COMPLETION OF HANKEL PARTIAL CONTRACTIONS OF EXTREMAL TYPE [b, c]

Raúl E. Curto [1, a], Sang Hoon Lee [2] and Jasang Yoon [3]

[1] Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242
[a] Corresponding author: raul-curto@uiowa.edu; http://www.math.uiowa.edu/~rcurto

[2] Department of Mathematics, Chungnam National University, Daejeon, 305-764, Republic of Korea

[3] Department of Mathematics, The University of Texas-Pan American, Edinburg, Texas 78539

Abstract. We find concrete necessary and sufficient conditions for the existence of contractive completions of Hankel partial contractions of size 4×4 in the extremal case. Along the way we introduce a new approach that allows us to solve, algorithmically, the contractive completion problem for 4×4 Hankel matrices. As an application, we obtain a concrete example of a partially contractive 4×4 Hankel matrix which does not admit a contractive completion.

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1. Introduction

Hankel and Toeplitz matrices have a long history [14], and have led to important recent applications in a variety of areas. On the other hand, matrix completion problems arise naturally in (i) probability and statistics (e.g. entropy methods for missing data; see, for instance, [11] and [12]); (ii) chemistry (e.g., the molecular conformation problem [5]); (iii) numerical analysis (e.g., optimization [17]); (iv) electrical engineering (e.g., data transmission, coding and image enhancement; see, for instance, [3]); and (v) geophysics (seismic reconstruction problems [13]).

A *Hankel partial contraction* is a Hankel matrix such that not all of its entries are determined, but in which every well-defined submatrix is a contraction. We address the problem of whether a Hankel partial contraction in which the upper left triangle is known can be completed to a contraction. Given real numbers $a_1, \cdots, a_n$, let

$$H_\triangle(a_1, a_2, \cdots, a_n) := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & \\ \vdots & \vdots & \ddots & & \\ a_{n-1} & a_n & & & \\ a_n & & & & \end{pmatrix}. \tag{1}$$

We say that $H_\triangle(a_1, a_2, \cdots, a_n)$ is a *partial contraction* if all submatrices are contractions (in the sense that their operator norms are at most 1). In this article, we study the following two problems.

Problem 1.1. *Given real numbers $a_1, a_2, \cdots, a_n$, find the necessary and sufficient conditions on the given data to guarantee the existence of a contractive Hankel completion.*

Problem 1.2. *Given real numbers $a_1, a_2, \cdots, a_n$, find a real number x such that*

$$H_\triangle(a_1, a_2, \cdots, a_n, x)$$

is partially contractive.

Let

$$H_n \equiv H(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1}) := \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & x_{n-3} & x_{n-2} \\ a_n & x_1 & \cdots & x_{n-2} & x_{n-1} \end{pmatrix}, \tag{2}$$

where $x_1, \cdots, x_{n-1}$ are real numbers to be determined.

Since $\|S\| \le \|T\|$ if S is a submatrix of the matrix T, it follows that each submatrix of a contraction is again a contraction. Thus, a necessary condition for a partial matrix T to be contraction is that each submatrix must be a contraction. We call a partial matrix meeting this necessary condition a *partial contraction* (*well-posed condition*). We say that Problem 1.1 is *well-posed* if

$$H_\triangle(a_1, a_2, \cdots, a_n)$$

is partially contractive, and that it is *soluble* if

$$H(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{n-1})$$

is contractive for some $x_1, \cdots, x_{n-1}$. We also say that $H_\triangle(a_1, a_2, \cdots, a_n)$ is *extremal* if $a_1^2 + \cdots + a_n^2 = 1$. When this happens, we will often say that $(a_1, a_2, \cdots, a_n)$ is extremal.

Similarly, we say that Problem 1.2 is *well-posed* if

$$H_\triangle(a_1, a_2, \cdots, a_n)$$

is partially contractive, and that it is *soluble* if $H_\triangle(a_1, a_2, \cdots, a_n, x)$ is partially contractive for some x.

For 2×2 *operator* matrices (with no required Hankel condition), a solution to the completion problem

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}$$

has been given by G. Arsene and A. Gheondea [1], by C. Davis, W. Kahan and H. Weinberger [9] (see also [8] and [4]), by C. Foiaş and A. Frazho [10] (using Redheffer products), by S. Parrott [18], and by Y. L. Shmul'yan and R. N. Yanovskaya [20]; a solution is also implicit in the work of W. Arveson [2] (see also [19] and [15]).

In this paper, we find necessary and sufficient conditions for the existence of contractive completions of Hankel partial contractions in the extremal 4×4 case. We do this by using a new technique that allows us to solve, algorithmically, the contractive completion problem for 4×4 Hankel matrices. To illustrate our approach, we first present a complete solution of Problem 1.1 for 3×3 matrices.

We conclude this section by recalling an observation in [6, Section 3, first paragraph], that is, the fact that it is straightforward to verify in these problems that once x_1 has been found we can easily obtain x_2, and a fortiori x_3. What happens is that, after finding x_1, one can consider new completion problems, incorporating x_1 as datum; these new problems are easier to solve, as described in [6]. Thus, in all our results we shall always aim to find x_1 first. However, as we shall see in the extremal 4×4 case in Section 4, the values of x_2 and x_3 are immediately determined once we know the value of x_1. Thus, our search for x_1 automatically yields values for x_2 and x_3.

2. Some technical lemmas

We begin by recalling that an $n \times n$ matrix M is a contraction if and only if the matrix

$$P \equiv P(M) := I - MM^* \tag{3}$$

is positive semi-definite (in symbols, $P \geq 0$), where I is the identity matrix and M^* is the adjoint of M. In order to check the positivity of P, we use the following version of Choleski's Algorithm.

Lemma 2.1. *Assume that*

$$P = \begin{pmatrix} u & \mathbf{t} \\ \mathbf{t}^* & P_0 \end{pmatrix},$$

where P_0 is an $(n-1) \times (n-1)$ matrix, $\mathbf{t}$ is a row vector, and u is a real number.
(i) If P_0 is invertible and positive, then $P \geq 0 \Longleftrightarrow (u - \mathbf{t}P_0^{-1}\mathbf{t}^*) \geq 0 \Longleftrightarrow \det P \geq 0$.
(ii) If $u > 0$ then $P \geq 0 \Leftrightarrow P_0 - \mathbf{t}^* u^{-1} \mathbf{t} \geq 0$.

We recall that for an $m \times n$ matrix A, a *Moore-Penrose inverse* of A is defined as an $n \times m$ matrix $A^\dagger$ satisfying all of the following four conditions:
(i) $AA^\dagger A = A$; (ii) $A^\dagger AA^\dagger = A^\dagger$; (iii) $\left(AA^\dagger\right)^* = AA^\dagger$; (iv) $\left(A^\dagger A\right)^* = A^\dagger A$.

The following result is a special form of Smul'jan's Lemma [21].

Lemma 2.2. *Let $P \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a finite matrix. Then $P \geq 0$ if and only if the following conditions hold:*
(i) $A \geq 0$;
(ii) ran $B \subseteq$ ran A (where, for a matrix T, ran T means the range of T as an operator); and
(iii) $C \geq B^* A^\dagger B$, where $A^\dagger$ is the Moore-Penrose inverse of A.

Proof. Since P is a finite matrix, it has closed range and hence A has a Moore-Penrose inverse $A^\dagger$. The desired result now follows from ([7, Lemma 1.2]). $\square$

Lemma 2.3. *(cf. [9], [18]) If* $\begin{pmatrix} A \\ C \end{pmatrix}$ *and* $\begin{pmatrix} A & B \end{pmatrix}$ *are rectangular contractions, then there exists a matrix D such that the matrix* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *is a contraction as well.*

For the case of a 3×3 Hankel matrix $H(a,b,c;d,e)$, we let

$$P_{13} := 1 - a^2 - b^2 - c^2,$$

$$P_{22} := I - \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1-a^2-b^2 & -ab-bc \\ -ab-bc & 1-b^2-c^2 \end{pmatrix},$$

$$P_{23} := I - \begin{pmatrix} a & b & c \\ b & c & d \end{pmatrix} \begin{pmatrix} a & b \\ b & c \\ c & d \end{pmatrix} = \begin{pmatrix} 1-a^2-b^2-c^2 & -ab-bc-cd \\ -ab-bc-cd & 1-b^2-c^2-d^2 \end{pmatrix} \text{ and}$$

$$P_{23}(x) := I - \begin{pmatrix} a & b & c \\ b & c & x \end{pmatrix} \begin{pmatrix} a & b \\ b & c \\ c & x \end{pmatrix} = \begin{pmatrix} 1-a^2-b^2-c^2 & -ab-bc-cx \\ -ab-bc-cx & 1-b^2-c^2-x^2 \end{pmatrix}.$$

Remark 2.4. While we focus on Hankel matrices, we mention in passing that there is a close connection between completion problems for Hankel matrices and those for Toeplitz matrices. An $m \times n$ matrix T is said to be *Toeplitz* if it is constant on diagonals, i.e., $T_{i+1,j+1} = T_{i,j}$ $(1 \le i \le m-1,\ 1 \le j \le n-1)$. Let T be an $m \times n$ partial Toeplitz matrix whose d specified diagonals from the lower left-hand corner of the matrix form a consecutive sequence. In [16], it was shown that every partial Toeplitz contraction has a Toeplitz contractive completion if and only if at least one of the following conditions is satisfied: (i) $m = 1$ or $n = 1$, (ii) $d \le 1$, (iii) $d \ge m+n-2$, or (iv) $m = n$ and $d = 2n-3$.

For $n \in \mathbb{N}$, let $\mathcal{T}_{n\times n}$ and $\mathcal{H}_{n\times n}$ denote the $n \times n$ Toeplitz and Hankel matrices with real entries, respectively, and let $T \in \mathcal{T}_{n\times n}$. Consider the (unitary) $n \times n$ permutation matrix U defined by $Ue_1 := e_n$, $Ue_2 := e_{n-1}, \cdots, Ue_n := e_1$. It easily follows that UT is Hankel, and since multiplication by a permutation preserves contractivity, we observe that finding a contractive completion of a finite Hankel matrix is equivalent to finding a contractive completion of a finite Toeplitz matrix.

The following result describes the approach taken in [6] to study the completion problem for Hankel matrices.

Lemma 2.5. *([6]) Let* $H := \begin{pmatrix} Q & \mathbf{r} \\ \mathbf{r}^* & x \end{pmatrix}$, *where Q is a $k \times k$ matrix , $\mathbf{r}$ is a column vector of length k over $\mathbb{R}$, and x is a real number to be determined. Let $P := I - HH^*$,* $A(\mathbf{r}) := I - \begin{pmatrix} Q & \mathbf{r} \end{pmatrix} \begin{pmatrix} Q^* \\ \mathbf{r}^* \end{pmatrix}$, $B(\mathbf{r}) := I - \begin{pmatrix} Q \\ \mathbf{r}^* \end{pmatrix} \begin{pmatrix} Q^* & \mathbf{r} \end{pmatrix}$, *and assume that $A(\mathbf{r})$ is positive and invertible. The following statements hold.*
(i) $P \ge 0$ *if and only if* $\det P \ge 0$.
(ii) $\det P = \alpha x^2 + \beta x + \gamma$, *where*

$$\alpha := -\det(I - QQ^*)$$

,

$$\beta := -\det A(\mathbf{r})(\mathbf{r}^* A(\mathbf{r})^{-1} Q\mathbf{r} + \mathbf{r}^* Q^* A(\mathbf{r})^{-1} Q\mathbf{r})$$

and

$$\gamma := \det A(\mathbf{r})(1 - \mathbf{r}^*\mathbf{r} - \mathbf{r}^* Q^* A(\mathbf{r})^{-1} Q\mathbf{r}).$$

(iii) The discriminant of $\det P$ *is* $\beta^2 - 4\alpha\gamma = 4 \det A(\mathbf{r}) \det B(\mathbf{r})$.
(iv) The graph of $\det P$ *is a downward parabola which meets the* x*-axis at points* $x_\ell \leq x_r$, *and any value of* x *between* x_ℓ *and* x_r *gives rise to a contractive* H.

3. Partially contractive Hankel matrices: The 3×3 case

Let $a_1, a_2, \cdots, a_n$ be given data for Problem 1.1 or Problem 1.2, and recall that $H_\Delta(a_1, a_2, \cdots, a_n)$ is extremal if $a_1^2 + \cdots + a_n^2 = 1$. We begin with a well known result. For the reader's convenience, we include a proof. The proof also shows how one might attempt to solve Problem 1.1 in the general case.

Lemma 3.1. *Let* $H_2 := H(a, b; x) = \begin{pmatrix} a & b \\ b & x \end{pmatrix} \in M_2(\mathbb{R})$. *If* H_2 *is well-posed then* H_2 *admits a contractive completion. Moreover,* x *is given by*

$$\begin{cases} x = -a, & \text{if } a^2 + b^2 = 1 \text{ and } b \neq 0 \\ -1 \leq x \leq 1, & \text{if } a^2 = 1 \text{ and } b = 0 \\ \frac{-1-a+b^2}{1+a} \leq x \leq \frac{1-a-b^2}{1-a}, & \text{if } a^2 + b^2 < 1. \end{cases}$$

Proof. A straightforward calculation shows that $I - H_2 H_2^* = \begin{pmatrix} 1 - a^2 - b^2 & -b(a + x) \\ -b(a + x) & 1 - b^2 - x^2 \end{pmatrix}$. Hence there exist x such that $\begin{pmatrix} 1 - a^2 - b^2 & -b(a + x) \\ -b(a + x) & 1 - b^2 - x^2 \end{pmatrix} \geq 0$ whenever $a^2 + b^2 \leq 1$. In fact, x is given by

$$\begin{cases} x = -a, & \text{if } a^2 + b^2 = 1 \text{ and } b \neq 0 \\ -1 \leq x \leq 1, & \text{if } a^2 = 1 \\ \frac{-1-a+b^2}{1+a} \leq x \leq \frac{1-a-b^2}{1-a}, & \text{if } a^2 + b^2 < 1. \end{cases}$$

□

In the following theorem, we formulate a result that uses the Moore-Penrose inverse of a matrix like the one in Lemma 2.2. In this result, we present a complete solution of Problem 1.1 for 3×3 matrices.

Theorem 3.2. *Let* $H_3 := H(a, b, c; x, y) = \begin{pmatrix} a & b & c \\ b & c & x \\ c & x & y \end{pmatrix} \in M_3(\mathbb{R})$. *If* H_3 *is well-posed, then* H_3 *has a contractive completion.*

Proof. Observe that H_3 has a contractive completion if and only if there exist x and y such that

$$P_{33}(x, y) := \begin{pmatrix} 1 - a^2 - b^2 - c^2 & -ab - bc - cx & -ac - bx - cy \\ -ab - bc - cx & 1 - b^2 - c^2 - x^2 & -bc - cx - xy \\ -ac - bx - cy & -bc - cx - xy & 1 - c^2 - x^2 - y^2 \end{pmatrix} \geq 0.$$

If (a,b,c) is extremal and $c \neq 0$, in order for $P_{33}(x,y) \geq 0$, we must have $-ab-bc-cx=0$ and $-ac-bx-cy=0$. These equations define $x=-\frac{b(a+c)}{c}$, and $y=\frac{b^2(a+c)-ac^2}{c^2}$. Since H_3 is well-posed, $P_{22} \geq 0$, that is, $\det P_{22} = a^2c^2 - b^2(a+c)^2 \geq 0$. Thus,

$$P_{33}\left(-\tfrac{b(a+c)}{c}, \tfrac{b^2(a+c)-ac^2}{c^2}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{a^2c^2-b^2(a+c)^2}{c^2} & -\frac{b(a^2c^2-b^2(a+c)^2)}{c^3} \\ 0 & -\frac{b(a^2c^2-b^2(a+c)^2)}{c^3} & \frac{b^2(a^2c^2-b^2(a+c)^2)}{c^4} \end{pmatrix}$$

$$= \frac{a^2c^2-b^2(a+c)^2}{c^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\frac{b}{c} \\ 0 & -\frac{b}{c} & \frac{b^2}{c^2} \end{pmatrix} \geq 0.$$

If (a,b,c) is extremal and $c=0$, in order for $P_{33}(x,y) \geq 0$, we must have $ab=0$ and $bx=0$. In this case,

$$P_{33}(x,y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2-x^2 & -xy \\ 0 & -xy & 1-x^2-y^2 \end{pmatrix}.$$

If $b \neq 0$, then $a=x=0$ and hence $P_{33}(0,y) \geq 0 \iff -1 \leq y \leq 1$. If $b=0$, then $P_{33}(x,y) \geq 0$ if and only if $|x| \leq |a|$ and $y_1 \leq y \leq y_2$, where $y_1 := -\sqrt{\left(1-\frac{x^2}{a^2}\right)(1-x^2)}$ and $y_2 := \sqrt{\left(1-\frac{x^2}{a^2}\right)(1-x^2)}$. Thus H_3 has a contractive completion.

If (a,b,c) is not extremal and $\det P_{22}=0$, then by Lemma 2.1 (iii), H_3 admits a contractive completion if and only if there exist x and y such that

$$D \equiv \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}$$

$$:= \begin{pmatrix} 1-b^2-c^2-x^2-\frac{(ab+bc+cx)^2}{1-a^2-b^2-c^2} & -(bc+cx+xy)-\frac{(ab+bc+cx)(ac+bx+cy)}{1-a^2-b^2-c^2} \\ -(bc+cx+xy)-\frac{(ab+bc+cx)(ac+bx+cy)}{1-a^2-b^2-c^2} & 1-c^2-x^2-y^2-\frac{(ac+bx+cy)^2}{1-a^2-b^2-c^2} \end{pmatrix} \geq 0.$$

Since $\det P_{22}=0$, we now prove that the $(1,1)$-entry of the above matrix is less than or equal to 0 for every x. Indeed, for every x,

$$\begin{aligned} 1-b^2-c^2-x^2-\frac{(ab+bc+cx)^2}{1-a^2-b^2-c^2} &= -\frac{(1-a^2-b^2)x^2+2bc(a+c)x+c^2(1-b^2-c^2)}{1-a^2-b^2-c^2} \\ &= -\frac{(1-a^2-b^2)\left(x+\frac{bc(a+c)}{1-a^2-b^2}\right)^2}{1-a^2-b^2-c^2} \leq 0. \end{aligned}$$

Thus, the only possible x is $x = -\frac{bc(a+c)}{1-a^2-b^2}$ and in that case we can choose y in the interval $y_3 \le y \le y_4$, where

$$y_3 := \frac{c^2(-a+a^3+2ab^2+b^2c)-(1-a^2-b^2)(1-a^2-b^2-c^2)}{(1-a^2-b^2)^2}$$

and

$$y_4 := \frac{c^2(-a+a^3+2ab^2+b^2c)+(1-a^2-b^2)(1-a^2-b^2-c^2)}{(1-a^2-b^2)^2}.$$

If (a,b,c) is not extremal and $\det P_{22} \neq 0$, then by Lemma 2.2 (ii), we have that H_3 admits a contractive completion if and only if $D \ge 0$. Since $(\det P_{22})P_{13} \ge 0$, assume first that $d_{11} = 0$; that is $x = x_1 := \frac{-c(ab+bc)\pm\sqrt{(1-a-b^2-c+ac)(1+a-b^2+c+ac)P_{13}}}{1-a^2-b^2}$, then we have

$$D \ge 0 \Longrightarrow d_{12} = 0 \text{ and } d_{22} \ge 0.$$

Note that

$$d_{12} = 0 \Longleftrightarrow y = y_5 := -\tfrac{(1-a^2-b^2-c^2)(bc+cx)-(ac+bx)(ab+bc+cx)}{-abc-bc^2-x+a^2x+b^2x}.$$

Thus, when $x = x_1$, we have

$$D \ge 0 \Longleftrightarrow y = y_5 \text{ and } d_{22} \ge 0.$$

Because $x = x_1$ and $y = y_5$ imply $d_{22} \ge 0$, we have that H_3 admits a contractive completion if and only if $x = x_1$ and $y = y_5$.

Now assume that $d_{11} > 0$. Then a direct calculation shows that the (Moore-Penrose) inverse of $\begin{pmatrix} 1-a^2-b^2-c^2 & -ab-bc-cx \\ -ab-bc-cx & 1-b^2-c^2-x^2 \end{pmatrix}$ is

$$\begin{pmatrix} \frac{1-b^2-c^2-x^2}{(1-a^2-b^2-c^2)d_{11}} & \frac{ab+bc+cx}{(1-a^2-b^2-c^2)d_{11}} \\ \\ \frac{ab+bc+cx}{(1-a^2-b^2-c^2)d_{11}} & \frac{1-a^2-b^2-c^2}{(1-a^2-b^2-c^2)d_{11}} \end{pmatrix}.$$

Thus, by Lemma 2.2 (ii) again, we have that H_3 admits a contractive completion if and only if $d_{11}d_{22} \ge d_{12}^2$. Let $x_2 := \frac{-bc(a+b)-\sqrt{P_{13}\det P_{22}}}{1-a^2-b^2}$ and $x_3 := \frac{-bc(a+b)+\sqrt{P_{13}\det P_{22}}}{1-a^2-b^2}$. We also let $y_6 := \frac{-1-a+b^2-c-ac+c^2+c^3-2bcx+x^2+ax^2}{1+a-b^2+c+ac}$ and $y_7 := \frac{1-a-b^2-c+ac-c^2+c^3-2bcx-x^2+ax^2}{1-a-b^2-c+ac}$. Since $y_7 - y_6 = \frac{2\det P_{23}}{\det P_{22}} \ge 0$, we get that

$$D \ge 0 \Longleftrightarrow d_{11} \ge 0 \text{ and } d_{11}d_{22} \ge d_{12}^2 \Longleftrightarrow x_2 \le x \le x_3 \text{ and } y_6 \le y \le y_7. \tag{4}$$

Thus, if $a^2+b^2+c^2 < 1$ and $\det P_{22} \neq 0$, we have

$$\{x = x_1,\ y = y_5\} \text{ or } \{x_2 \le x \le x_3,\ y_6 \le y \le y_7\}.$$

Therefore, by the argument above, we conclude that H_3 has a contractive completion. $\square$

Remark 3.3. (i) From the Proof of Theorem 3.2, we can provide the exact values x and y given below when H_3 admits a contractive completion:

$$\begin{cases} x=0 \text{ and } -1\le y\le 1, & \text{if } a^2+b^2+c^2=1,\ b\neq 0 \text{ and } c=0 \\ |x|\le |a| \text{ and } y_1\le y\le y_2, & \text{if } a^2+b^2+c^2=1 \text{ and } b=c=0 \\ x=-\frac{b(a+c)}{c} \text{ and } y=\frac{b^2(a+c)-ac^2}{c^2}, & \text{if } a^2+b^2+c^2=1 \text{ and } c\neq 0 \\ x=-\frac{bc(a+c)}{1-a^2-b^2} \text{ and } y_3\le y\le y_4, & \text{if } a^2+b^2+c^2<1 \text{ and } \det P_{22}=0 \\ \{x=x_1,\ y=y_5\} \text{ or } \{x_2\le x\le x_3,\ y_6\le y\le y_7\}, & \text{if } a^2+b^2+c^2<1 \text{ and } \det P_{22}>0. \end{cases} \tag{5}$$

(ii) By (5), we can see that the solution set of Problem 1.1,

$$\left\{(x,y)\in\mathbb{R}^2 : H(a,b,c;x,y) \text{ is a contraction}\right\},$$

is a rectangle in $\mathbb{R}^2$.

4. Partially contractive Hankel matrices of extremal type

In this section we focus attention on the extremal case for 4×4 Hankel matrices of the form $H_4 := H(a,b,c,d;x,y,z)$, i.e., $a^2+b^2+c^2+d^2=1$. We introduce a new approach using Choleski's Algorithm that allows us to solve completely the contractive problem for 4×4 Hankel matrices. Consider the solubility of Problem 1.1 for the Hankel matrix H_4, which is well-posed and with (a,b,c,d) extremal.

The case $d=0$. We first consider the solubility of Problem 1.1 when the Hankel matrix is of the form $H_4 := H(a,b,c,0;x,y,z)$. Observe that the Proof of Theorem 4.1 below shows that the solution, if it exists, is generally *not* unique.

Theorem 4.1. *Assume $d=0$. Then Problem 1.1 is soluble for H_4 if and only if $b(a+c)=0$.*

Proof. We recall that Problem 1.1 is soluble for H_4 if and only if there exist x such that we simultaneously have

$$\left\|\begin{pmatrix} a & b & c & 0 \\ b & c & 0 & x \end{pmatrix}\right\|\le 1 \text{ and } \left\|\begin{pmatrix} a & b & c \\ b & c & 0 \\ c & 0 & x \end{pmatrix}\right\|\le 1 \text{ (by Lemma 2.3).} \tag{6}$$

By a direct calculation, we have

$$\left\|\begin{pmatrix} a & b & c & 0 \\ b & c & 0 & x \end{pmatrix}\right\|\le 1 \iff \begin{pmatrix} 0 & -b(a+c) \\ -b(a+c) & a^2-x^2 \end{pmatrix}\ge 0 \iff \begin{cases} x^2\le a^2 \\ b(a+c)=0. \end{cases}$$

We see at once that $b(a+c)=0$ is a necessary condition for solubility. On the other hand, if $b(a+c)=0$, it suffices to choose $|x|\le|a|$ to ensure that $\left\|\begin{pmatrix} a & b & c & 0 \\ b & c & 0 & x \end{pmatrix}\right\|\le 1.$

Looking now at the 3×3 matrix, we have

$$\left\|\begin{pmatrix} a & b & c \\ b & c & 0 \\ c & 0 & x \end{pmatrix}\right\|\le 1 \iff \begin{pmatrix} 0 & -b(a+c) & -c(a+x) \\ -b(a+c) & a^2 & -bc \\ -c(a+x) & -bc & a^2+b^2-x^2 \end{pmatrix}\ge 0$$

which is equivalent to the conditions

$$\begin{cases} b(a+c)=0 \\ c(a+x)=0 \\ \begin{pmatrix} a^2 & -bc \\ -bc & a^2+b^2-x^2 \end{pmatrix} \geq 0. \end{cases} \tag{7}$$

Thus, if $c=0$, these conditions reduce to $ab=0$ and $x^2 \leq 1$, so choosing $x^2 \leq a^2$ fulfills (6). On the other hand, if $c \neq 0$, then we must choose $x=-a$ to meet (7) and with this choice we also have (6).

From the above analysis, it follows that Problem 1.1 is soluble for H_4 if and only if $b(a+c)=0$. □

The case $d \neq 0$. Let $t := -(ab+bc+cd+ad)(ab+bc+cd-ad)$. We break the study of this case into three subcases: $t=0$ and $a \neq 0$; $t=0$ and $a=0$; and $t>0$. Observe that the Proof of Theorem 4.2 shows that Problem 1.1 admits two solutions when $t=0$ and $a \neq 0$. Similarly, the Proofs of Theorems 4.3 and 4.4 show that Problem 1.1 admits the same solutions when $t=0$ and $a=0$, and when $t>0$, the solution is always unique.

Consider the 4×4 matrix

$$P \equiv P(x,y,z) \equiv (p_{ij})_{i,j=1}^4 := I - H_4 H_4^* (p_{ij})_{i,j=1}^4.$$

Since $p_{11}=0$ (recall that (a,b,c,d) is extremal), the positive semi-definiteness of P requires $p_{12}=p_{13}=p_{14}=0$ (and of course $p_{21}=p_{31}=p_{41}=0$), so under the assumption $d \neq 0$, there can be at most one triple $(x,y,z) \in \mathbb{R}^3$ such that $P(x,y,z) \geq 0$.

Observe that $p_{12}=0 \iff x = -\frac{ab+bc+cd}{d}$, and that $p_{22} \equiv a^2-x^2$ satisfies the identity $d^2 p_{22} - t = 0$ for x as above, so $p_{22}=0 \iff t=0 \iff a^2d^2-(ab+bc+cd)^2=0$.

Theorem 4.2. *Assume $d \neq 0$, $t=0$ and $a \neq 0$. Then Problem 1.1 is soluble for H_4 if and only if one of the following two conditions hold:*
(i) $(a+c)(b+d)=ac+bd=0$;
(ii) $ab+bc+cd-ad=0$.
(Observe that (i) and (ii) cannot occur simultaneously, since $a \neq 0$ and $d \neq 0$.)

Proof. We have that

$$P \equiv P(x,y,z)$$

$$:= \begin{pmatrix} 0 & -ab-bc-cd-dx & -ac-bd-cx-dy & -ad-bx-cy-dz \\ -ab-bc-cd-dx & a^2-x^2 & -bc-cd-dx-xy & -bd-cx-dy-xz \\ -ac-bd-cx-dy & -bc-cd-dx-xy & a^2-x^2+b^2-y^2 & -cd-dx-xy-yz \\ -ad-bx-cy-dz & -bd-cx-dy-xz & -cd-dx-xy-yz & a^2-x^2+b^2-y^2+c^2-z^2 \end{pmatrix}.$$

Assume first that Problem 1.1 is soluble, that is, $P(x,y,z) \geq 0$ for a triple $(x,y,z) \in \mathbb{R}^3$. Recall that $t=0 \iff p_{22} \equiv a^2-x^2=0$. Thus, it follows from Lemma 2.5 (iv), that $p_{12}=p_{13}=p_{14}=p_{23}=p_{24}=0$ and

$$R \equiv R(x,y,z) := \begin{pmatrix} b^2-y^2 & -cd-dx-xy-yz \\ -cd-dx-xy-yz & b^2-y^2+c^2-z^2 \end{pmatrix}$$

$$\equiv \begin{pmatrix} b^2-y^2 & ab-xy+bc-yz \\ ab-xy+bc-yz & b^2-y^2+c^2-z^2 \end{pmatrix} \geq 0,$$

because $p_{12} = -ab - bc - cd - dx = 0$. Since $p_{22} = 0$, two cases arise.

Subcase 1: $x = a$. Here we have

$$\begin{cases} p_{12} = 0 \iff ab + bc + cd + da = 0, \\ p_{13} = 0 \iff 2ac + bd + dy = 0, \\ p_{14} = 0 \iff ab + ad + cy + dz = 0, \\ p_{23} = 0 \iff bc + cd + da + ay = 0 \text{ and} \\ p_{24} = 0 \iff bd + ac + dy + az = 0. \end{cases}$$

From $p_{12} = 0$, $p_{23} = 0$ and $a \neq 0$ we obtain at once $y = b$, which when combined with $p_{13} = 0$ yields $ac + bd = 0$. From the last equation together with $p_{24} = 0$, we obtain $z = c$. Thus, $P \geq 0$ implies $ab + bc + cd + da$ $(= p_{14}) = ac + bd$ $(= \frac{p_{24}}{2}) = 0$ (this is condition (i)), and moreover $y = b$ and $z = c$.

Subcase 2: $x = -a$. Here $p_{12} = 0 \iff ab + bc + cd - ad = 0$ (this is already condition (ii)) and $p_{23} = 0 \iff bc + cd - ad - ay = 0$, so that $P \geq 0$ implies $y = -b$. Looking now at $p_{14} = 0$, we see that $ad - ab - bc + dz = 0$, which together with $p_{12} = 0$ yields $z = -c$.

Conversely, if (i) holds, we can let $x := a$, $y := b$ and $z := c$, and verify that $p_{12} = p_{13} = p_{14} = p_{23} = p_{24} = 0$ and that $R = 0$, so that $P = 0$, and therefore Problem 1.1 is soluble. If (ii) holds, we can let $x := -a, y := -b$ and $z := -c$, and verify that $P = 0$. □

Theorem 4.3. *Assume $d \neq 0$, $t = 0$ and $a = 0$. Then Problem 1.1 is soluble for H_4 if and only if $c(b + d) = 0$.*

Proof. By the same argument given in the Proof of Theorem 4.2, we have that

$$P(x, y, z) = \begin{pmatrix} 0 & -bc - cd & -bd - dy & -cy - dz \\ -bc - cd & 0 & -bc - cd & -bd - dy \\ -bd - dy & -bc - cd & b^2 - y^2 & -cd - yz \\ -cy - dz & -bd - dy & -cd - yz & b^2 - y^2 + c^2 - z^2 \end{pmatrix}.$$

Thus, it follows at once that $P \geq 0 \Longrightarrow p_{12} \equiv -c(b + d) = 0$. Conversely, if $c(b + d) = 0$, we let $y := -b$ and $z := -c$, and we easily verify that $P = 0$. □

Theorem 4.4. *Assume $d \neq 0$ and $t > 0$. Then Problem 1.1 is soluble if and only if*

$$q := [(a + c)^2(b + d)^2 - (ac + bd)^2](ab + bc + cd - ad)^2 \geq 0.$$

Proof. Assume now that $P(x, y, z) \geq 0$ for a unique triple $(x, y, z) \in \mathbb{R}^3$ with $p_{22} \equiv a^2 - x^2 > 0$. We certainly have $p_{12} = p_{13} = p_{14} = 0$, but this time we cannot assume that $p_{23} = p_{24} = 0$. Instead, we observe, via a calculation using *Mathematica* [23], that

$$p_{22}p_{33} = p_{23}^2 + \frac{q}{d^4}, \; p_{22}p_{34} = p_{23}p_{24} - \frac{cq}{d^5} \text{ and } p_{22}p_{44} = p_{24}^2 + \frac{c^2 q}{d^6}.$$

If we now apply Choleski's Algorithm (or Lemma 2.2 (ii)) to P, then we see at once that

$$P \geq 0 \iff \begin{pmatrix} p_{33} - \frac{p_{23}^2}{p_{22}} & p_{34} - \frac{p_{23}p_{24}}{p_{22}} \\ p_{34} - \frac{p_{23}p_{24}}{p_{22}} & p_{44} - \frac{p_{24}^2}{p_{22}} \end{pmatrix} = \frac{q}{d^6 p_{22}} \begin{pmatrix} d^2 & -cd \\ -cd & c^2 \end{pmatrix} \geq 0. \tag{8}$$

Since the last 2×2 matrix in (8) is always positive semi-definite of rank at least 1, it follows that $q \geq 0$.

Conversely, if $q \geq 0$ we can use the equations for p_{12}, p_{13} and p_{14} to define x, y and z, respectively; for instance, $x := \frac{-(ab+bc+cd)}{d}$. With this choice of x we verify that $p_{22} \equiv a^2 - x^2 = -(ab+bc+cd+ad)(ab+bc+cd-ad) = t > 0$, and therefore $P \geq 0$ from (8). $\square$

Remark 4.5. (i) We know that once x is given in $H(a,b,c,d;x,y,z)$, we can easily obtain y and a fortiori z which make $H(a,b,c,d;x,y,z)$ a contractive Hankel completion. From the Proofs of Theorem 4.1, 4.2, 4.3 and 4.4, we can provide the exact value x given below such that $H_{\triangle}(a,b,c,d,x)$ is partially contractive.

$$\begin{cases} |x| \leq |a|, & \text{if } c = d = 0 \text{ and } b(a+c) = 0 \\ x = -a, & \text{if } c = b(a+c) = 0 \text{ and } d \neq 0 \\ x = a, & \text{if } t = (a+c)(b+d) = ac + bd = 0,\ a \neq 0 \text{ and } d \neq 0 \\ x = -a, & \text{if } t = ab + bc + cd - ad = 0,\ a \neq 0 \text{ and } d \neq 0 \\ x = 0, & \text{if } a = t = c(b+d) = 0 \text{ and } d \neq 0 \\ x = -\frac{ab+bc+cd}{d}, & \text{if } d \neq 0,\ q \geq 0 \text{ and } t > 0. \end{cases} \tag{9}$$

(ii) By (5), (9), and imitating the calculations in the Proof of Theorem 3.2, we have that the solution set of Problem 1.1 for $H_4(a,b,c,d;x,y,z)$,

$$\left\{(x,y,z) \in \mathbb{R}^3 : H_4(a,b,c,d;x,y,z) \text{is contractive}\right\},$$

is a prism in $\mathbb{R}^3$, when $a^2+b^2+c^2+d^2 = 1$.
(iii) By (5), (9) and (ii), we conjecture that the above mentioned solution set is also a prism in $\mathbb{R}^3$, when (a,b,c,d) is not extremal.

5. Applications: Hankel extensions and an example

While we have described a complete solution to Problem 1.1 for 3×3 Hankel partial contractions and for 4×4 extremal Hankel partial contractions, much less can be said of Problem 1.2. As the reader may have gleaned from the discussion in Section 1, and from the Proof of Theorem 4.1, Problem 1.2 is significantly more difficult, since we must search for a value of x which simultaneously yields various Hankel partial contractions of bigger size. In this section, we apply the results from Section 4 to solve Problem 1.2 when $n = 3$. We recall that $x_2 = \frac{-bc(a+b)-\sqrt{P_{13}\det P_{22}}}{1-a^2-b^2}$ and $x_3 = \frac{-bc(a+b)+\sqrt{P_{13}\det P_{22}}}{1-a^2-b^2}$ in the Proof of Theorem 3.2. Then we have:

Theorem 5.1. *Suppose that H_3 is well-posed. Then there exists x such that (a,b,c,x) is extremal and $H_{\triangle}(a,b,c,x)$ has a contractive completion if and only if one of the following four conditions hold:*
(i) $b(a+c) = 0$ *and* $a^2+b^2+c^2 = 1$;
(ii) $\frac{bc(a+c)}{1-a^2-b^2} = \mp\sqrt{1-a^2-b^2-c^2}$, $a^2+b^2+c^2 < 1$ *and* $\det P_{22} = 0$;
(iii) $\frac{-c(ab+bc)\pm\sqrt{(1-a-b^2-c+ac)(1+a-b^2+c+ac)P_{13}}}{1-a^2-b^2} = \pm\sqrt{1-a^2-b^2-c^2}$, $a^2+b^2+c^2 < 1$ *and* $\det P_{22} > 0$;
(iv) $\pm\sqrt{1-a^2-b^2-c^2} \in [x_2, x_3]$, $a^2+b^2+c^2 < 1$ *and* $\det P_{22} > 0$.

Proof. ($\Longrightarrow$) Suppose that there exists x such that (a,b,c,x) is extremal and $H_{\triangle}(a,b,c,x)$ has a contractive completion. Then $x = \pm\sqrt{1-a^2-b^2-c^2}$. If $a^2+b^2+c^2 = 1$, then x should be zero, and hence by Theorem 4.1, we have $b(a+c) = 0$. Assume now that $a^2+b^2+c^2 < 1$ and $\det P_{22} = 0$. By Theorem 3.2 we must have $x = -\frac{bc(a+c)}{1-a^2-b^2}$. Thus, we have the condition (ii). If $a^2+b^2+c^2 < 1$ and $\det P_{22} > 0$, then by Theorem 3.2, we have $x = \frac{-c(ab+bc)\pm\sqrt{(1-a-b^2-c+ac)(1+a-b^2+c+ac)P_{13}}}{1-a^2-b^2}$ or $\pm\sqrt{1-a^2-b^2-c^2} \in [x_2, x_3]$. Hence we have the conditions (iii) or (iv).

($\Longleftarrow$) The converses are all straightforward, using Theorems 3.2, 4.1, 4.2, 4.3, 4.4 and the calculations given above to define the values of x. $\square$

There is a formulation of Problem 1.1 for Toeplitz matrices which has been studied by C. R. Johnson and L. Rodman [16], and by H. J. Woerdeman [22]. [16, Theorem 1] implies that the 3×3 case is always soluble, and that there exist real numbers a,b,c,d such that $H(a,b,c,d;x,y,z)$ is partially contractive but not contractive for any choice of x,y,z; in both instances, the results are theoretical in nature. In Example 5.2 below, we provide concrete real numbers a,b,c such that $H(a,b,c,0;x,y,z)$ is partially contractive but not contractive for any choice of x,y,z.

Example 5.2. For $a \in \left(0, \sqrt{\frac{2}{3}}\right)$, let $b := \sqrt{\frac{1}{3}}$ and $c := \sqrt{\frac{2-3a^2}{3}}$. Then $H(a,b,c;x,y)$ is contractive but $H(a,b,c,0;x,y,z)$ is not contractive for any choice of x,y,z.

Proof. Note that $a^2+b^2+c^2 = 1$. The solubility of $H(a,b,c;x,y)$ is clear from Theorem 3.2. A direct calculation shows that $b(a+c) = \frac{3a+\sqrt{3(2-3a^2)}}{3\sqrt{2}} \neq 0$. Thus, by Theorem 4.1, we have that $H(a,b,c,0;x,y,z)$ is not contractive for any x,y,z. $\square$

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