Hyponormality of Bounded-Type Toeplitz Operators

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Abstract. In this paper we deal with the hyponormality of Toeplitz operators with matrix-valued symbols. The aim of this paper is to provide a tractable criterion for the hyponormality of *bounded-type* Toeplitz operators T_Φ (i.e., the symbol $\Phi \in L^\infty_{M_n}$ is a matrix-valued function such that Φ and Φ^* are of bounded type). In particular, we get a much simpler criterion for the hyponormality of T_Φ when the co-analytic part of the symbol Φ is a left divisor of the analytic part.

Keywords. Toeplitz operators, Hardy spaces, matrix-valued symbols, functions of bounded type, rational functions, hyponormal, pseudo-hyponormal, interpolation problems.

1. Introduction

An elegant theorem of C. Cowen [Co] characterizes the hyponormality of Toeplitz operators T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} \subset \mathbb{C}$ in terms of their symbols $\varphi \in L^\infty(\mathbb{T})$. Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution with specified properties of a certain functional equation involving the operator's symbol φ. Today, this theorem is referred as *Cowen's Theorem.* In 2006, Gu, Hendricks and Rutherford [GHR] extended Cowen's Theorem for block Toeplitz operators T_Φ on the matrix-valued Hardy space $H^2_{M_n}(\mathbb{T})$. Their characterization resembles Cowen's Theorem, except for an additional condition - the normality of the symbol $\Phi \in L^\infty_{M_n}$. However, the hyponormality of T_Φ with matrix-valued symbol Φ, though solved in principle by the characterization given in [GHR], is in practice very complicated - in fact it may not even be possible to find tractable conditions for the hyponormality of T_Φ in terms of their symbols Φ unless certain assumptions are made about Φ.

To date, explicit criteria for the hyponormality of Toeplitz operators T_Φ have been established via interpolation problems when Φ is a matrix-valued trigonometric polynomial or a rational function (cf. [GHR], [HL1], [HL2]). Very recently, in [CHL], the hyponormality of Toeplitz operators T_Φ was investigated when Φ is a matrix-valued function such that Φ and Φ^* are of bounded type (a "bounded type" function means a quotient of two bounded analytic functions). A sufficient condition for the hyponormality was given by an interpolation involving the H^∞-functional calculus via a triangular representation for compressions of the unilateral shift operator T_z. The aim of this paper is to provide a tractable criterion for the hyponormality of *bounded-type* Toeplitz operators T_Φ (i.e., Φ and Φ^* are of bounded type). In particular, we get a much simpler criterion for the hyponormality of T_Φ when the co-analytic part of the symbol is a left divisor of the analytic part. To do so, we provide a definition of "divisor" for matrix-valued analytic functions whose adjoints are of bounded type.

We first review a few essential facts for (block) Toeplitz operators and (block) Hankel operators (cf. [BS], [Do], [Ni], [Pe]). Let $\mathcal{H}$ denote an infinite dimensional separable complex Hilbert space

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and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H})$, A^* and $\ker A$ denote the adjoint and the kernel, respectively, of A. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if its self-commutator $[A^*, A] \equiv A^*A - AA^*$ is positive semi-definite. For a set $\mathcal{M}$, $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of square-integrable measurable functions on the unit circle $\mathbb{T} \equiv \partial\mathbb{D}$ in the complex plane and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $L^\infty \equiv L^\infty(\mathbb{T})$ be the set of bounded measurable functions on $\mathbb{T}$ and let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty \cap H^2$. For a Hilbert space $\mathcal{X}$, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. Let $M_{n\times m}$ denote the set of $n \times m$ complex matrices and write $M_n := M_{n\times n}$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty \otimes M_n)$ then the block Toeplitz operator T_Φ and the block Hankel operator H_Φ on $H^2_{\mathbb{C}^n}$ are defined by

$$T_\Phi f = P_n(\Phi f) \quad \text{and} \quad H_\Phi f = JP_n^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}), \tag{1}$$

where P_n and $P_n^\perp$ denote the orthogonal projections that map from $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\big(H^2_{\mathbb{C}^n}\big)^\perp$, respectively and J denotes the unitary operator from $L^2_{\mathbb{C}^n}$ to $L^2_{\mathbb{C}^n}$ given by $J(g)(z) = \overline{z}I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ ($I_n :=$ the $n \times n$ identity matrix). If $n = 1$, T_Φ and H_Φ are called the (scalar) Toeplitz operator and the (scalar) Hankel operator, respectively. For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{2}$$

For $\Phi \in L^\infty_{M_n}$, we also write

$$\Phi_+ := P_n\Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \big(P_n^\perp \Phi\big)^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_-^* + \Phi_+$. However, it will often be convenient to allow the constant term in Φ_-. Hence, if there is no confusion we may assume that Φ_- shares the constant term with Φ_+: in this case, $\Phi(0) = \Phi_+(0) + \Phi_-(0)^*$. A matrix function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if Θ is isometric almost everywhere on $\mathbb{T}$. The following facts are clear from the definition:

$$T^*_\Phi = T_{\Phi^*}, \quad H^*_\Phi = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{3}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); \tag{4}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \quad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n},\ \Psi \in H^\infty_{M_n}). \tag{5}$$

For matrix-valued functions

$$A(z) := \sum_{j=-\infty}^{\infty} A_j z^j \in L^2_{M_n} \text{ and } B(z) := \sum_{j=-\infty}^{\infty} B_j z^j \in L^2_{M_n},$$

we define the inner product of A and B by

$$\langle A, B\rangle := \int_{\mathbb{T}} \operatorname{tr}(B^*A)\, d\mu = \sum_{j=-\infty}^{\infty} \operatorname{tr}(B_j^* A_j),$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix and define $||A||_2 := \langle A, A\rangle^{\frac{1}{2}}$. We also define, for $A \in L^\infty_{M_n}$,

$$||A||_\infty := \text{ess sup}_{z\in\mathbb{T}} ||A(z)|| \quad (||\cdot|| \text{ denotes the spectral norm of a matrix}).$$

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We would remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det\Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

For notational convenience, we write

$$H_0^2 := zI_n H^2_{M_n}\,.$$

Suppose $\Phi \equiv \Phi_-^* + \Phi_+ = [\varphi_{ij}] \in L^\infty_{M_n}$ is of bounded type, in other words, each entry φ_{ij} is of the form $\varphi_{ij}(z) = \psi_{ij}^{(1)}(z)/\psi_{ij}^{(2)}(z)$ for almost all $z \in \mathbb{T}$, where $\psi_{ij}^{(1)}, \psi_{ij}^{(2)} \in H^\infty$. Then it was ([Ab]) known that φ_{ij} can be written as the form $\varphi_{ij} = \overline{\theta_{ij}} b_{ij}$, where θ_{ij} is an inner function, $b_{ij} \in H^\infty$, and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common multiple of θ_{ij}'s then we can write

$$\Phi = [\varphi_{ij}] = [\overline{\theta_{ij}} b_{ij}] = [\overline{\theta} c_{ij}] = C\Theta^* \quad (\Theta \equiv \theta I_n,\ C \equiv [c_{ij}] \in H^\infty_{M_n}).$$

Thus we have

$$\Phi_- = \Theta(C - \Phi_+\Theta)^* \equiv \Theta A^* \quad (\Theta \equiv \theta I_n,\ A := C - \Phi_+\Theta \in H^2_{M_n}). \tag{6}$$

If Ω is the greatest common left inner divisor of A and Θ in the representation (6):

$$\Phi_- = \Theta A^* = A^*\Theta \quad (\Theta \equiv \theta I_n \text{ for an inner function } \theta),$$

then $\Theta = \Omega\Omega_l$ and $A = \Omega A_l$ for some inner matrix Ω_l (where $\Omega_l \in H^2_{M_n}$ because $\det\Theta$ is not identically zero) and some $A_l \in H^2_{M_n}$. Thus we can write

$$\Phi_- = {A_l}^*\Omega_l, \quad \text{where } A_l \text{ and } \Omega_l \text{ are left coprime:} \tag{7}$$

in this case, $A_l^*\Omega_l$ is called the *left coprime factorization* of F and similarly, we can write

$$\Phi_- = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime:} \tag{8}$$

in this case, $\Omega_r A_r^*$ is called the *right coprime factorization* of Φ_-.

On the other hand, we note that by (5), the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator T_{zI_n} on $H^2_{\mathbb{C}^n}$. Thus if $\ker H_\Phi \neq \{0\}$ then by the Beurling-Lax-Halmos Theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ. In general, Θ need not be a square matrix function.

We however have:

Lemma 1.1. ([GHR]) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

(i) Φ is of bounded type;
(ii) $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;
(iii) $\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.

In general, the condition "right coprime" for matrix-valued functions is not easy to check. It was also known [CHKL] that if $A, B \in H^2_{M_n}$ and B is a rational function such that $\det B$ is not identically zero then

$$A \text{ and } B \text{ are right coprime } \iff \ker A(\alpha) \cap \ker B(\alpha) = \{0\} \text{ for any } \alpha \in \mathbb{D}. \tag{9}$$

On the other hand, recently, Gu, Hendricks and Rutherford [GHR] characterized the hyponormality of block Toeplitz operators in terms of their symbols:

Lemma 1.2. (Hyponormality of Block Toeplitz Operators) [GHR] For each $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \Big\{ K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

Observe that for $\Phi \in L^\infty_{M_n}$, by (4),

$$[T^*_\Phi, T_\Phi]_p := H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi + T_{\Phi^*\Phi - \Phi\Phi^*}.$$

Since the normality of Φ is a necessary condition for the hyponormality of T_Φ, the positivity of $H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi$ is an essential condition for the hyponormality of T_Φ. Thus we isolate this property as a new notion, weaker than hyponormality. The reader will notice at once that this notion is meaningful for non-scalar symbols. Now a block Toeplitz operator T_Φ is said to be *pseudo-hyponormal* if

$$H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi \geq 0.$$

We thus have that

$$T_\Phi \text{ is hyponormal } \iff T_\Phi \text{ is pseudo-hyponormal and } \Phi \text{ is normal}$$

and that (via [GHR, Theorem 3.3])

$$T_\Phi \text{ is pseudo-hyponormal } \iff \mathcal{E}(\Phi) \neq \emptyset.$$

Note that for each $M \in M_n$,

$$T_\Phi \text{ is pseudo-hyponormal} \iff T_{\Phi+M} \text{ is pseudo-hyponormal.} \tag{10}$$

Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (6) we can write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i $(i = 1, 2)$ and $A, B \in H^2_{M_n}$. For $F = [f_{ij}] \in H^\infty_{M_n}$, we say that F is *rational* if each entry f_{ij} is a rational function. Also if given $\Phi \in L^\infty_{M_n}$, Φ_+ and Φ_- are rational then we say that T_Φ has a rational symbol Φ.

The organization of this paper is as follows. In Section 2, we prove the main theorem - a criterion for the hyponormality of bounded-type Toeplitz operators T_Φ. In Section 3, we consider the rational symbol case. In Section 4, we provide revealing examples to illustrate how much more it is gained by our criterion.

2. A criterion for hyponormality of bounded-type Toeplitz operators

Let $\lambda \in \mathbb{D}$ and write

$$b_\lambda(z) := \xi \frac{z - \lambda}{1 - \bar{\lambda} z} \ (\xi \in \mathbb{T}) :$$

b_λ is called a *Blaschke factor* and $\theta := e^{i\theta} \prod_{m=1}^d b_m$ is called a finite Blaschke product. For an inner matrix function $\Theta \in H^\infty_{M_n}$, we write

$$\mathcal{H}(\Theta) := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}, \quad \mathcal{H}_\Theta := H^2_{M_n} \ominus \Theta H^2_{M_n} \quad \text{and} \quad \mathcal{K}_\Theta := H^2_{M_n} \ominus H^2_{M_n} \Theta.$$

If $\Theta = \theta I_n$ for an inner function θ, then $\mathcal{H}_\Theta = \mathcal{K}_\Theta$ and if $n = 1$, then $\mathcal{H}(\Theta) = \mathcal{H}_\Theta = \mathcal{K}_\Theta$. Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type: in this case, we shall say that T_Φ is a *bounded-type* Toeplitz operator. Then in view of (6) we can write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \tag{11}$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i $(i = 1, 2)$. If $\Phi \in L^\infty_{M_n}$ is rational then the θ_i are chosen as finite Blaschke products. Moreover it is known (cf. [CHL, Lemma 3.2]) that if T_Φ is pseudo-hyponormal then Θ_2 is an inner divisor of Θ_1 if the representations in (11) are right coprime factorizations even though the Θ_i are arbitrary inner functions. Thus, when we consider the pseudo-hyponormality of bounded-type Toeplitz operators T_Φ, we may assume that the symbol $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \Theta_1 \Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad \text{(right coprime factorizations)}. \tag{12}$$

For $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$, write

$$\mathcal{C}(\Phi) := \Big\{ K \in H^\infty_{M_n} :\ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Thus if $\Phi \in L^\infty_{M_n}$ then $K \in \mathcal{E}(\Phi)$ if and only if $K \in \mathcal{C}(\Phi)$ and $||K||_\infty \le 1$.

To prove the main theorem we need several auxiliary lemmas.

We begin with:

Lemma 2.1. If Θ_1 and Θ_2 are inner matrix functions in $H^\infty_{M_n}$, then

(a) $\widetilde{\mathcal{K}_{\Theta_1}} = \mathcal{H}_{\widetilde{\Theta}_1}$,

(b) $\mathcal{K}_{\Theta_1\Theta_2} = \mathcal{K}_{\Theta_1}\Theta_2 \oplus \mathcal{K}_{\Theta_2}$,

(c) $\mathcal{H}_{\Theta_1\Theta_2} = \Theta_1\mathcal{H}_{\Theta_2} \oplus \mathcal{H}_{\Theta_1}$.

Proof. (a) Let $C \in H^2_{M_n}$ be arbitrary. Then

$$\begin{aligned} A \in \mathcal{K}_{\Theta_1} &\iff \int_{\mathbb{T}} \mathrm{tr}((C\Theta_1)^* A) d\mu = \langle A, C\Theta_1\rangle = 0 \\ &\iff \int_{\mathbb{T}} \mathrm{tr}(\widetilde{A}(\widetilde{\Theta}_1\widetilde{C})^*) d\mu = \int_{\mathbb{T}} \mathrm{tr}(\widetilde{(C\Theta_1)^* A}\ ) d\mu = 0 \\ &\iff \langle \widetilde{A}, \widetilde{\Theta}_1\widetilde{C}\rangle = \int_{\mathbb{T}} \mathrm{tr}((\widetilde{\Theta}_1\widetilde{C})^*\widetilde{A}) d\mu = 0 \\ &\iff \widetilde{A} \in \mathcal{H}_{\widetilde{\Theta}_1}, \end{aligned}$$

which gives the result.

(b) Suppose $A \in \mathcal{K}_{\Theta_1}$ and $B \in \mathcal{K}_{\Theta_2}$. Firstly, we will show that $A\Theta_2 + B \in \mathcal{K}_{\Theta_1\Theta_2}$. Indeed, if $C \in H^2_{M_n}$ is arbitrary then

$$\begin{aligned} \langle A\Theta_2 + B, C\Theta_1\Theta_2\rangle &= \int_{\mathbb{T}} \mathrm{tr}\left(\Theta_2^*\Theta_1^* C^*(A\Theta_2 + B)\right) d\mu \\ &= \int_{\mathbb{T}} \mathrm{tr}\left((A\Theta_2 + B)\Theta_2^*\Theta_1^* C^*\right) d\mu \\ &= \int_{\mathbb{T}} \mathrm{tr}\,(A\Theta_1^* C^*) d\mu + \int_{\mathbb{T}} \mathrm{tr}\,(B\Theta_2^*\Theta_1^* C^*) d\mu \\ &= \langle A, C\Theta_1\rangle + \langle B, (C\Theta_1)\Theta_2\rangle \\ &= 0, \end{aligned}$$

which gives $\mathcal{K}_{\Theta_1}\Theta_2 \oplus \mathcal{K}_{\Theta_2} \subseteq \mathcal{K}_{\Theta_1\Theta_2}$. For the reverse inclusion, let $A \in \mathcal{K}_{\Theta_1\Theta_2}$ and write $B := P_{\mathcal{K}_{\Theta_2}} A$. Then $P_{\mathcal{K}_{\Theta_2}}(A - B) = 0$ and hence $A - B \in H^2_{M_n}\Theta_2$. Thus it suffices to show that $(A - B)\Theta_2^* \in \mathcal{K}_{\Theta_1}$. Indeed, if $C \in H^2_{M_n}$ is arbitrary, then

$$\begin{aligned} \langle (A - B)\Theta_2^*, C\Theta_1\rangle &= \int_{\mathbb{T}} \mathrm{tr}\left(\Theta_1^* C^*(A - B)\Theta_2^*\right) d\mu \\ &= \int_{\mathbb{T}} \mathrm{tr}\left((A - B)\Theta_2^*\Theta_1^* C^*\right) d\mu \\ &= \int_{\mathbb{T}} \mathrm{tr}\left(A(C\Theta_1\Theta_2)^*\right) d\mu - \int_{\mathbb{T}} \mathrm{tr}\left(B(C\Theta_1\Theta_2)^*\right) d\mu \\ &= \langle A, C\Theta_1\Theta_2\rangle - \langle B, C\Theta_1\Theta_2\rangle \\ &= 0, \end{aligned}$$

which implies $(A - B)\Theta_2^* \in \mathcal{K}_{\Theta_1}$.

(c) Observe by (a) and (b) that

$$A \in \mathcal{H}_{\Theta_1\Theta_2} \iff \widetilde{A} \in \mathcal{K}_{\widetilde{\Theta}_2\widetilde{\Theta}_1} \iff \widetilde{A} \in \mathcal{K}_{\widetilde{\Theta}_2}\widetilde{\Theta}_1 \oplus \mathcal{K}_{\widetilde{\Theta}_1} \iff A \in \Theta_1\mathcal{H}_{\Theta_2} \oplus \mathcal{H}_{\Theta_1},$$

which gives the result. □

Lemma 2.2. Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then we may write

$$\Phi_+ = \Theta_1\Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^*,$$

where $\Theta_1 = \theta_1 I_n$ for an inner function θ_1 and Θ_2 is inner. Let $\Theta_2 A^* = A_1^*\Theta$, where A_1 and Θ are left coprime. For each (scalar) inner function θ_3, put

$$\Phi_C := \Phi_-^* + \Theta_1\Theta_3\,(P_{\mathcal{K}_{\Theta_1}} A_1)^* + \Theta_3 C^* \quad (\Theta_3 := \theta_3 I_n,\ C \in \mathcal{K}_{\Theta_3}).$$

Then

$$T_\Phi \text{ is pseudo-hyponormal} \Longleftrightarrow T_{\Phi_C} \text{ is pseudo-hyponormal}.$$

In particular, $\mathcal{E}(\Phi_C) = \{K\Theta^*\Theta_3 : K \in \mathcal{E}(\Phi)\}$, where $K' \equiv K\Theta^* \in H^2_{M_n}$.

Proof. Suppose T_Φ is pseudo-hyponormal. Then there exists a matrix function $K \in \mathcal{E}(\Phi)$. We will show that

$$K = K'\Theta \quad \text{for some } K' \in H^2_{M_n}. \tag{13}$$

Indeed if $K \in \mathcal{E}(\Phi)$, then $B\Theta_1^* - KA\Theta_2^*\Theta_1^* \in H^2_{M_n}$, so that $K\Theta^* A_1 \in H^2_{M_n}$. We thus have that by (5),

$$0 = H^*_{K\Theta^* A_1} = H_{\widetilde{A}_1\widetilde{\Theta}^*\widetilde{K}} = H_{\widetilde{A}_1\widetilde{\Theta}^*}T_{\widetilde{K}},$$

which implies that $\widetilde{K}H^2_{\mathbb{C}^n} \subseteq \ker H_{\widetilde{A}_1\widetilde{\Theta}^*} = \widetilde{\Theta}H^2_{\mathbb{C}^n}$ since A_1 and Θ are left coprime, and hence $\widetilde{A}_1$ and $\widetilde{\Theta}$ are right coprime. It thus follows (cf. [FF, Corollary IX.2.2]) that $\widetilde{\Theta}$ is a left inner divisor of $\widetilde{K}$, so that $\widetilde{K} = \widetilde{\Theta}\widetilde{K}'$ for some $\widetilde{K}' \in H^2_{M_n}$, and hence $K = K'\Theta$. This proves (13). Now if T_Φ is pseudo-hyponormal then $B\Theta_1^* - (K'\Theta)A\Theta_2^*\Theta_1^* \in H^2_{M_n}$, and hence $B\Theta_1^* - K'A_1\Theta_1^* \in H^2_{M_n}$. Thus $B\Theta_1^* - (K'\Theta_3)(P_{\mathcal{K}_{\Theta_1}}A_1 + C\Theta_1)\Theta_1^*\Theta_3^* \in H^2_{M_n}$ for some $C \in \mathcal{K}_{\Theta_3}$, which implies that T_{Φ_C} is pseudo-hyponormal. This argument is reversible. The last assertion is evident from the above proof. □

Lemma 2.3. Suppose that $\Theta_1 = \theta_1 I_n$ for an inner function θ_1 and Θ_2 is an inner matrix function in $H^\infty_{M_n}$. If θ_1 has a Blaschke factor, then

$$\mathcal{K}_{\Theta_2} \subseteq \mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_n\Theta_2} \subseteq \mathcal{K}_{\Theta_1\Theta_2}\,, \tag{14}$$

or equivalently,

$$\mathcal{H}_{\Theta_2} \subseteq \mathcal{H}_{zI_n\Theta_2} \cdot \mathcal{H}_{\Theta_1} \subseteq \mathcal{H}_{\Theta_1\Theta_2}\,. \tag{15}$$

In particular,

$$\operatorname{span}\big(\mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_n\Theta_2}\big) = \mathcal{K}_{\Theta_1\Theta_2} \quad \text{and} \quad \operatorname{span}\big(\mathcal{H}_{zI_n\Theta_2} \cdot \mathcal{H}_{\Theta_1}\big) = \mathcal{H}_{\Theta_1\Theta_2}\,. \tag{16}$$

Proof. Let $A \in \mathcal{K}_{\Theta_1}$ and $B \in \mathcal{K}_{zI_n\Theta_2}$. Then for arbitrary $D \in H^2_{M_n}$,

$$0 = \langle A,\, D\Theta_1\rangle = \int_{\mathbb{T}} \operatorname{tr}(\Theta_1^* D^* A)\,d\mu = \langle A\Theta_1^*,\, D\rangle,$$

which implies that $\Theta_1 A^* \in H^2_0$, and similarly, $\Theta_2 B^* \in H^2_{M_n}$. Thus we have $C\Theta_2 B^* \in H^2_{M_n}$ for arbitrary $C \in H^\infty_{M_n}$. If $C \in H^\infty_{M_n}$ is arbitrary, then

$$\begin{aligned}
\langle AB,\, C\Theta_1\Theta_2\rangle &= \int_{\mathbb{T}} \operatorname{tr}\big((C\Theta_1\Theta_2)^* AB\big)\,d\mu \\
&= \int_{\mathbb{T}} \operatorname{tr}(AB\Theta_2^*\Theta_1^* C^*)\,d\mu \\
&= \int_{\mathbb{T}} \operatorname{tr}\big(\Theta_1^*(C\Theta_2 B^*)^* A\big)\,d\mu \quad (\text{since } \Theta_1 = \theta_1 I_n \text{ is diagonal-constant}) \\
&= 0 \quad (\text{since } C\Theta_2 B^* \in H^2_{M_n} \text{ and } A \in \mathcal{K}_{\Theta_1}),
\end{aligned}$$

which implies $AB \in \mathcal{K}_{\Theta_1\Theta_2}$. Thus we can see that $\mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_n\Theta_2} \subseteq \mathcal{K}_{\Theta_1\Theta_2}$, which gives the second inclusion of (14). For the first inclusion of (14), suppose θ_1 has a Blaschke factor b_α, so that $\theta_1(\alpha) = 0$. If $A \in \mathcal{K}_{\Theta_2}$, then $\Theta_2 A^* \in H_0^2$. Thus

$$zI_n\Theta_2\big((1-\overline{\alpha}z)I_nA\big)^* = zI_n\Theta_2A^* - \alpha I_n\Theta_2A^* \in H_0^2,$$

which implies that $(1-\overline{\alpha}z)I_nA \in \mathcal{K}_{zI_n\Theta_2}$. But since $\Theta_1 = \theta_1 I_n$ and $\frac{1}{1-\overline{\alpha}z}I_n \in \mathcal{K}_{\Theta_1}$, it follows that

$$A \in \frac{1}{1-\overline{\alpha}z}I_nK_{zI_n\Theta_2} \subseteq \mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_n\Theta_2},$$

which says that the first inclusion of (14) holds if θ_1 has a Blaschke factor. The statement (15) follows from (14) together with Lemma 2.1(a).

For (16), observe that by Lemma 2.1(b),

$$\mathcal{K}_{\Theta_1\Theta_2} = \mathcal{K}_{\Theta_1}\Theta_2 \oplus \mathcal{K}_{\Theta_2}$$

and

$$\mathcal{K}_{\Theta_1}\Theta_2 \subseteq \mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_n\Theta_2}.$$

But since $\mathcal{K}_{\Theta_1\Theta_2}$ is a subspace of $H^2_{M_n}$ and $\mathcal{K}_{\Theta_1}\Theta_2 \cup \mathcal{K}_{\Theta_2} \subseteq \mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_n\Theta_2}$, it follows from (14) that $\text{span}\big(\mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_n\Theta_2}\big) = \mathcal{K}_{\Theta_1\Theta_2}$, and similarly, $\text{span}\big(\mathcal{H}_{zI_n\Theta_2} \cdot \mathcal{H}_{\Theta_1}\big) = \mathcal{H}_{\Theta_1\Theta_2}$, which proves (16). □

From Lemma 2.3, we are tempted to guess that

$$\Phi = \Psi\Upsilon \ (\Phi \in \mathcal{K}_{\Theta_1\Theta_2},\ \Psi \in \mathcal{K}_{\Theta_1},\ \Upsilon \in H^\infty_{M_n}) \Longrightarrow \Upsilon \in \mathcal{K}_{zI_n\Theta_2}. \tag{17}$$

But this is not the case. In fact, (17) does not hold for even scalar-valued functions. Indeed, if $f = 2z^3 + z^2$, $g = z^2 + 2z$, and $h = \frac{z+\frac{1}{2}}{1+\frac{1}{2}z} \cdot z$, then $f = gh$, but (17) fails.

On the other hand, in view of Lemma 2.3, we might define the notion of "divisor" of matrix-valued analytic functions as follows: if $\Phi \in \mathcal{K}_{\Theta_1\Theta_2}$, $\Psi \in \mathcal{K}_{\Theta_1}$, $\Upsilon \in \mathcal{K}_{zI_n\Theta_2}$ satisfies $\Phi = \Psi\Upsilon$, then we say that Ψ is a left divisor of Φ. However, we must consider another aspect. Let

$$\Phi = \begin{bmatrix} z^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} z & 0 \\ 0 & z^3 \end{bmatrix}, \quad \text{and} \quad \Upsilon = \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix}. \tag{18}$$

If we regard Φ as an element in $\mathcal{K}_{\Theta_1\Theta_2}$ ($\Theta_1 = z^4I_2,\ \Theta_2 = zI_2$) then

$$\Phi = \Psi\Upsilon \in \mathcal{K}_{\Theta_1} \cdot \mathcal{K}_{zI_2\Theta_2}.$$

Thus Ψ is a left divisor of Φ. But if we regard Φ as an element in $\mathcal{K}_{\Theta_1\Theta_2}$ ($\Theta_1 = z^4I_2$, $\Theta_2 = I_2$), then Ψ cannot be a left divisor of Φ. Based on this observation, we should be careful when defining the notion of "divisor" for matrix-valued functions.

Before we define the notion of "divisor," we need to observe:

Lemma 2.4. Let $\Phi \in H^2_{M_n}$ be of the form

$$\Phi = \Theta A^* \quad \text{(right coprime factorization)}.$$

Then $A \in \mathcal{K}_{zI_n\Theta}$ and $\Phi \in \mathcal{H}_{zI_n\Theta}$. In particular, if $\Phi \in H_0^2$, then $A \in \mathcal{K}_\Theta$.

Proof. Since $\Phi = \Theta A^* \in H^2_{M_n}$, it follows that for any $C \in H^2_{M_n}$,

$$0 = \langle \overline{z}I_nC^*,\ \Phi\rangle = \int_{\mathbb{T}} \text{tr}\,(A\Theta^*\overline{z}I_nC^*)d\mu = \int_{\mathbb{T}} \text{tr}\,(\Theta^*\overline{z}I_nC^*A)d\mu = \langle A,\ CzI_n\Theta\rangle,$$

which implies that $A \in \mathcal{K}_{zI_n\Theta}$. Also, for any $C \in H^2_{M_n}$,

$$\langle \Phi,\ zI_n\Theta C\rangle = \int_{\mathbb{T}} \text{tr}\,\big(C^*\Theta^*\overline{z}I_n\Theta A^*\big)d\mu = \langle A^*,\ CzI_n\rangle = 0,$$

which implies $\Phi \in \mathcal{H}_{zI_n\Theta}$. Similarly we also have that if $\Phi \in H_0^2$, then $A \in \mathcal{K}_\Theta$. □

We now define the notion of "divisor" for matrix-valued analytic functions whose adjoints are of bounded type.

Definition 2.5. Let $\Phi, \Psi \in H^2_{M_n}$ be such that Φ^* and Ψ^* are of bounded type. Then we can write

$$\Phi = \Theta_1 A^* \quad \text{and} \quad \Psi = \Theta_2 B^* \quad \text{(right coprime factorizations)},$$

where the Θ_i $(i = 1, 2)$ are inner, $A \in \mathcal{K}_{zI_n\Theta_1}$ and $B \in \mathcal{K}_{zI_n\Theta_2}$. If $\Theta_1 = \Theta\Theta_2$ for some inner function $\Theta \in H^2_{M_n}$, and

$$\Phi = \Psi\Gamma \text{ for some } \Gamma \in \mathcal{H}_{zI_n\Theta}\,, \tag{19}$$

then we say that Ψ is a *left divisor* of Φ. If $\widetilde{\Psi}$ is a left divisor of $\widetilde{\Phi}$ then we say that Ψ is a *right divisor* of Φ. We note that if $\Theta_i = \theta_i I_n$ $(i = 1, 2)$, then (19) can be also written as

$$\Phi = \Psi\Gamma \text{ for some } \Gamma \in \mathcal{K}_{zI_n\Theta}\,.$$

Lemma 2.6. Let $\Phi, \Psi \in H^2_{M_n}$ be of the form

$$\Phi = \Theta_1\Theta_2 A^* \quad \text{and} \quad \Psi = \Theta_1 B^* \quad \text{(right coprime factorizations)},$$

where $\Theta_i = \theta_i I_n$ $(i = 1, 2)$, $A \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$ and $B \in \mathcal{K}_{zI_n\Theta_1}$. Then we have

$$\Psi \text{ is a left divisor of } \Phi \iff A = EB \text{ for some } E \in \mathcal{K}_{zI_n\Theta_2}.$$

Proof. If Ψ is left divisor of Φ then there exists $\Gamma \in \mathcal{K}_{zI_n\Theta_2}$ such that $\Phi = \Psi\Gamma$. Thus $\Theta_1\Theta_2A^* = \Theta_1 B^*\Gamma$, and hence $A = \Theta_2\Gamma^* B$. It suffices to show that

$$E \equiv \Theta_2\Gamma^* \in \mathcal{K}_{zI_n\Theta_2}.$$

Indeed, since $\Gamma \in \mathcal{K}_{zI_n\Theta_2}$, it follows that for any $C \in H^\infty_{M_n}$,

$$0 = \langle \Gamma,\, CzI_n\Theta_2\rangle = \int_{\mathbb{T}} \operatorname{tr}\,(\Theta_2^*\overline{z}I_nC^*\Gamma)d\mu = \int_{\mathbb{T}} \operatorname{tr}\,\big(\overline{z}I_nC^*(\Theta_2\Gamma^*)^*\big)d\mu = \langle \Theta_2\Gamma^*,\, (zI_nC)^*\rangle,$$

which implies that $\Theta_2\Gamma^* \in H^2_{M_n}$. Thus by Lemma 2.4, $E \equiv \Theta_2\Gamma^* \in \mathcal{K}_{zI_n\Theta_2}$.

Conversely, if $A = EB$ for some $E \in \mathcal{K}_{zI_n\Theta_2}$ then

$$\Phi = \Theta_1\Theta_2A^* = (\Theta_1B^*)(\Theta_2E^*) = \Psi\Gamma.$$

Since $E \in \mathcal{K}_{zI_n\Theta_2}$, it follows that $\Theta_2E^* \in H^2_{M_n}$, and hence by Lemma 2.4, $\Gamma \equiv \Theta_2E^* \in \mathcal{K}_{zI_n\Theta_2}$. Thus Ψ is a left divisor of Φ. This completes the proof. □

The following proposition provides a criterion for the hyponormality of bounded-type Toeplitz operators T_Φ when the co-analytic part of Φ is a left divisor of the analytic part.

Proposition 2.7. Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Thus in view of (12), we may write

$$\Phi_+ = \Theta_1\Theta_2A^* \quad \text{and} \quad \Phi_- = \Theta_1B^* \quad \text{(right coprime factorizations)}\,.$$

Assume that $\Theta_i = \theta_i I_n$ for inner functions θ_i $(i = 1, 2)$. If Φ_- is a left divisor of Φ_+ (or equivalently, in view of Lemma 2.6 , $A = EB$ for some $E \in \mathcal{K}_{zI_n\Theta_2}$), then the following are equivalent:

(i) T_Φ is pseudo-hyponormal;
(ii) There exists a function $Q \in H^\infty_{M_n}$ with $||Q||_\infty \le 1$ such that $QE \in I_n + \Theta_1 H^2_{M_n}$;
(iii) T_Ψ is pseudo-hyponormal, where $\Psi = \Theta_1^* + \Theta_1\big(P_{\mathcal{K}_{\Theta_1}}E\big)^*$.

Moreover, if $\theta_1 = \theta_2$ then T_Φ is pseudo-hyponormal if and only if $T_{\Theta_1^*+\Theta_1E^*}$ is pseudo-hyponormal.

Proof. For the equivalence (i) $\Leftrightarrow$ (ii), let $\Phi' = \Phi_-^* + \Theta_1\big(P_{\mathcal{K}_{\Theta_1}}(A)\big)^*$. Then by Lemma 2.2 we have $\mathcal{E}(\Phi) = \{Q\Theta_2 : Q \in \mathcal{E}(\Phi')\}$. We then have

$$\begin{aligned} T_\Phi \text{ is pseudo-hyponormal} &\Longleftrightarrow \Theta_1^* B - (Q\Theta_2)\Theta_1^*\Theta_2^* A \in H^2_{M_n} \text{ and } Q \in \mathcal{E}(\Phi') \\ &\Longleftrightarrow \Theta_1^* B - Q\Theta_1^* A \in H^2_{M_n} \text{ and } ||Q||_\infty \le 1 \\ &\Longleftrightarrow B - QA \in \Theta_1 H^2_{M_n} \text{ and } ||Q||_\infty \le 1 \\ &\Longleftrightarrow (I_n - QE)B \in \Theta_1 H^2_{M_n} \text{ and } ||Q||_\infty \le 1 \\ &\Longleftrightarrow I_n - QE \in \Theta_1 H^2_{M_n} \text{ and } ||Q||_\infty \le 1 \\ &\qquad\qquad \text{(since } B \text{ and } \Theta_1 \text{ are coprime)} \\ &\Longleftrightarrow QE \in I_n + \Theta_1 H^2_{M_n} \text{ and } ||Q||_\infty \le 1\,, \end{aligned}$$

which proves the equivalence (i) $\Leftrightarrow$ (ii). The equivalence (ii) $\Leftrightarrow$ (iii) follows at once from the following equivalence:

$$\begin{aligned} QE \in I_n + \Theta_1 H^2_{M_n} &\Longleftrightarrow \Theta_1^* - Q\Theta_1^* E \in H^2_{M_n} \\ &\Longleftrightarrow \Theta_1^* - Q(P_{H^2_0}(\Theta_1 E^*))^* \in H^2_{M_n} \\ &\Longleftrightarrow \Theta_1^* - Q\big(P_{\mathcal{K}_{\Theta_1}} E\big)\Theta_1^* \in H^2_{M_n} \\ &\Longleftrightarrow \;\; T_\Psi \text{ is pseudo-hyponormal.} \end{aligned}$$

For the second assertion, we first observe that if $\theta_1 = \theta_2$ then $E \in \mathcal{K}_{zI_n\Theta_1}$. But since $\mathcal{K}_{zI_n\Theta_1} = \mathcal{K}_{\Theta_1} \oplus \mathcal{K}_{zI_n}\Theta_1$, it follows that $P_{\mathcal{K}_{\Theta_1}} E = E + M\Theta_1$ ($M \in M_n$), so that $\Theta_1\big(P_{\mathcal{K}_{\Theta_1}} E\big)^* = \Theta_1 E^* + M$. Since by (10), $T_{\Theta_1^* + \Theta_1 E^*}$ is pseudo-hyponormal if and only if $T_{\Theta_1^* + \Theta_1 E^* + M}$ is pseudo-hyponormal, it follows from the first assertion that T_Φ is pseudo-hyponormal if and only if $T_{\Theta_1^* + \Theta_1 E^*}$ is. This completes the proof. □

Before we go on, we shall introduce a "reverse pull-back symbol" $\Phi^\sharp$ associated to a symbol $\Phi \in L^\infty_{M_n}$, with Φ and Φ^* of bounded type. Suppose that $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \Theta_1\Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad \text{(right coprime factorizations)}.$$

Assume that $\Theta_i = \theta_i I_n$ for inner functions θ_i ($i = 1, 2$). We write

$$\Phi^\sharp := \Theta_1^*(P_{\mathcal{K}_{\Theta_1}} A) + \Phi_- \tag{20}$$

($\Phi^\sharp$ is a pull-back of Φ^* - i.e., pulling back the co-analytic part of Φ^* to have the same degree as that of the analytic part). We then claim that

$$A_1 := P_{\mathcal{K}_{\Theta_1}} A \text{ and } \Theta_1 \text{ are right coprime:} \tag{21}$$

indeed, if we write $A = A_1 + \Theta_1 A_2$ for some $A_2 \in H^2_{M_n}$ and assume to the contrary that Θ_1 and A_1 have a common right inner divisor Ω, then $A = A_1 + A_2\Theta_1 = A_1'\Omega + A_2\Theta_1'\Omega = (A_1' + A_2\Theta_1')\Omega$ for some $A_1', \Theta_1' \in H^2_{M_n}$, which implies that A and Θ_1 have a common right inner divisor Ω, a contradiction.

The following result provides a core idea for our main theorem.

Proposition 2.8. Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Thus in view of (12), we may write

$$\Phi_+ = \Theta_1\Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad \text{(right coprime factorizations)}.$$

Assume that $\Theta_i = \theta_i I_n$ for inner functions θ_i ($i = 1, 2$) and write

$$\Phi^\sharp := \Theta_1^*(P_{\mathcal{K}_{\Theta_1}} A) + \Phi_-.$$

Then the set $\{P_{\mathcal{K}_{\Theta_1}} K : \; K \in \mathcal{C}(\Phi^\sharp)\}$ is either a singleton or the empty set.

Proof. Write $A_1 := P_{\mathcal{K}_{\Theta_1}} A$, and hence $\Phi^\sharp = \Theta_1^* A_1 + \Phi_-$. Assume $K_1, K_2 \in \mathcal{C}(\Phi^\sharp)$. Then

$$\Theta_1^* A_1 - K_1 \Phi_-^* \in H^2_{M_n} \quad \text{and} \quad \Theta_1^* A_1 - K_2 \Phi_-^* \in H^2_{M_n},$$

which implies that $(K_1 - K_2) B \Theta_1^* \in H^2_{M_n}$, so that $(K_1 - K_2) B \in \Theta_1 H^2_{M_n}$. If we write $K := P_{\mathcal{K}_{\Theta_1}}(K_1 - K_2)$, then $KB \in \Theta_1 H^2_{M_n}$, and hence, $KB\Theta_1^* \in H^2_{M_n}$, which implies that $H_{KB\Theta_1^*} = 0$. Thus by (5), $T^*_{\widetilde{K}} H_{B\Theta_1^*} = 0$, so that $H_{\widetilde{B}\widetilde{\Theta_1}^*} T_{\widetilde{K}} = 0$ (with $\widetilde{\Theta_1} := I_{\widetilde{\theta}_1}$), which implies that

$$\widetilde{K} H^2_{\mathbb{C}^n} \subseteq \ker H_{\widetilde{B}\widetilde{\Theta}_1^*}.$$

Since Θ_1 and B are left coprime, so that $\widetilde{\Theta_1}$ and $\widetilde{B}$ are right coprime, it follows from Lemma 1.1 that

$$\widetilde{K} H^2_{\mathbb{C}^n} \subseteq \ker H_{\widetilde{B}\widetilde{\Theta_1}^*} = \widetilde{\Theta_1} H^2_{\mathbb{C}^n},$$

which implies that $\widetilde{\Theta_1}$ is a left inner divisor of $\widetilde{K}$. Therefore $\widetilde{K} = \widetilde{\Theta_1} E$ for some $E \in H^2_{M_n}$, and hence $K = \widetilde{E}\Theta_1 \in H^2_{M_n}\Theta_1$. But since $K \in \mathcal{K}_{\Theta_1}$, we should have $K = 0$, i.e., $P_{\mathcal{K}_{\Theta_1}} K_1 = P_{\mathcal{K}_{\Theta_1}} K_2$, which says that $\{P_{\mathcal{K}_{\Theta_1}} K : K \in \mathcal{C}(\Phi^\sharp)\}$ is a singleton set. □

Our main theorem now follows:

Theorem 2.9. (A Criterion for Hyponormality of Bounded-Type Toeplitz Operators) Let $\Phi \in L^\infty_{M_n}$ be a normal matrix function such that Φ and Φ^* are of bounded type. Thus in view of (12), we may write

$$\Phi_+ = \Theta_1 \Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad \text{(right coprime factorizations)}.$$

Assume that $\Theta_i = \theta_i I_n$ for inner functions θ_i $(i = 1, 2)$. Write

$$\Phi^\sharp := \Theta_1^*(P_{\mathcal{K}_{\Theta_1}} A) + \Phi_-.$$

If $\mathcal{C}(\Phi^\sharp)$ is nonempty, we may, in view of Proposition 2.8, write $K^\sharp := P_{\mathcal{K}_{\Theta_1}} K$ (where $K \in \mathcal{C}(\Phi^\sharp)$). Then the following are equivalent:

(i) T_Φ is hyponormal;
(ii) There exists a function $Q \in H^\infty_{M_n}$ with $||Q||_\infty \le 1$ such that $QK^\sharp \in I_n + \Theta_1 H^2_{M_n}$;
(iii) T_Ψ is pseudo-hyponormal, where $\Psi = \Theta_1^* + \Theta_1 (K^\sharp)^*$.

Moreover, if $A = EB$ for some $E \in \mathcal{K}_{zI_n\Theta_2}$, then $K^\sharp$ can be chosen as E.

Proof. Write

$$\Phi_C := \Phi_-^* + \Theta_1^2 (P_{\mathcal{K}_{\Theta_1}} A)^* + \Theta_1 C^* \quad (C \in \mathcal{K}_{\Theta_1}).$$

Then it follows from Lemma 2.2 that

$$T_\Phi \text{ is pseudo-hyponormal} \Longleftrightarrow T_{\Phi_C} \text{ is pseudo-hyponormal.} \tag{22}$$

Put $A_1 := P_{\mathcal{K}_{\Theta_1}} A$. Thus we can write

$$\Phi_C = \Theta_1^* B + \Theta_1^2 (A_1 + \Theta_1 C)^*.$$

Now we will show that if $K \in \mathcal{C}(\Phi^\sharp)$, then

$$A_1 + \Theta_1 C = K^\sharp B \text{ for some } C \in \mathcal{K}_{\Theta_1}. \tag{23}$$

Indeed, if $K \in \mathcal{C}(\Phi^\sharp)$, then

$$\Theta_1^* A_1 - K\Theta_1^* B \in H^2_{M_n}, \quad \text{so that} \quad A_1 - KB \in \Theta_1 H^2_{M_n}.$$

It thus follows that $P_{\mathcal{K}_{\Theta_1}}(A_1 - KB) = 0$, so that $P_{\mathcal{K}_{\Theta_1}}(A_1 - (P_{\mathcal{K}_{\Theta_1}} K)B) = 0$, and hence $A_1 - K^\sharp B \in \Theta_1 H^2_{M_n}$. Thus

$$A_1 + \Theta_1 C = K^\sharp B \text{ for some } C \in H^2_{M_n}.$$

Now we will show that $C \in \mathcal{K}_{\Theta_1}$. To see this we note that $\Theta_1^2(A_1 + \Theta_1 C)^* = \Theta_1^2 B^*(K^\sharp)^*$. But since $B \in \mathcal{K}_{\Theta_1}$ and $K^\sharp \in \mathcal{K}_{\Theta_1}$, it follows that

$$\Theta_1^2 A_1^* + \Theta_1 C^* = (\Theta_1 B^*)(\Theta_1 (K^\sharp)^*) \in H_0^2,$$

which implies $\Theta_1 C^* \in H_0^2$, and hence, $C \in \mathcal{K}_{\Theta_1}$. This proves (23). Then by Lemma 2.6 and (23), $(\Phi_C)_-$ is a left divisor of $(\Phi_C)_+$. Thus all assertions follow at once from (22) and Proposition 2.7. □

Theorem 2.9 is often useful for the cases of even scalar-valued symbols.

Example 2.10. Let δ be a singular inner function of the form

$$\delta(z) = \exp\left(\frac{z+1}{z-1}\right)$$

and consider the function

$$\varphi = \overline{z}\Big(\overline{\delta} - \frac{1}{2}\Big) + 4z\Big(\delta - \frac{1}{2}\Big)\Big(\delta - \frac{1}{3}\Big).$$

Then T_φ is hyponormal.

Proof. Observe that

$$\varphi_- = z\delta\overline{\Big(1 - \frac{1}{2}\delta\Big)} \quad \text{and} \quad \varphi_+ = z\delta^2\overline{4\Big(1 - \frac{1}{2}\delta\Big)\Big(1 - \frac{1}{3}\delta\Big)}.$$

Then under the notations of Theorem 2.9, $A = 4\Big(1 - \frac{1}{2}\delta\Big)\Big(1 - \frac{1}{3}\delta\Big)$, $B = 1 - \frac{1}{2}\delta$, so that E can be given by

$$E = 4\Big(1 - \frac{1}{3}\delta\Big).$$

Put

$$Q := E^{-1} = \frac{1}{4(1 - \frac{1}{3}\delta)}.$$

Then $Q \in H^\infty$ with $||Q||_\infty \leq 1$ and $QE = 1 \in 1 + z\delta H^2$. Therefore by Theorem 2.9, T_Φ is hyponormal. □

3. The cases of rational symbols

To describe the cases of rational symbols, we review the classical Hermite-Fejér interpolation problem (cf. [FF]).

Given the sequence $\{K_{ij} : 1 \leq i \leq n,\ 0 \leq j < n_i\}$ of $n \times n$ complex matrices and a set of distinct complex numbers $\alpha_1, \dots, \alpha_n$ in $\mathbb{D}$, the classical Hermite-Fejér interpolation problem is to find necessary and sufficient conditions for the existence of a contractive analytic function K in $H^\infty_{M_n}$ satisfying

$$\frac{K^{(j)}(\alpha_i)}{j!} = K_{i,j} \qquad (1 \leq i \leq n,\ 0 \leq j < n_i). \tag{24}$$

To construct a polynomial $K(z) \equiv P(z)$ satisfying (24), let $p_i(z)$ be the polynomial of order $d - n_i$ defined by

$$p_i(z) := \prod_{k=1, k\neq i}^{n} \Big(\frac{z - \alpha_k}{\alpha_i - \alpha_k}\Big)^{n_k}.$$

Consider the polynomial $P(z)$ of degree $d-1$ defined by

$$P(z) := \sum_{i=1}^{n}\left(K'_{i,0} + K'_{i,1}(z-\alpha_i) + K'_{i,2}(z-\alpha_i)^2 + \cdots + K'_{i,n_i-1}(z-\alpha_i)^{n_i-1}\right)p_i(z), \tag{25}$$

where the $K'_{i,j}$ are obtained by the following equations:

$$K'_{i,j} = K_{i,j} - \sum_{k=0}^{j-1} \frac{K'_{i,k}\, p_i^{(j-k)}(\alpha_i)}{(j-k)!} \quad (1 \le i \le n;\ 0 \le j < n_i)$$

and $K'_{i,0} = K_{i,0}$ $(1 \le i \le n)$. Then $P(z)$ satisfies (24). We call P the *Hermite-Fejér polynomial with respect to* $\{\alpha_1, \cdots, \alpha_n\}$. Note that $P(z)$ may not be contractive.

The following lemma guarantees that $\mathcal{C}(\Phi^\sharp)$ is nonempty if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function.

Lemma 3.1. Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. Thus in view of (12), we may write

$$\Phi_+ = \Theta_1\Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad \text{(right coprime factorizations)}.$$

Assume that $\Theta_i = \theta_i I_n$ for inner functions θ_i $(i = 1, 2)$. If $\Phi^\sharp := \Theta_1^*(P_{\mathcal{K}_{\Theta_1}} A) + \Theta_1 B^*$, then $\mathcal{C}(\Phi^\sharp)$ is nonempty.

Proof. Since Φ is a matrix-valued rational function, θ_1 is a finite Blaschke product. Thus we can write

$$\theta_1(z) \equiv \prod_{i=1}^N \left(\frac{z - \alpha_i}{1 - \overline{\alpha_i} z}\right)^{p_i},$$

where $d = \sum_{i=1}^N p_i$. Write $A_1 := P_{\mathcal{K}_{\Theta_1}} A$ and $\Phi^\sharp = \Theta_1^* A_1 + \Phi_-$. Then

$$\begin{aligned} K \in \mathcal{C}(\Phi^\sharp) &\iff \Theta_1^* A_1 - K\Theta_1^* B \in H^2_{M_n} \\ &\iff A_1 - KB \in \Theta_1 H^2_{M_n} \\ &\iff \widetilde{A}_1 - \widetilde{B}\widetilde{K} \in \widetilde{\Theta}_1 H^2_{M_n}. \end{aligned} \tag{26}$$

Note that

(i) $\widetilde{\Theta}_1^{(n)}(\overline{\alpha}_i) = 0 \quad (0 \le n < p_i)$;

(ii) $\widetilde{B}(\overline{\alpha}_i)$ is invertible for each $i = 1, 2, \cdots, N$; and

(iii) $\widetilde{A}^{(j)}(\overline{\alpha}_i) = \widetilde{A}_1^{(j)}(\overline{\alpha}_i) \quad (1 \le i \le N,\ 0 \le j < p_i)$.

Thus the last statement in (26) is equivalent to the following equation:

$$\frac{\widetilde{K}^{(j)}(\overline{\alpha}_i)}{j!} = d_{i,j} \quad (1 \le i \le N,\ 0 \le j < p_i), \tag{27}$$

where the $d_{i,j}$ are determined by the following equation: for each $i = 1, \cdots, N$,

$$\begin{bmatrix} d_{i,0} \\ d_{i,1} \\ d_{i,2} \\ \vdots \\ d_{i,p_i-2} \\ d_{i,p_i-1} \end{bmatrix} := \begin{bmatrix} b_{i,0} & 0 & 0 & 0 & \cdots & 0 \\ b_{i,1} & b_{i,0} & 0 & 0 & \cdots & 0 \\ b_{i,2} & b_{i,1} & b_{i,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b_{i,p_i-2} & b_{i,p_i-3} & \ddots & \ddots & b_{i,0} & 0 \\ b_{i,p_i-1} & b_{i,p_i-2} & \cdots & b_{i,2} & b_{i,1} & b_{i,0} \end{bmatrix}^{-1} \begin{bmatrix} a_{i,0} \\ a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,p_i-2} \\ a_{i,p_i-1} \end{bmatrix}, \tag{28}$$

where

$$a_{i,j} := \frac{\widetilde{A}^{(j)}(\overline{\alpha}_i)}{j!} \quad \text{and} \quad b_{i,j} := \frac{\widetilde{B}^{(j)}(\overline{\alpha}_i)}{j!}.$$

This is exactly the classical Hermite-Fejér interpolation problem except for the contractivity condition for K. Thus if P is the Hermite-Fejér polynomial with respect to $\{\alpha_1, \cdots, \alpha_N\}$, then $K \equiv P$ satisfies (27). Thus by (26), we must have $P \in \mathcal{C}(\Phi^\sharp)$, and therefore $\mathcal{C}(\Phi^\sharp)$ is nonempty. This completes the proof. $\square$

If $\Phi, \Psi \in H^2_{M_n}$ are matrix-valued rational functions then the notion of divisor can be somewhat relaxed in the sense that the quotient of the division may belong to a larger class.

Lemma 3.2. Let $\Phi, \Psi \in H^2_{M_n}$ be matrix valued rational functions of the form

$$\Phi = \Theta_1\Theta_2 A^* \quad \text{and} \quad \Psi = \Theta_1 B^* \quad \text{(right coprime factorizations)},$$

where $\Theta_i = \theta_i I_n$ for some finite Blaschke product θ_i $(i = 1, 2)$. If $\Phi = \Psi\Gamma$ for some $\Gamma \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$, then we have $\Gamma \in \mathcal{K}_{zI_n\Theta_2}$, so that Ψ is a left divisor of Φ.

Proof. By Lemma 2.4, we see that $A \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$ and $B \in \mathcal{K}_{zI_n\Theta_1}$. Suppose $\Phi = \Psi\Gamma$ for some $\Gamma \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$. We want to show $\Gamma \in \mathcal{K}_{zI_n\Theta_2}$. Assume to the contrary that $\Gamma \notin \mathcal{K}_{zI_n\Theta_2}$. Since $\Theta_i = \theta_i I_n$ for some finite Blaschke product θ_i $(i = 1, 2)$ and $\Gamma \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$, it follows from the observation $\mathcal{K}_{zI_n\Theta_1\Theta_2} = \mathcal{K}_{zI_n\Theta_2} \oplus \mathcal{K}_{\Theta_1}(zI_n\Theta_2)$ that

$$\Gamma = \Gamma_0 + \Gamma_1(zI_n\Theta_2),$$

where $\Gamma_0 = P_{\mathcal{K}_{zI_n\Theta_2}}\Gamma$ and $\Gamma_1 \in \mathcal{K}_{\Theta_1}$ with $\Gamma_1 \neq 0$. Thus

$$\Phi = \Psi\Gamma = \Psi\Gamma_0 + \Psi\Gamma_1(zI_n\Theta_2).$$

But since $\Gamma_0 \in \mathcal{K}_{zI_n\Theta_2}$, it follows from Lemmas 2.3 and 2.4 that $\Psi\Gamma_0 \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$. Since also $\Phi \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$, it follows that $\Psi\Gamma_1(zI_n\Theta_2) \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$, so that $\Psi\Gamma_1 \in \mathcal{K}_{\Theta_1}$, and hence $\overline{z}I_n\Gamma_1^* B \in H^2_{M_n}$. This implies that $H_{\overline{z}I_n\Gamma_1^*}T_B = 0$, so that

$$BH^2_{\mathbb{C}^n} \subseteq \ker H_{\overline{z}I_n\Gamma_1^*}. \tag{29}$$

Write

$$zI_n\Gamma_1 = \Delta D^* \quad \text{(right coprime factorization)},$$

where Δ is inner and $D \in H^2_{M_n}$. Then by (29), $BH^2_{\mathbb{C}^n} \subset \Delta H^2_{\mathbb{C}^n}$ and hence $B = \Delta E$ for some $E \in H^2_{M_n}$. But since $\Gamma_1 \in \mathcal{K}_{\Theta_1}$, we have $\Theta_1(zI_n\Gamma_1)^* \in H^2_{M_n}$. Thus $\Theta_1 D\Delta^* \in H^2_{M_n}$, and hence $\Theta_1 D = F\Delta$ for some $F \in H^2_{M_n}$. Therefore for each $\alpha \in \mathcal{Z}(\theta_1)$, it follows that $(F\Delta)(\alpha) = 0$. Since B and Θ_1 are right coprime, so that by (9), $B(\alpha)$ is invertible, and hence so is $\Delta(\alpha)$, it follows that $F(\alpha) = 0$. Thus we can write $F = (z-\alpha)I_n F' = b_\alpha I_n(1-\overline{\alpha}z)I_n F'$ for some $F' \in H^2_{M_n}$, so that $\Theta_1\overline{b}_\alpha I_n D = F\overline{b}_\alpha I_n\Delta = (1-\overline{\alpha}z)I_n F'\Delta$, and hence, $\Theta_1\overline{b}_\alpha I_n\Gamma_1^* = z(1-\overline{\alpha}z)I_n F' \in H^2_0$, which implies $\Gamma_1 \in \mathcal{K}_{\Theta_1^{(1)}}$ with $\Theta_1^{(1)} := (\theta_1\overline{b}_\alpha)I_n$. Repeating this argument we have

$$\Gamma_1 \in \mathcal{K}_{\Theta_1^{(2)}},$$

where $\Theta_1^{(2)} = \Theta_1\overline{b}_\alpha I_n\overline{b}_\beta I_n$ for $\beta \in \mathcal{Z}(\theta_1\overline{b}_\alpha)$. Continuing this process we get $\Gamma_1 = 0$, a contradiction. This completes the proof. □

We are ready for:

Theorem 3.3. (A Criterion for Hyponormality of Rational Toeplitz Operators) Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued normal rational function. Thus in view of (12), we may write

$$\Phi_+ = \Theta_1\Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad \text{(right coprime factorizations)}.$$

Assume that $\Theta_i = \theta_i I_n$ for finite Blaschke products θ_i $(i = 1, 2)$. Put

$$D := P_{\mathcal{K}_{\Theta_1}} P\,,$$

where P is is the Hermite-Fejér polynomial with respect to the zeros of θ_1. Then the following are equivalent:

(i) T_Φ is hyponormal;
(ii) There exists a function $Q \in H^\infty_{M_n}$ with $||Q||_\infty \leq 1$ such that $QD \in I_n + \Theta_1 H^2_{M_n}$;
(iii) T_Ψ is pseudo-hyponormal, where $\Psi = \Theta_1^* + \Theta_1 D^*$.

Moreover, if $A = EB$ for some $E \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$, then D can be chosen as E.

Proof. If P is is the Hermite-Fejér polynomial with respect to the zeros of θ_1, then from the proof of Lemma 3.1 we can see that $P \in \mathcal{C}(\Phi^\sharp)$. Thus if we take $D \equiv K^\sharp := P_{\mathcal{K}_{\Theta_1}} P$, then the first assertion follows at once from Theorem 2.9. The second assertion follows from Lemma 2.6, Lemma 3.2 and Theorem 2.9. □

Corollary 3.4. (A Necessary Condition for Hyponormality) Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued normal rational function. Thus in view of (12), we may write

$$\Phi_+ = \Theta_1 \Theta_2 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad \text{(right coprime factorizations)}\,.$$

Assume that $\Theta_i = \theta_i I_n$ for finite Blaschke products θ_i $(i = 1, 2)$ and that $A = EB$ for some $E \in \mathcal{K}_{zI_n\Theta_1\Theta_2}$. If T_Φ is hyponormal then $||B(\alpha)A(\alpha)^{-1}|| \le 1$ for each zero α of θ_1.

Proof. Suppose T_Φ is hyponormal and $\theta_1(\alpha) = 0$. By (9), $A(\alpha)$ and $B(\alpha)$ are invertible. By Theorem 3.3 (ii), $Q(\alpha)E(\alpha) = I_n$, so that $||B(\alpha)A(\alpha)^{-1}|| = ||E(\alpha)^{-1}|| = ||Q(\alpha)|| \le 1$. □

4. Revealing examples

In this section, we provide revealing examples to illustrate that Theorem 3.3 is much simpler than the criteria due to the interpolation problems given in [HL2] and [HL3] when the co-analytic part of the symbol is a left divisor of the analytic part. To see this we recall the criterion by the classical Hermite-Fejér interpolation problem (cf. [HL2]). Let

$$\theta := e^{i\xi} \prod_{i=1}^{n} b_i^{n_i},$$

where

$$b_i(z) := \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}, \quad (|\alpha_i| < 1), \quad n_i \ge 1, \quad \text{and} \quad \sum_{i=1}^{n} n_i = d.$$

Let $q_j := (1 - |\alpha_j|^2)^{\frac{1}{2}}$ $(1 \le j \le d)$ and let M be the matrix on $\mathbb{C}^d$ of the form

$$M := \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ q_1 q_2 & \alpha_2 & 0 & 0 & \cdots & \cdots & 0 \\ -q_1 \overline{\alpha_1} q_3 & q_2 q_3 & \alpha_3 & 0 & \cdots & \cdots & 0 \\ q_1 \overline{\alpha_2 \alpha_3} q_4 & -q_2 \overline{\alpha_3} q_4 & q_3 q_4 & \alpha_4 & \cdots & \cdots & 0 \\ -q_1 \overline{\alpha_2 \alpha_3 \alpha_4} q_5 & q_2 \overline{\alpha_3 \alpha_4} q_5 & -q_3 \overline{\alpha_4} q_5 & q_4 q_5 & \ddots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ (-1)^d q_1 \left(\prod_{j=2}^{d-1} \overline{\alpha_j} \right) q_d & (-1)^{d-1} q_2 \left(\prod_{j=3}^{d-1} \overline{\alpha_j} \right) q_d & \cdots & \cdots & \cdots & q_{d-1} q_d & \alpha_d \end{bmatrix}. \tag{30}$$

If $P(z)$ is given by (25) and the K_{ij} are given by the equation (28) with $d_{ij} \equiv K_{i,j}$ and $B \equiv \Theta_2 B$, then the matrix $P(M)$ on $\mathbb{C}^{n \times d}$ is defined by

$$P(M) := \sum_{i=0}^{d-1} P_i \otimes M^i, \quad \text{where } P(z) = \sum_{i=0}^{d-1} P_i z^i.$$

Then $P(M)$ is called the *Hermite-Fejér matrix* determined by (24) (cf. [FF]). It follows from [HL2, Proof of Theorem 2.1] that if Φ is given as in Theorem 3.3, then we have (with $\theta \equiv \theta_1 \theta_2$)

$$T_\Phi \text{ is pseudo-hyponormal} \iff P(M) \text{ is contractive}\,. \tag{31}$$

Example 4.1. (A comparison of two criteria). Let $b(z) := \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$ and consider

$$\Phi := \begin{bmatrix} 2b+2\overline{z} & \overline{z}+b+3zb \\ \overline{z}+b+3zb & 2b+2\overline{z} \end{bmatrix} \in L^\infty_{M_2}.$$

Then Φ is normal and

$$\Phi_+ = zb \begin{bmatrix} 2z & z+3 \\ z+3 & 2z \end{bmatrix}^* \quad \text{and} \quad \Phi_- = z \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^*.$$

Thus we can write

$$\Theta_1 = zI_2, \quad \Theta_2 = bI_2, \quad A = \begin{bmatrix} 2z & z+3 \\ z+3 & 2z \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(i) By the criterion (31): By (30) (with $\theta = zb$) and (25), we observe

$$M = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{bmatrix};$$

$$p_1(z) = -2z+1, \quad p_2(z) = 2z;$$

$$K_{1,0} = -\frac{1}{6} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \; K_{2,0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

and

$$P(z) = K'_{1,0}\, p_1(z) + K'_{2,0}\, p_2(z) = -\frac{1}{6} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} (-2z+1) = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} z - \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Therefore the Hermite-Fejér matrix $P(M)$ is given by

$$\begin{aligned} P(M) &= \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \bigotimes \frac{1}{2} \begin{bmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \bigotimes I_2 \\ &= \frac{1}{6} \begin{bmatrix} -1 & -2 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ \sqrt{3} & 2\sqrt{3} & 0 & 0 \\ 2\sqrt{3} & \sqrt{3} & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence a straightforward calculation shows that

$$I - P(M)^* P(M) = \begin{bmatrix} \frac{4}{9} & -\frac{4}{9} & 0 & 0 \\ -\frac{4}{9} & \frac{4}{9} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \geq 0 \qquad (\text{eigenvalues} : 1, 0, \frac{8}{9}),$$

which shows that T_Φ is hyponormal.

(ii) By the criterion (2) of Theorem 3.3: Observe

$$E := AB^{-1} = \begin{bmatrix} z-1 & 2 \\ 2 & z-1 \end{bmatrix}.$$

If $Q \in H^\infty_{M_2}$ is arbitrary then a straightforward calculation shows that

$$QE \in I_2 + zH^2_{M_2} \iff Q \in \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + zH^2_{M_2}.$$

Thus if we take $Q := \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ then since $||Q||_\infty = 1$, it follows from Theorem 3.3 that T_Φ is hyponormal.

Example 4.2. Let $b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}$ and consider

$$\Phi := \begin{bmatrix} 3\overline{b_{\frac{1}{2}}} + 3b_{\frac{1}{2}} & \overline{z} + z \\ \overline{z} + zb_{\frac{1}{3}} & 3\overline{b_{\frac{1}{2}}} + 3b_{\frac{1}{2}}b_{\frac{1}{3}} \end{bmatrix} \in L^\infty_{M_2}.$$

Then

$$\Phi_+ = zb_{\frac{1}{2}}b_{\frac{1}{3}} \begin{bmatrix} 3zb_{\frac{1}{3}} & b_{\frac{1}{2}}b_{\frac{1}{3}} \\ b_{\frac{1}{2}} & 3z \end{bmatrix}^* \quad \text{and} \quad \Phi_- = zb_{\frac{1}{2}} \begin{bmatrix} 3z & b_{\frac{1}{2}} \\ b_{\frac{1}{2}} & 3z \end{bmatrix}^*.$$

Thus under the notations of Corollary 3.4, we can write

$$\Theta_1 := zb_{\frac{1}{2}}I_2, \quad \Theta_2 := b_{\frac{1}{3}}I_2, \quad A := \begin{bmatrix} 3zb_{\frac{1}{3}} & b_{\frac{1}{2}}b_{\frac{1}{3}} \\ b_{\frac{1}{2}} & 3z \end{bmatrix}, \quad B := \begin{bmatrix} 3z & b_{\frac{1}{2}} \\ b_{\frac{1}{2}} & 3z \end{bmatrix}.$$

Then

$$B(0)A(0)^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{6} \\ -\frac{1}{2} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}.$$

But since $||B(0)A(0)^{-1}|| = 3 > 1$, we can, by Corollary 3.4, conclude that T_Φ is not hyponormal.

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