(\alpha,\beta)}$ does not automatically follow from the commutativity of $W_{(\alpha,\beta)}$.

There is a second plausible definition of Aluthge transform, which uses a joint polar decomposition. Assume that we have a decomposition of the form $(T_1, T_2) = (U_1P, U_2P)$, where $P := \sqrt{T_1^*T_1 + T_2^*T_2}$

and ker $U_1 \cap$ ker $U_2 =$ ker $P$. Now let $\widehat{W}_{(\alpha,\beta)} = (\widehat{T}_1, \widehat{T}_2) := \left(\sqrt{P}U_1\sqrt{P}, \sqrt{P}U_2\sqrt{P}\right)$. We refer to $\widehat{W}_{(\alpha,\beta)}$ as the *spherical* Aluthge transform of $W_{(\alpha,\beta)}$. Even though $\widehat{T}_1 = \sqrt{P}U_1\sqrt{P}$ is not the Aluthge transform of $T_1$, we observe that $Q := \sqrt{U_1^*U_1 + U_2^*U_2}$ is a (joint) partial isometry; for, $PQ^2P = P^2$, from which it follows that $Q$ is isometric on the range of $P$.

We will prove in Section 2 that this particular definition of the Aluthge transform preserves commutativity, and it also behaves well in terms of hyponormality, for a large class of 2-variable weighted shifts. There is also another useful aspect of the spherical Aluthge transform, which we now mention. If we consider the fixed points of this transform acting on 2-variable weighted shifts, we obtain an appropriate generalization of the concept of quasinormality. More precisely, if a 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ satisfies $W_{(\alpha,\beta)} = (\widehat{T}_1, \widehat{T}_2)$, then $T_1^*T_1 + T_2^*T_2$ is, up to scalar multiple, a spherical isometry. (We recall that a commuting pair $\mathbf{T} \equiv (T_1, T_2)$ is called a *spherical isometry* if $P^2 \equiv T_1^*T_1 + T_2^*T_2 = I$ [13].) It follows that we can then study some properties of the spherical Aluthge transform using well known results about spherical isometries. In this paper, we also focus on the following basic problem.

**Problem 1.1** *(i) For $k \geq 1$, if $W_{(\alpha,\beta)}$ is k-hyponormal, does it follow that the toral Aluthge transform $\widetilde{W}_{(\alpha,\beta)}$ k-hyponormal? What about the case of the spherical Aluthge transform? When does either Aluthge transform preserve hyponormality?;*
*(ii) When do we have the equality $\widetilde{W}_{(\alpha,\beta)} = \widehat{W}_{(\alpha,\beta)}$?;*
*(iii) Is the toral Aluthge transform $(T_1, T_2) \to \left(\widetilde{T}_1, \widetilde{T}_2\right)$ continuous in the uniform topology? Similarly, does continuity hold for the spherical Aluthge transform?*

## 2. Main Results

We first consider (joint) hyponormality for the toral and spherical Aluthge transforms.

**Proposition 2.1** *(i) For $W_{(\alpha,\beta)}$, we have: (a) $\widetilde{W}_{(\alpha,\beta)}$ is a commuting pair if and only if*

$$\alpha_{(k_1,k_2+1)}\alpha_{(k_1+1,k_2+1)} = \alpha_{(k_1+1,k_2)}\alpha_{(k_1,k_2+2)} \text{ for all } k_1, k_2 \geq 0$$

*(with similar conditions holding for the weight sequence $\{\beta_{(k_1,k_2)}\}$); and (b) $\widehat{W}_{(\alpha,\beta)}$ is always a commuting pair.*
*(ii) If $W_{(\alpha,\beta)}$ is a commuting pair of hyponormal operators, so is $\widehat{W}_{(\alpha,\beta)}$.*

We next show that there exists a subnormal $W_{(\alpha,\beta)}$ such that $\widetilde{W}_{(\alpha,\beta)}$ is not hyponormal; we also prove that there exists a non-hyponormal $W_{(\alpha,\beta)}$ such that $\widehat{W}_{(\alpha,\beta)}$ is hyponormal. We start with some definitions. The *core* $c(W_{(\alpha,\beta)})$ of $W_{(\alpha,\beta)}$ is the restriction of $W_{(\alpha,\beta)}$ to the invariant subspace $\mathcal{M} \cap \mathcal{N}$, where $\mathcal{M} \equiv \bigvee \left\{e_{(k_1,k_2)} \in \ell^2(\mathbb{Z}_+^2) : k_1 \geq 0 \text{ and } k_2 \geq 1\right\}$ and $\mathcal{N} \equiv \bigvee \left\{e_{(k_1,k_2)} \in \ell^2(\mathbb{Z}_+^2) : k_1 \geq 1 \text{ and } k_2 \geq 0\right\}$. Given a 1-variable unilateral weighted shift $W_\omega$, consider the 2-variable weighted shift $\Theta(W_\omega) \equiv W_{(\alpha,\beta)}$ on $\ell^2(\mathbb{Z}_+^2)$ given by the double-indexed weight sequences $\alpha_{(k_1,k_2)} = \beta_{(k_1,k_2)} := \omega_{k_1+k_2}$ for $k_1, k_2 \geq 0$. The shift $\Theta(W_\omega)$ can be represented by the weight diagram in Figure 1(ii).

**Lemma 2.2** *([8]) Let $\Theta(W_\omega)$ be given by Figure 1(ii), and let $k \geq 1$. $W_\omega$ is k-hyponormal if and only if $\Theta(W_\omega)$ is (jointly) k-hyponormal.*

We next show that hyponormality is not stable under the toral Aluthge transform.

**Proposition 2.3** *For $0 < x, y < 1$, let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift in Figure 1(i), where $\omega_0 = \omega_1 = \omega_2 = \cdots := 1$, $\alpha_{(0,0)} = \beta_{(0,0)} := x$, $\alpha_{(0,k_2)} = \beta_{(k_1,0)} := y$ $(k_1, k_2 \geq 1)$ and all remaining weights are equal to $1$. Then, we have:*

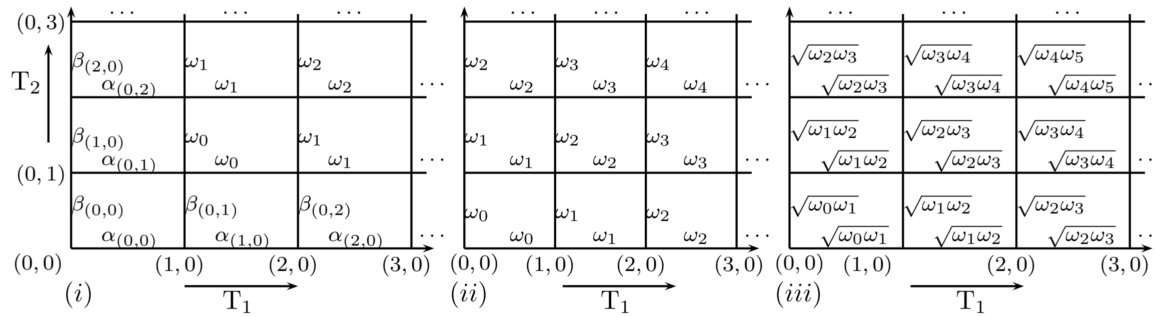

Figure 1. Weight diagram of the 2-variable weighted shift with $c(W_{(\alpha,\beta)}) = \Theta(W_\omega)$, weight diagram of a generic 2-variable weighted shift $\Theta(W_\omega) \equiv (T_1, T_2)$, and weight diagram of the Aluthge transform $\widetilde{\Theta}(W_\omega) = \widehat{\Theta}(W_\omega)$ of $\Theta(W_\omega)$, respectively.

*(i)* $W_{(\alpha,\beta)}$ *is subnormal if and only if* $x \le s(y) := \sqrt{\frac{1}{2-y^2}}$;

*(ii)* $W_{(\alpha,\beta)}$ *is hyponormal if and only if* $x \le h(y) := \sqrt{\frac{1+y^2}{2}}$;

*(iii)* $\widetilde{W}_{(\alpha,\beta)}$ *is hyponormal if and only if* $x \le CA(y) := \frac{1+y}{2}$;

*(iv)* $\widehat{W}_{(\alpha,\beta)}$ *is hyponormal if and only if* $x \le PA(y) := \frac{2\left(1+y^2-y^4\right)}{(1+\sqrt{2})(1+y^2)\left(\sqrt{1+y^2}-y^2\right)}$.

*Clearly,* $s(y) \le h(y) \le PA(y)$ *and* $CA(y) \le h(y)$ *for all* $0 < y < 1$, *while* $CA(y) < s(y)$ *on* $(0, q)$ *and* $CA(y) > s(y)$ *on* $(q, 1)$, *where* $q \cong 0.52138$. *Thus,* $W_{(\alpha,\beta)}$ *is hyponormal but* $\widetilde{W}_{(\alpha,\beta)}$ *is not hyponormal if* $0 < CA(y) < x \le s(y)$, *and* $\widehat{W}_{(\alpha,\beta)}$ *is hyponormal but* $W_{(\alpha,\beta)}$ *is not hyponormal if* $0 < h(y) < x \le PA(y)$.

Next, we consider the invariance of hyponormality under the two Aluthge transforms. We describe a large class of 2-variable weighted shifts for which both transforms coincide. By Figure 1(ii) and (iii), and direct calculations, we observe that $\widetilde{\Theta}(W_\omega) = \widehat{\Theta}(W_\omega)$. We now have:

**Theorem 2.4** *If* $\Theta(W_\omega)$ *is hyponormal, then* $\widetilde{\Theta}(W_\omega)$ *and* $\widehat{\Theta}(W_\omega)$ *are both hyponormal.*

**Remark 2.5** *As in the* 1*-variable case, we use* $W_\omega \equiv \mathrm{shift}\left(\frac{1}{2}, 2, \frac{1}{2}, 2, \cdots\right)$ *to build an example of a* 2*-variable weighted shift which is not subnormal, but whose Aluthge transforms are both subnormal.*

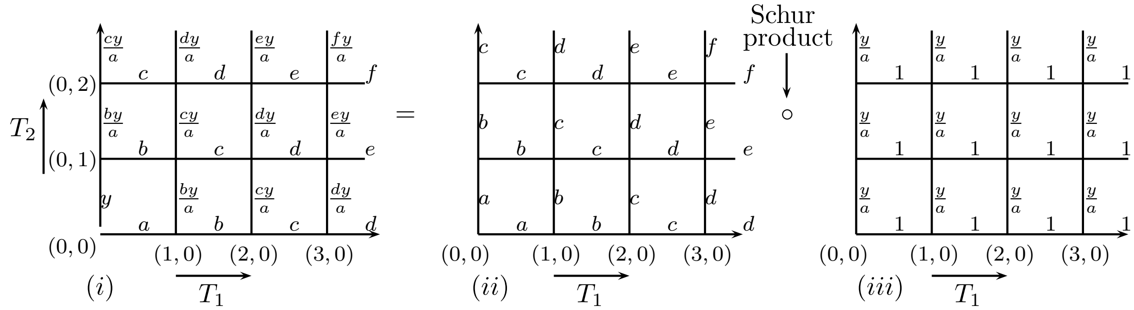


Figure 2. Weight diagram of the 2-variable weighted shift in Theorem 2.6.

**Theorem 2.6** *For* $W_{(\alpha,\beta)}$, $\widehat{W}_{(\alpha,\beta)} = \widetilde{W}_{(\alpha,\beta)}$ *if and only if* $W_{(\alpha,\beta)}$ *is given by Figure 2(i). Furthermore, if* $\mathrm{shift}(a, b, c, \cdots)$ *is subnormal, then* $W_{(\alpha,\beta)}$ *is subnormal with Berger measure* $\mu = \upsilon * \delta_{\left(1, \sqrt{\frac{y}{a}}\right)}$, *where* $\upsilon$ *is the diagonal Berger measure of the* 2*-variable weighted shift given by Figure 2(ii) (see [8]) and* $*$ *is the convolution product (see [19]).*

We now turn our attention to the continuity properties of the toral (resp. spherical) Aluthge transform of a commuting pair. Since the continuity of the toral Aluthge transform is straightforward, we focus on the spherical case. The following result is well known: for a single operator $T \in \mathcal{B}(\mathcal{H})$, the Aluthge transform map $T \to \widetilde{T}$ is $(\|\cdot\|, \|\cdot\|)-$ continuous on $\mathcal{B}(\mathcal{H})$ ([12]). We extend this to the multivariable case.

**Lemma 2.7** *(cf. [12, Lemma 2.1]) Let $\mathbf{T} \equiv (T_1, T_2)$ be a pair of commuting operators, and let $(T_1, T_2) \equiv (U_1 P, U_2 P)$ be its joint polar decomposition; recall that $P = \sqrt{T_1^* T_1 + T_2^* T_2}$. For $n \in \mathbb{N}$ and $t > 0$, let $f_n(t) := \sqrt{\max\left(\frac{1}{n}, t\right)}$ and let $A_n := f_n(\mathbf{T})$. Then we have:*
*(i) $\|A_n\| \leq \max\left(n^{-\frac{1}{2}}, \|P\|^{\frac{1}{2}}\right)$; (ii) $\left\|PA_n^{-1}\right\| \leq \|P\|^{\frac{1}{2}}$; (iii) $\left\|A_n - P^{\frac{1}{2}}\right\| \leq n^{-\frac{1}{2}}$;*
*(iv) $\left\|PA_n^{-1} - P^{\frac{1}{2}}\right\| \leq \frac{1}{4} n^{-\frac{1}{2}}$; (v) for $i = 1, 2$, $\left\|A_n T_i A_n^{-1} - P^{\frac{1}{2}} U_i P^{\frac{1}{2}}\right\| \leq \frac{5}{4} n^{-\frac{1}{2}} \|T_i\|^{\frac{1}{2}}$.*

From Lemma 2.7 we obtain:

**Lemma 2.8** *(cf. [12, Lemma 2.2]) Given $R \geq 1$ and $\epsilon > 0$, there are real polynomials $p$ and $q$ such that for every commuting pair $\mathbf{T} \equiv (T_1, T_2)$ with $\|T_i\| \leq R$ $(i = 1, 2)$, we have*

$$\left\|P^{\frac{1}{2}} U_i P^{\frac{1}{2}} - p\left(T_1^* T_1 + T_2^* T_2\right) T_i q\left(T_1^* T_1 + T_2^* T_2\right)\right\| < \epsilon.$$

By Lemmas 2.7 and 2.8, we have:

**Theorem 2.9** *The spherical Aluthge transform $(T_1, T_2) \to \widehat{(T_1, T_2)}$ is $(\|\cdot\|, \|\cdot\|)-$ continuous on $\mathcal{B}(\mathcal{H})$.*

We now study the class of spherically quasinormal (resp. spherically isometric) commuting pairs of operators ([2], [3], [4], [13], [15], [16]). In the literature, spherical quasinormality of a commuting $n$-tuple $\mathbf{T} \equiv (T_1, \cdots, T_n)$ is associated with the commutativity of each $T_i$ with $P^2$. It is not hard to prove that, for 2-variable weighted shifts, this is equivalent to requiring that $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ be a fixed point of the spherical Aluthge transform, that is, $\widehat{W_{(\alpha,\beta)}} = W_{(\alpha,\beta)}$. A straightforward calculation shows that this is equivalent to requiring that each $U_i$ commutes with $P$. In particular, $(U_1, U_2)$ is commuting whenever $(T_1, T_2)$ is commuting. Also, recall from Section 1 that a commuting pair $\mathbf{T}$ is a spherical isometry if $P^2 = I$. Thus, in the case of spherically quasinormal 2-variable weighted shifts, we always have $U_1^* U_1 + U_2^* U_2 = I$. In the following result, the key new ingredient is the equivalence of (i) and (ii).

**Theorem 2.10** *For $W_{(\alpha,\beta)} \equiv (T_1, T_2)$, the following statements are equivalent:*
*(i) $W_{(\alpha,\beta)}$ is spherically quasinormal; (ii) for all $(k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2 = C > 0$; (iii) $T_1^* T_1 + T_2^* T_2 = C \cdot I$.*

**Sketch of Proof.** Briefly stated, our strategy is as follows: By the continuous functional calculus, we can assume that $T_1$ and $T_2$ commute with $P$. It follows that for all $(k_1, k_2) \in \mathbb{Z}_+^2$, $\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2$ is constant. Next, we compute $T_1^* T_1 + T_2^* T_2$. $\square$

By the proof of Theorem 2.10, we remark that once the zero-th row of $T_1$, call it $W_0$, is given, then the entire 2-variable weighted shift is fully determined.

We briefly pause to recall the construction of Stampfli's shift $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge} \equiv \text{shift}\left((\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge\right)$, where $0 < \sqrt{a} < \sqrt{b} < \sqrt{c}$. From [6], $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is subnormal, with 2-atomic Berger measure $\xi = \rho_0 \delta_{s_0} + \rho_1 \delta_{s_1}$, where $\varphi_0 := -\frac{ab(c-b)}{b-a}$, $\varphi_1 := \frac{b(c-a)}{b-a}$, $s_0 := \frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_0}}{2}$, $s_1 := \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_0}}{2}$, $\rho_0 := \frac{s_1 - a}{s_1 - s_0}$, and $\rho_1 := \frac{a - s_0}{s_1 - s_0}$. We are now ready for

**Theorem 2.11** *Consider a spherically quasinormal $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ given by Figure 1(i), where $W_0 := \text{shift}(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots) = W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$. For given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, we let $\alpha_{(k_1,k_2)}^2 + \beta_{(k_1,k_2)}^2 = \varphi_1$, where $\varphi_1$ is given as above. Then, $W_{(\alpha,\beta)}$ is subnormal with Berger measure $\left(\frac{s_1 - a}{s_1 - s_0}\right) \delta_{(s_0,s_1)} + \left(\frac{a - s_0}{s_1 - s_0}\right) \delta_{(s_1,s_0)}$.*

The subnormality of $W_{(\alpha,\beta)}$ in Theorem 2.11 is a special case of the following result.

**Lemma 2.12** *([13]) Any spherical isometry is subnormal.*

By Theorem 2.10 and Lemma 2.12, we obtain:

**Corollary 2.13** *Any spherically quasinormal* 2*-variable weighted shift is subnormal.*

**Remark 2.14** *(i) A. Athavale and S. Poddar have recently proved that a commuting spherically quasinormal pair is always subnormal [3, Proposition2.1]; this provides a different proof of Corollary 2.13.*
*(ii) In a different direction, let* $Q_{\mathbf{T}}(X) := T_1^* X T_1 + T_2^* X T_2$. *By induction, it is easy to prove that if* $\mathbf{T}$ *is spherically quasinormal, then* $Q_{\mathbf{T}}^n(I) = (Q_{\mathbf{T}}(I))^n$ $(n \geq 0)$*; by [5, Remark 4.6],* $\mathbf{T}$ *is subnormal.*

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