Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols

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Abstract. In this paper we deal with the subnormality and the quasinormality of Toeplitz operators with matrix-valued rational symbols. In particular, in view of Halmos's Problem 5, we focus on the question: Which subnormal Toeplitz operators are normal or analytic? We first prove: Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function having a "matrix pole," i.e., there exists $\alpha \in \mathbb{D}$ for which $\ker H_\Phi \subseteq (z-\alpha)H^2_{\mathbb{C}^n}$, where H_Φ denotes the Hankel operator with symbol Φ. If

(i) T_Φ is hyponormal;

(ii) $\ker\,[T^*_\Phi, T_\Phi]$ is invariant for T_Φ,

then T_Φ is normal. Hence in particular, if T_Φ is subnormal then T_Φ is normal.
Next, we show that every pure quasinormal Toeplitz operator with a matrix-valued rational symbol is unitarily equivalent to an analytic Toeplitz operator.

Keywords. Toeplitz operators, matrix-valued rational functions, Abrahamse's Theorem, Amemiya, Ito and Wong's Theorem, subnormal, quasinormal, hyponormal.

1. Introduction: Halmos's Problem 5

In 1970, P.R. Halmos addressed a problem on the subnormality of Toeplitz operators T_φ on the Hardy space $H^2 \equiv H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$. This is the so-called Halmos's Problem 5, presented in his lectures, *Ten problems in Hilbert space* [Hal1], [Hal2]:

Halmos's Problem 5. Is every subnormal Toeplitz operator either normal or analytic?

A subnormal operator is one that has a normal extension. A Toeplitz operator T_φ (with symbol $\varphi \in L^\infty \equiv L^\infty(\mathbb{T})$) is defined by the expression $T_\varphi f := P(\varphi f)$ for each $f \in H^2$, where P is the orthogonal projection from $L^2 \equiv L^2(\mathbb{T})$ onto H^2. A Toeplitz operator T_φ is called *analytic* if $\varphi \in H^\infty \equiv L^\infty \cap H^2$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, M_φ is a normal extension of T_φ, where M_φ is the normal operator of multiplication by φ on L^2. Thus, the question is natural because the two classes, the normal and analytic Toeplitz operators, are well understood and are subnormal. In 1984, Halmos's Problem 5 was answered in the negative by C. Cowen and J. Long [CoL]. However, Cowen and Long's idea does not give any general connection between subnormality and Toeplitz operators. Until now researchers have been unable to characterize subnormal Toeplitz operators T_φ in terms of their symbols φ. In fact, it may not even be possible to find tractable necessary and sufficient condition for the subnormality of T_φ in

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terms of their symbols unless certain assumptions are made about φ. Thus Halmos's Problem 5 may be reformulated as:

$$\text{Which Toeplitz operators are subnormal?} \tag{1}$$

The notion of subnormality was introduced by P.R. Halmos in 1950; the study of subnormal operators has been highly successful and fruitful (we refer to the book [Con] for details). Indeed, the theory of subnormal operators has made significant contributions to a number of problems in functional analysis, operator theory, mathematical physics, and other fields. However, ironically, the question "Which operators are subnormal?" is difficult to answer, as we will see below after we introduce a formal definition of subnormality of a Hilbert space operator. On the other hand, Toeplitz operators arise in a variety of problems in several fields of mathematics and physics, and nowadays the theory of Toeplitz operators is a very wide area. Thus it is natural and significant to elucidate the subnormality of Toeplitz operators.

To proceed, we introduce some basic definitions. Throughout this paper, let $\mathcal{H}$ denote a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, T^* denotes the adjoint of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if its self-commutator $[T^*, T] \equiv T^*T - TT^*$ is positive semi-definite, and *quasinormal* if T commutes with T^*T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *pure* if it has no nonzero reducing subspace on which it is normal. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $T = N|_{\mathcal{H}}$. In this case, N is called a *normal extension* of T. In general, it is quite difficult to examine whether such a normal extension exists for an operator. Of course, there are a couple of constructive methods for determining subnormality; one of them is the Bram-Halmos criterion of subnormality ([Br]), which states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{bmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{bmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{2}$$

Thus the Bram-Halmos criterion can be stated as follows: T is subnormal if and only if the positivity condition (2) holds for all $k \geq 1$. But it may not still be possible to test the positivity condition (2) for *every* positive integer k, in general. Hence the following question is interesting and challenging:

Which operators are subnormal?

As we remarked before, the class of Toeplitz operators is a nice test ground for this question. Directly connected with Halmos's Problem 5 is the following question:

$$\text{Which subnormal Toeplitz operators are normal or analytic?} \tag{3}$$

Partial answers to question (3) have been obtained by many authors (cf. [Abr], [AIW], [Co1], [CoL], [CHL1], [CHL2], [CL1], [CL2], [CL3], [ItW], [NT]). The best answers are obtained in one of two ways: (i) by strengthening the assumption of "subnormality," and (ii) by restricting the symbol to a special class of L^∞. Indeed, in 1975, I. Amemiya, T. Ito and T.K. Wong showed that the answer to Halmos's Problem 5 is affirmative for quasinormal operators ([AIW]):

Amemiya, Ito and Wong's Theorem ([AIW, Theorem]). Every quasinormal Toeplitz operator is either normal or analytic.

On the other hand, a function $\varphi \in L^\infty$ is said to be of bounded type if there are analytic functions $\psi_1, \psi_2 \in H^\infty$ such that $\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$ for almost all $z \in \mathbb{T}$. Evidently, rational functions are of bounded type. In 1976, M.B. Abrahamse showed that the answer is affirmative for Toeplitz operators with bounded type symbols ([Abr]):

Abrahamse's Theorem ([Abr, Theorem]). Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If
(i) T_φ is hyponormal;
(ii) $\ker\,[T_\varphi^*, T_\varphi]$ is invariant for T_φ,
then T_φ is normal or analytic.

Consequently, since $\ker\,[T^*, T]$ is invariant for every subnormal operator T, it follows that if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then every subnormal Toeplitz operator T_φ must be either normal or analytic.

The aim of this paper is to consider the following matrix-valued version of Halmos's Problem 5:

$$\text{Which subnormal Toeplitz operators with matrix-valued symbols are normal or analytic?} \tag{4}$$

In particular, we examine to what extent Abrahamse's Theorem and Amemiya, Ito and Wong's Theorem remain valid for Toeplitz operators with matrix-valued symbols.

To state our main theorems we define (block) Toeplitz operators and (block) Hankel operators. For $\mathcal{X}$ a Hilbert space, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$, and let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ and $H^\infty_{\mathcal{X}} \equiv H^\infty_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy spaces. Let $M_{m\times n} \equiv M_{m\times n}(\mathbb{C})$ denote the set of $m \times n$ complex matrices and write $M_n := M_{n\times n}$. If Φ is a matrix-valued function in $L^\infty_{M_n}$, then the (*block*) *Toeplitz operator* T_Φ and the (*block*) *Hankel operator* H_Φ on $H^2_{\mathbb{C}^n}$ are defined by

$$T_\Phi f := P(\Phi f) \quad \text{and} \quad H_\Phi f := JP^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}), \tag{5}$$

where P and $P^\perp$ denote the orthogonal projections that map $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\left(H^2_{\mathbb{C}^n}\right)^\perp$, respectively, and J denotes the unitary operator from $L^2_{\mathbb{C}^n}$ to $L^2_{\mathbb{C}^n}$ given by $(Jg)(z) := \overline{z} I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ ($I_n :=$ the $n \times n$ identity matrix). For $\Phi \in L^\infty_{M_{m\times n}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{6}$$

In general, question (4) is more difficult to answer, in comparison with the scalar-valued case. Indeed, Abrahamse's Theorem does not hold for block Toeplitz operators (even with matrix-valued *trigonometric polynomial* symbol): For instance, if

$$\Phi := \begin{bmatrix} z+\overline{z} & 0 \\ 0 & z \end{bmatrix},$$

then

$$T_\Phi = \begin{bmatrix} U_+ + U_+^* & 0 \\ 0 & U_+ \end{bmatrix} \quad (U_+ := \text{the unilateral shift on } H^2)$$

is neither normal nor analytic, although T_Φ is evidently subnormal. We believe this is due to the absence of a "matrix pole" in the symbol Φ (see Definition 3.5). That is, once we assume that a rational symbol has a matrix pole, we can get a version of Abrahamse's Theorem (Theorem 1.1 below). This concept is different from the classical notion of "pole" for matrix-valued rational functions (i.e., some entry in the matrix has a pole). The two notions coincide for scalar-valued rational functions.

Theorem 1.1. (Abrahamse's Theorem for Matrix-Valued Rational Symbols) Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function having a "matrix pole," i.e., there exists $\alpha \in \mathbb{D}$ for which $\ker H_\Phi \subseteq (z-\alpha)H^2_{\mathbb{C}^n}$. If

(i) T_Φ is hyponormal;
(ii) $\ker\,[T_\Phi^*, T_\Phi]$ is invariant for T_Φ,

then T_Φ is normal. Hence in particular, if T_Φ is subnormal then T_Φ is normal.

Remark 1.2. The assumption "Φ has a matrix pole" in Theorem 1.1 is automatically satisfied if Φ is scalar-valued (i.e., when $n = 1$). Thus, if $n = 1$, Theorem 1.1 is a special case of [Abr, Theorem].

On the other hand, Amemiya, Ito and Wong's Theorem does not also hold for the cases of matrix-valued symbols: indeed, if

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z}+2z \\ \overline{z}+2z & \overline{z} \end{bmatrix}, \tag{7}$$

then a straightforward calculation shows that

$$T_\Phi = \begin{bmatrix} U_+^* & U_+^*+2U_+ \\ U_+^*+2U_+ & U_+^* \end{bmatrix} \tag{8}$$

commutes with $T_\Phi^* T_\Phi$, i.e., T_Φ is quasinormal, but T_Φ is neither normal nor analytic. However if $W := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then W is unitary and

$$W^* T_\Phi W = 2 \begin{bmatrix} U_+^* + U_+ & 0 \\ 0 & -U_+ \end{bmatrix}, \tag{9}$$

which says that T_Φ is unitarily equivalent to a direct sum of a normal operator, say $2(U_+^* + U_+)$ and an analytic Toeplitz operator, say $-2U_+$. This phenomenon is not an accident. Indeed, we have:

Theorem 1.3. (Amemiya, Ito and Wong's Theorem for Matrix-Valued Rational Symbols) Every pure quasinormal Toeplitz operator with a matrix-valued rational symbol is unitarily equivalent to an analytic Toeplitz operator.

In Section 2, we give definitions and preliminaries which will be needed in the sequel. In Section 3 and Section 4, we give proofs of Theorem 1.1 and Theorem 1.3, respectively.

2. Preliminaries

We review a few essential facts for (block) Toeplitz operators and (block) Hankel operators, and for that we will use [BS], [Do1], [Do2], [Ni], and [Pe]. A matrix function $\Theta \in H^\infty_{M_{m\times n}}$ is called an *inner* function if Θ is isometric a.e. on $\mathbb{T}$. The following basic relations can be easily derived from the definition:

$$T_\Phi^* = T_{\Phi^*}, \quad H_\Phi^* = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{10}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); \tag{11}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \quad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}). \tag{12}$$

For a matrix-valued function $\Phi = [\phi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry ϕ_{ij} is of bounded type and that Φ is *rational* if each entry ϕ_{ij} is a rational function. For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We would remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero, then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$ (cf. [FF]).

In 1988, the hyponormality of Toeplitz operators T_φ was completely characterized in terms of their symbols φ via an elegant theorem of C. Cowen [Co2]. Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution with specified properties to a certain functional equation involving the symbol φ. Today, this theorem is referred as *Cowen's Theorem.*

Cowen's Theorem ([Co2], [NT]). For $\varphi \in L^\infty$, write

$$\mathcal{E}(\varphi) := \Big\{ k \in H^\infty : \ ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty \Big\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

In 2006, Gu, Hendricks and Rutherford [GHR] extended Cowen's Theorem to block Toeplitz operators. Their characterization for hyponormality of block Toeplitz operators resembles Cowen's Theorem except for an additional condition - the normality of the symbol.

Lemma 2.1. (Hyponormality of Block Toeplitz Operators) [GHR] For each $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \Big\{ K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

On the other hand, we note that by (12), the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator T_{zI_n} on $H^2_{\mathbb{C}^n}$. Thus if $\ker H_\Phi \ne \{0\}$ then by the Beurling-Lax-Halmos Theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ. In general, Θ need not be a square matrix function.

We nevertheless have:

Lemma 2.2. ([GHR]) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

1. Φ is of bounded type;
2. $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some *square* inner matrix function Θ;
3. $\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.

For an inner matrix function $\Theta \in H^2_{M_n}$, we write

$$\mathcal{H}_\Theta := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}.$$

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n(\Phi) \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \big(P_n^\perp(\Phi)\big)^* \in H^2_{M_n},$$

where P_n and $P_n^\perp$ denote the orthogonal projections from $L^2_{M_n}$ onto $H^2_{M_n}$ and $(H^2_{M_n})^\perp$, respectively. Thus, we can write $\Phi = \Phi_-^* + \Phi_+$. In view of Lemma 2.2, if $\Phi \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type then Φ_+ and Φ_- can be written in the form

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \tag{13}$$

where Θ_1 and Θ_2 are inner, $A, B \in H^2_{M_n}$, Θ_1 and A are right coprime, and Θ_2 and B are right coprime. In (13), $\Theta_1 A^*$ and $\Theta_2 B^*$ will be called *right coprime factorizations* of Φ_+ and Φ_-, respectively.

3. Proof of Theorem 1.1

Let $\lambda \in \mathbb{D}$ and write

$$b_\lambda(z) := \xi \frac{z-\lambda}{1-\overline{\lambda} z} \quad (\xi \in \mathbb{T})\,;$$

b is called a *Blaschke factor*. If M is a nonzero closed subspace of $\mathbb{C}^n$ then the matrix function of the form

$$b_\lambda P_M + (I - P_M) \quad (P_M := \text{the orthogonal projection of } \mathbb{C}^n \text{ onto } M)$$

is called a *Blaschke-Potapov factor*; an $n \times n$ matrix function D is called *a finite Blaschke-Potapov product* if D is of the form

$$D = \nu \prod_{m=1}^{d} \Big(b_m P_m + (I - P_m) \Big), \tag{14}$$

where ν is an $n \times n$ unitary constant matrix, b_m is a Blaschke factor, and P_m is an orthogonal projection in $\mathbb{C}^n$ for each $m = 1, \cdots, d$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D = \nu \prod_{m=1}^{d} b_m$, where $\nu = e^{i\omega}$. It is known (cf. [Po]) that an $n \times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product. Thus if $\Phi \in L^\infty_{M_n}$ is rational then Θ_1 and Θ_2 can be chosen as finite Blaschke-Potapov products in the right coprime factorizations of (13).

The condition "(left/right) coprime" for two matrix-valued functions is not easy to check in general. However, if one of them is a rational function whose determinant is not identically zero then we can obtain a more tractable criterion on their (left/right) coprime-ness. To see this, we first observe:

Lemma 3.1. If $F \in H^2_{M_n}$ and M is a non-zero closed subspace of $\mathbb{C}^n$ then

$$b_\lambda P_M + (I - P_M) \text{ is a right inner divisor of } F \iff M \subseteq \ker F(\lambda). \tag{15}$$

Proof. To see this we first observe that if $\lambda \in \mathbb{D}$, then we have

$$F(z) - F(\lambda) = G(z)\,(z-\lambda) I_n = G(z)\,(1 - \overline{\lambda} z) b_\lambda(z) I_n \quad \text{for some } G \in H^2_{M_n}.$$

Thus, we can write

$$F = F(\lambda) + b_\lambda\, F_1 \quad \text{for some } F_1 \in H^2_{M_n}\,.$$

If $M \subseteq \ker F(\lambda)$, then we can see that $F(\lambda) = F(\lambda)\big(b_\lambda P_M + (I - P_M)\big)$. We thus have

$$\begin{aligned} F &= F(\lambda) + b_\lambda F_1 \\ &= F(\lambda)\big(b_\lambda P_M + (I-P_M)\big) + F_1\big(P_M + b_\lambda (I - P_M)\big)\big(b_\lambda P_M + (I - P_M)\big) \\ &= \Big(F(\lambda) + F_1\big(P_M + b_\lambda(I-P_M)\big)\Big)\big(b_\lambda P_M + (I - P_M)\big), \end{aligned}$$

which implies that $b_\lambda P_M + (I - P_M)$ is a right inner divisor of F. Conversely, if $F = G\,\big(b_\lambda P_M + (I - P_M)\big)$ for some $G \in H^2_{M_n}$, then

$$F(\lambda) = G(\lambda)(I - P_M)\,,$$

which implies that $M \subseteq \ker F(\lambda)$. This proves the lemma. □

Corollary 3.2. If $A, B \in H^2_{M_n}$ and B is a rational function such that $\det B$ is not identically zero then

$$A \text{ and } B \text{ are right coprime} \iff \ker A(\alpha) \cap \ker B(\alpha) = \{0\} \text{ for any } \alpha \in \mathbb{D}.$$

Proof. We first observe that if $\det B$ is not identically zero then each right inner divisor F of B is square, i.e., $F \in H^\infty_{M_n}$. Since B is rational, F is a finite Blaschke-Potapov product. Thus if A and B are not right coprime then A and B have a common non-constant Blaschke-Potapov factor $b_\lambda P_M + (I - P_M)$ as a right inner divisor. By (15), $\ker A(\lambda) \cap \ker B(\lambda) \supseteq M \neq \{0\}$. Conversely, if $\ker A(\lambda) \cap \ker B(\lambda) = M \neq \{0\}$ for some $\lambda \in \mathbb{D}$ then, again by (15), A and B have a common non-constant Blaschke-Potapov factor $b_\lambda P_M + (I - P_M)$ as a right inner divisor. This proves the corollary. □

From Corollary 3.2, we can see that if $\Theta = \theta I_n$ for a finite Blaschke product θ, then for any $A \in H^2_{M_n}$,

$$A \text{ and } \Theta \text{ are right coprime} \iff A(\alpha) \text{ is invertible for each zero } \alpha \text{ of } \theta \tag{16}$$

(cf. [CHL2, Lemma 3.3]).

For an operator $T \in \mathcal{B}(\mathcal{H})$, the *essential norm* $||T||_e$ is defined by

$$||T||_e := \inf \big\{||T - K|| : \ K \text{ is compact}\big\}.$$

It is known (cf. [Pe, Theorem I.5.3]) that if $\varphi \in L^\infty$ then the essential norm of a Hankel operator H_φ can be computed from the formula

$$||H_\varphi||_e = \text{dist}_{L^\infty}\Big(\varphi,\ H^\infty + C\Big),$$

where C is the set of continuous functions on $\mathbb{T}$. In particular, since H_φ is compact if and only if $\varphi \in H^\infty + C$, it follows that $||H_\varphi||_e = 0$ if $\varphi \in H^\infty + C$.

The following proposition provides important information on $\mathcal{E}(\Phi)$ if Φ is a matrix-valued rational function such that T_Φ is hyponormal.

Proposition 3.3. Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function such that T_Φ is hyponormal. Then $\mathcal{E}(\Phi)$ contains an inner matrix function.

Proof. If T_Φ is hyponormal, then by Lemma 2.1 there exists a matrix function $K_1 \in \mathcal{E}(\Phi)$. Since Φ^* is a matrix-valued rational function, it follows from Kronecker's Lemma that H_{Φ^*} is of finite rank, and hence $\ker H_{\widetilde{\Phi}^*} = \widetilde{\Theta} H^2_{\mathbb{C}^n}$ for some finite Blaschke-Potapov product Θ (cf. (6)). We claim that

$$\mathcal{E}(\Phi) = \Big\{K_1 + F\Theta : F \in H^\infty_{M_n} \text{ and } \|K_1 + F\Theta\|_\infty \le 1\Big\}. \tag{17}$$

To see this, suppose that $K_2 \in \mathcal{E}(\Phi)$. Then $(K_2 - K_1)\Phi^* \in H^\infty_{M_n}$, so that $\widetilde{\Phi}^*(\widetilde{K_2} - \widetilde{K_1}) \in H^\infty_{M_n}$. Thus, $H_{\widetilde{\Phi}^*(\widetilde{K_2} - \widetilde{K_1})} = 0$, so that

$$(\widetilde{K_2} - \widetilde{K_1}) H^2_{\mathbb{C}^n} \subseteq \ker H_{\widetilde{\Phi}^*} = \widetilde{\Theta} H^2_{\mathbb{C}^n}.$$

Thus, $\widetilde{\Theta}$ is a left inner divisor of $\widetilde{K_2} - \widetilde{K_1}$ (cf. [FF, Corollary IX.2.2]). Hence $\widetilde{K_2} \in \widetilde{K_1} + \widetilde{\Theta} H^\infty_{M_n}$, so that $K_2 = K_1 + F\Theta$ for some $F \in H^\infty_{M_n}$, which implies

$$\mathcal{E}(\Phi) \subseteq \Big\{K_1 + F\Theta : F \in H^\infty_{M_n} \text{ and } \|K_1 + F\Theta\|_\infty \le 1\Big\};$$

the reverse inclusion is evident. This proves (17). Observe

$$||H_{K_1\Theta^*}|| \le ||K_1\Theta^*||_\infty = ||K_1||_\infty \le 1. \tag{18}$$

Since Θ is a rational function and $\ker H_{K_1\Theta^*} \supseteq \Theta H^2_{\mathbb{C}^n}$, so that $\text{ran}\, H^*_{K_1\Theta^*} \subseteq \mathcal{H}_\Theta$, it follows that $H^*_{K_1\Theta^*}$ is of finite rank and hence so is $H_{K_1\Theta^*}$, which implies that

$$||H_{K_1\Theta^*}||_e = 0. \tag{19}$$

Now we recall a matrix-valued version of the Adamyan-Arov-Krein Theorem (cf. [Pe, Theorem 14.14.1]): if $||H_\Phi||_e < 1$, then $||H_\Phi|| \le 1$ if and only if there exists a unitary-valued matrix function W such that $H_\Phi = H_W$ and $||H_{W^*}||_e < 1$. Thus by (18) and (19), we can find a function $F_1 \in H^\infty_{M_n}$

such that $K_1\Theta^* + F_1$ is a unitary-valued matrix function in $L^\infty_{M_n}$. Hence, $K_1 + F_1\Theta$ is an inner matrix function in $\mathcal{E}(\Phi)$. □

Remark 3.4. (a) From the proof of Proposition 3.3, we can see that if $\Phi \in L^\infty_{M_n}$ is such that $\mathcal{E}(\Phi)$ contains a matrix function K_1 with $||K_1||_\infty < 1$, then $\mathcal{E}(\Phi)$ contains an inner matrix function: indeed, this follows at once from the observation that if $||K_1||_\infty < 1$, then $||H_{K_1\Theta^*}||_e \le ||K_1\Theta^*||_\infty \le ||K_1||_\infty < 1$, so that the same argument as in the proof of Proposition 3.3 gives the result.

(b) If we consider the formula (19) in the proof of Proposition 3.3, we might suspect that the rationality assumption of the symbol is too strong because it is enough to have $||H_{K_1\Theta^*}||_e < 1$. However, it may happen that for even scalar-valued cases, there is no gap between them. For example, let $k_1 := \alpha \in \mathbb{C}$ with $|\alpha| = 1$ and θ be inner. Suppose $||H_{k_1\overline{\theta}}||_e < 1$, and hence $\text{dist}_{L^\infty}(k_1\overline{\theta},\ H^\infty + C) < 1$. Since $H^\infty + C$ is a closed subalgebra of L^∞, it follows that if $h \in H^\infty + C$ and $||k_1\overline{\theta} - h||_\infty = ||1 - \frac{1}{\alpha}\theta h||_\infty < 1$, then $k_1\overline{\theta} = h\sum_{n=0}^\infty (1 - \frac{1}{\alpha}\theta h)^n \in H^\infty + C$, and hence $||H_{k_1\overline{\theta}}||_e = 0$ (cf. [Ni, p.323]).

(c) If $|c_0| = 1$ and $k(z) := \sum_{n=0}^\infty c_n z^n$ ($c_n \in \mathbb{C}$ for $n = 0, 1, \cdots$), then evidently, $||k||_\infty > 1$ whenever $k \ne c_0$. However this is not the case for matrix functions. In particular, Proposition 3.3 guarantees that if $\Phi \in L^\infty_{M_n}$ is rational and $C_0 \in M_n$ with $C_0 \in \mathcal{E}(\Phi)$ and $||C_0||_\infty = 1$, but C_0 is not unitary, then we can always find an inner matrix function $K \in \mathcal{E}(\Phi)$. For example, consider

$$\Phi := \begin{bmatrix} \overline{z} & \overline{z} + 2z \\ \overline{z} + 2z & \overline{z} \end{bmatrix}.$$

If we put $K := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $\Phi - K\Phi^* \in H^\infty_{M_2}$ and $||K||_\infty = 1$, which implies $K \in \mathcal{E}(\Phi)$, but K is not inner. However if we take $K' := \frac{1}{2}\begin{bmatrix} 1+z & 1-z \\ 1-z & 1+z \end{bmatrix}$, then we have: (i) $K' = K + \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}z$; (ii) $\Phi - K'\Phi^* \in H^\infty_{M_2}$; and (iii) K' is inner, which illustrates Proposition 3.3. □

We now introduce the notion of a "matrix pole" for matrix-valued rational functions. To do so, we first consider a representation for poles of scalar-valued rational functions. Let $\varphi \in L^\infty$ be a rational function. Then we may write

$$\varphi_- = \theta\overline{a} \quad \text{(coprime factorization)},$$

where θ is a nonconstant finite Blaschke product and $a \in H^2$. Since $\varphi = \frac{a}{\theta} + \varphi_+$, it follows that $\varphi(z)$ has a pole at $z = \alpha \in \mathbb{D}$ if and only if θ has a zero at $z = \alpha$ if and only if the Blaschke factor b_α is an inner divisor of θ. Observe that $\ker H_\varphi = \ker H_{\overline{\varphi_-}} = \theta H^2$ and that $(z - \alpha)H^2 = b_\alpha H^2$ because $1 - \overline{\alpha}z$ is an outer function, and hence $(1 - \overline{\alpha}z)H^2 = H^2$. We thus have

$$\varphi(z) \text{ has a pole at } z = \alpha \iff \ker H_\varphi \subseteq (z - \alpha)H^2. \tag{20}$$

On the other hand, block Hankel operators have been extensively exploited when considering properties of matrix-valued functions in $L^\infty_{M_n}$ (e.g., matrix-valued versions of Nehari's Theorem, Hartman's Theorem and Kronecker's Lemma). In particular, if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function, then it is known (cf. [Pe, p. 81]) that rank H_Φ is equal to the McMillan degree of Φ_-.

For the definition of matrix poles for matrix-valued rational functions, we will adopt the idea in (20).

Definition 3.5. Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. Then we say that Φ has a *matrix pole* at $\alpha \in \mathbb{D}$ if

$$\ker H_\Phi \subseteq (z - \alpha)H^2_{\mathbb{C}^n}.$$

We shall say that an inner matrix function $\Theta \in H^\infty_{M_n}$ is *diagonal-constant* if Θ is of the form θI_n, where θ is an inner function. We then have:

Lemma 3.6. Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Thus in view of (13), we may write

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization)}.$$

Then Φ has a matrix pole if and only if Θ has a nonconstant diagonal-constant inner divisor.

Proof. We first observe that by Lemma 2.2,

$$\ker H_\Phi = \ker H_{\Phi_-^*} = \ker H_{B\Theta^*} = \Theta H^2_{\mathbb{C}^n}\,.$$

Since $(z-\alpha)H^2_{\mathbb{C}^n} = b_\alpha H^2_{\mathbb{C}^n}$, it follows that Φ has a matrix pole at $z=\alpha$ if and only if $\Theta H^2_{\mathbb{C}^n} \subseteq b_\alpha H^2_{\mathbb{C}^n}$ if and only if $b_\alpha I_n$ is an inner divisor of Θ. This proves the lemma. □

Remark 3.7. (i) Recall that if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function then Φ is said to have a *pole* at $\alpha \in \mathbb{D}$ if some entry of $\Phi(z)$ has a pole at $z=\alpha$. We now claim that for $\alpha \in \mathbb{D}$,

$$\alpha \text{ is a matrix pole of } \Phi \implies \alpha \text{ is a pole of } \Phi\,. \tag{21}$$

Towards (21) we write

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization)}.$$

Suppose α is a matrix pole of Φ. Then by Lemma 3.6, $\Theta = b_\alpha I_n \Theta_1$ for some inner function Θ_1. Thus by (16), $B(\alpha)$ is invertible. Since $\Phi \equiv \Phi_-^* + \Phi_+ = (B + \Phi_+\Theta)\Theta^*$ and $\det\Theta$ is inner, we have

$$\det\,\Phi = \frac{\det\,(B+\Phi_+\Theta)}{\det\,\Theta} = \frac{\det\,(B+\Phi_+\Theta)}{b_\alpha^n\,\det\Theta_1}.$$

But since $(B+\Phi_+\Theta)(\alpha) = B(\alpha)$ is invertible, it follows that α is a pole of $\det\Phi$, which implies that some entry of $\Phi(z)$ has a pole at $z=\alpha$. This proves (21). However the converse of (21) is not true. For example if

$$\Phi := \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{bmatrix},$$

then Φ has a pole at $z=0$. But since

$$\Phi_- = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^* \quad \text{(right coprime factorization)}$$

and $\Theta \equiv \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$ has no inner divisor of the form $b_\alpha I_2$, it follows that Φ has no matrix pole. Of course, by definition, if $n=1$ then a matrix pole reduces to a pole.

(ii) From the viewpoint of scalar-valued rational functions, we are tempted to guess that if a matrix-valued rational function $\Phi \in L^\infty_{M_n}$ has a matrix pole at $z=\alpha \in \mathbb{D}$, then Φ can be written as

$$\Phi(z) = \sum_{k=-N}^{\infty} A_k (z-\alpha)^k \quad (N \ge 1;\ A_{-N} \text{ is invertible}), \tag{22}$$

where "nonzero" in the scalar-valued case is interpreted as "invertible" in the matrix-valued case. But this guess is not true. For example, consider the function

$$\Phi(z) = \begin{bmatrix} \frac{1}{z^2} + z^2 & 0 \\ 0 & \frac{1}{z} + z \end{bmatrix}.$$

Then since $\Phi_-(z) = \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix}$, it follows from Lemma 3.6 that Φ has a matrix pole at $z=0$, while $\Phi(z) = \sum_{k=-2}^{2} A_k z^k$ with $A_{-2} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ non-invertible. However we can easily check that (22) is a sufficient condition for Φ to have a matrix pole at $z=\alpha$. □

To prove Theorem 1.1, we recall ([CHL2, Lemma 3.2]) that if $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type, so that we may write, as in (13),

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta B^* \quad \text{(right coprime factorizations)}$$

and T_Φ is hyponormal, then

$$\Theta_1 = \Theta\Theta_2 \quad \text{for some inner matrix function } \Theta_2; \tag{23}$$

in other words, Θ is a left inner divisor of Θ_1.

We are ready for:

Proof of Theorem 1.1**.** Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Thus in view of (13), we may write

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization)}.$$

Suppose Φ has a matrix pole at $z = \alpha \in \mathbb{D}$. Then by Lemma 3.6, Θ has an inner divisor of the form θI_n, where θ is a Blaschke factor of the form $\theta(z) := \frac{z-\alpha}{1-\overline{\alpha}z}$ $(\alpha \in \mathbb{D})$, and we can write

$$\Theta = \theta I_n \Theta_1 \text{ for some finite Blaschke-Potapov product } \Theta_1.$$

Now we assume to the contrary that T_Φ is not normal. For the reader's convenience, we split the proof into three steps.

STEP 1 (An analysis of $\text{ran}\,[T_\Phi^*, T_\Phi]$): Since T_Φ is hyponormal we may write, in view of (23),

$$\Phi_+ = \Theta\Theta_2 A^* \quad \text{(right coprime factorization)}\,,$$

where Θ_2 is a finite Blaschke-Potapov product. By Proposition 3.3, there exists an inner matrix function $K \in \mathcal{E}(\Phi)$. Thus by (11) and (12),

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_{K\Phi^*}^* H_{K\Phi^*} = H_{\Phi^*}^* H_{K^*} H_{K^*}^* H_{\Phi^*}. \tag{24}$$

Observe

$$\big(\text{ran}\,[T_\Phi^*, T_\Phi]\big)^\perp = \ker [T_\Phi^*, T_\Phi] = \ker H_{K^*}^* H_{\Phi^*} = \ker H_{K^*}^* H_{\Phi_+^*} \supseteq \Theta\Theta_2 H^2_{\mathbb{C}^n},$$

so that

$$\text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}_{\Theta\Theta_2}\,. \tag{25}$$

Let Θ_3 be a diagonal-constant inner function of the form

$$\Theta_3 = \theta_3 I_n = \Delta\Theta\Theta_2, \quad \text{so that } \ \text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}_{\Theta_3}, \tag{26}$$

where θ_3 is an inner function and Δ is constructed as follows. If we write, using (14), $\Theta\Theta_2 = \nu \prod_{j=1}^m b_j P_j + (I - P_j)$, then Δ is obtained by

$$\Delta = \Delta_m \cdots \Delta_2 \Delta_1,$$

where

$$\Delta_1 := \big(P_1 + b_1(I - P_1)\big)\nu^* \ \text{ and } \ \Delta_j := P_j + b_j(I - P_j) \ (j \geq 2),$$

and eventually, $\theta_3 = \prod_{j=1}^m b_{\lambda_j}$. In view of (26) we can define

$$M := \bigcap \Big\{ \mathcal{H}_{\delta I_n} : \ \text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}_{\delta I_n}, \ \delta \text{ is a scalar inner function } \Big\}.$$

Since each $\mathcal{H}_{\delta I_n}$ is an invariant subspace for $T_{\overline{z}I_n}$, M is also invariant for $T_{\overline{z}I_n}$. Therefore by the Beurling-Lax-Halmos Theorem, $M = \mathcal{H}_\Omega$ for an inner function $\Omega \in H_{M_{n\times m}}$. One can easily verify that $\Omega = \omega I_n$, i.e., $M = \mathcal{H}_{\omega I_n}$, where

$$\omega := GCD\Big\{ \delta : \ \text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}_{\delta I_n}, \ \delta \text{ is an inner function } \Big\}.$$

Evidently, ω is a nonconstant inner function because T_Φ is not normal. Also it is clear (from the way that Ω was constructed) that Ω is a "minimal" diagonal-constant inner function such that

$$\text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}_\Omega, \quad \text{or equivalently,} \quad \Omega^* \left(\text{ran}\,[T_\Phi^*, T_\Phi]\right) \subseteq (H^2_{\mathbb{C}^n})^\perp. \tag{27}$$

STEP 2 (A relationship between θ and ω): We claim that

$$\theta \text{ is an inner divisor of } \omega. \tag{28}$$

Toward (28), we first argue that

$$\text{ran}\,[T_\Phi^*, T_\Phi] \nsubseteq \mathcal{H}_{\Theta_1\Theta_2}. \tag{29}$$

Assume to the contrary that $\text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}_{\Theta_1\Theta_2}$, or equivalently, $\Theta_1\Theta_2 H^2_{\mathbb{C}^n} \subseteq \ker[T_\Phi^*, T_\Phi]$, so that $\Theta_1\Theta_2 H^2_{\mathbb{C}^n} \subseteq \ker H^*_{K^*} H_{\Phi_+^*}$, which implies that

$$H^*_{K^*} H_{A\Theta_2^*\Theta^*}(\Theta_1\Theta_2 H^2_{\mathbb{C}^n}) = 0 \ \text{ and hence, } \quad H^*_{K^*} H_{A\overline{\theta} I_n}(H^2_{\mathbb{C}^n}) = 0. \tag{30}$$

By (16), $\widetilde{A}$ and $\widetilde{\theta} I_n$ are right coprime, so that by Lemma 2.2,

$$\text{ran}(H_{A\overline{\theta} I_n}) = \left(\ker H_{\widetilde{A}(\widetilde{\theta} I_n)^*}\right)^\perp = \left((\widetilde{\theta} I_n) H^2_{\mathbb{C}^n}\right)^\perp = \mathcal{H}_{\widetilde{\theta} I_n}.$$

It follows from (30) that

$$\ker H_{\widetilde{K^*}} \supseteq \mathcal{H}_{\widetilde{\theta} I_n}. \tag{31}$$

Write $K \equiv [k_{ij}]_{1\le i,j\le n}$. Since $\frac{1}{1-\alpha z}$ is the reproducing kernel for $\overline{\alpha}$ and $\widetilde{\theta}(z) = \frac{z-\overline{\alpha}}{1-\alpha z}$, it follows that $\langle \widetilde{\theta} f,\ \frac{1}{1-\alpha z}\rangle = \widetilde{\theta}(\overline{\alpha}) f(\overline{\alpha}) = 0$ for any $f \in H^2$, which implies that $\frac{1}{1-\alpha z} \in \mathcal{H}_{\widetilde{\theta}}$. Thus, by (31), we have

$$k_{ij}(\overline{z}) \frac{1}{1-\alpha z} \in H^2 \quad \text{for each } 1 \le i, j \le n.$$

Therefore, $k_{ij}(\overline{z}) \in (1-\alpha z) H^2 \subseteq H^2$, which forces that each k_{ij} is constant, and hence K is constant. Thus by (24), $[T_\Phi^*, T_\Phi] = 0$, i.e., T_Φ is normal, which is a contradiction. This proves (29). Observe that

$$\mathcal{H}_{\Theta\Theta_2} = \mathcal{H}_{\Theta_1\Theta_2\theta I_n} = \mathcal{H}_{\Theta_1\Theta_2} \oplus \Theta_1\Theta_2 \mathcal{H}_{\theta I_n}.$$

Thus, by (25), (27), and (29), there exist a nonzero $f \in \mathcal{H}_{\theta I_n}$ and $g \in \mathcal{H}_{\Theta_1\Theta_2}$ such that

$$g + \Theta_1\Theta_2 f \in \text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}_\Omega.$$

Observe that

$$\begin{aligned} g + \Theta_1\Theta_2 f \in \mathcal{H}_\Omega &\Longrightarrow \langle g + \Theta_1\Theta_2 f,\ \Omega h\rangle = 0 \ \text{ for each } h \in H^2_{\mathbb{C}^n} \\ &\Longrightarrow \langle g + \Theta_1\Theta_2 f,\ \Omega\Theta_1\Theta_2 h\rangle = 0 \ \text{ for each } h \in H^2_{\mathbb{C}^n} \\ &\Longrightarrow \langle f,\ \Omega h\rangle = 0 \ \text{ for each } h \in H^2_{\mathbb{C}^n} \quad (\Omega \equiv \omega I_n \text{ and } g \in \mathcal{H}_{\Theta_1\Theta_2}) \\ &\Longrightarrow f \in \mathcal{H}_\Omega. \end{aligned} \tag{32}$$

Since $0 \ne f \in \mathcal{H}_{\theta I_n}$ and $\mathcal{H}_\theta = \left\{\frac{d}{1-\overline{\alpha} z} : d \in \mathbb{C}\right\}$, we can write

$$f = \begin{bmatrix} \frac{d_1}{1-\overline{\alpha} z} \\ \vdots \\ \frac{d_n}{1-\overline{\alpha} z} \end{bmatrix} \quad (\text{where } d_i \in \mathbb{C},\ \ d_{i_0} \ne 0 \ \text{ for some } i_0).$$

By (32), we know that $f \in \mathcal{H}_\Omega$, and therefore

$$\langle f,\ \Omega\, \mathbf{e}_{i_0}\rangle = 0 \quad (\text{where } \mathbf{e}_{i_0} \text{ is 1 on the } i_0\text{-th component and 0 otherwise}),$$

so that $d_{i_0}\omega(\alpha) = 0$, and hence $\omega(\alpha) = 0$, which proves (28).

STEP 3 (Deriving a contradiction): Let $h \in \text{ran}\,[T_\Phi^*, T_\Phi]$. Since by assumption, $\ker[T_\Phi^*, T_\Phi]$ is invariant for T_Φ, we have $T_\Phi^*(\text{ran}\,[T_\Phi^*, T_\Phi]) \subseteq \text{ran}\,[T_\Phi^*, T_\Phi]$, and hence,

$$T_\Phi^* h \in \text{ran}\,[T_\Phi^*, T_\Phi].$$

From (27), $\Omega^* T_\Phi^* h \in (H^2_{\mathbb{C}^n})^\perp$, so that $\Omega^*\Phi^* h \in (H^2_{\mathbb{C}^n})^\perp$, and hence

$$\Theta_1^*\Omega^*\Phi^* h \in (H^2_{\mathbb{C}^n})^\perp. \tag{33}$$

Note that since $\Theta \equiv \theta I_n \Theta_1$ and B are right coprime, it follows from (16) that $B(\alpha)$ is invertible. We write

$$B(z) - B(\alpha) = \theta I_n B_1 \quad \text{for some } B_1 \in H^2_{M_n}.$$

Keeping in mind that $\Omega \equiv \omega I_n$ is diagonal-constant, we get

$$\begin{aligned}\Theta_1^*\Omega^*\Phi^*h &= \Theta_1^*\Omega^*(\Phi_+^* + \theta I_n\Theta_1 B^*)h\\ &= \Theta_1^*\Phi_+^*(\Omega^*h) + B^*\Omega^*\theta I_n h\\ &= \Theta_1^*\Phi_+^*(\Omega^*h) + (B_1^*\overline{\theta} I_n + B(\alpha)^*)\Omega^*\theta I_n h\\ &= \Theta_1^*\Phi_+^*(\Omega^*h) + B_1^*\Omega^*h + B(\alpha)^*\Omega^*\theta I_n h.\end{aligned} \tag{34}$$

But since by (27), $\Omega^*h \in (H^2_{\mathbb{C}^n})^\perp$, it follows from (33) and (34) that

$$B(\alpha)^*\Omega^*\theta I_n h \in (H^2_{\mathbb{C}^n})^\perp,$$

so that

$$\Omega^*\theta I_n h \in (H^2_{\mathbb{C}^n})^\perp$$

since $B(\alpha)$ is invertible. Therefore we see that

$$(\Omega\overline{\theta} I_n)^*h \in (H^2_{\mathbb{C}^n})^\perp \quad \text{for each } h \in \text{ran}\,[T_\Phi^*, T_\Phi]\,.$$

This is a contradiction, because by (28), $\Omega\overline{\theta} I_n = (\omega\overline{\theta})I_n \in H^\infty_{M_n}$, but $\Omega \equiv \omega I_n$ was chosen to be a *minimal* diagonal-constant inner function satisfying $\Omega^*h \in (H^2_{\mathbb{C}^n})^\perp$ for each $h \in \text{ran}\,[T_\Phi^*, T_\Phi]$. Therefore T_Φ should be normal.

The second assertion follows at once from the fact that if T_Φ is subnormal then $\ker\,[T_\Phi^*, T_\Phi]$ is invariant for T_Φ. This completes the proof of Theorem 1.1. □

The positivity condition (2) provides a measure of the gap between hyponormality and subnormality. In fact, condition (2) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (2) for all $k \ge 1$. Recall ([cf. CL2]) that for $k \ge 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *k-hyponormal* if T satisfies the positivity condition (2) for a fixed k. Thus the Bram-Halmos criterion can be stated as: T is subnormal if and only if T is k-hyponormal for all $k \ge 1$. k-hyponormality has been considered by many authors with an aim at understanding the gap between hyponormality and subnormality. For instance, the Bram-Halmos criterion on subnormality indicates that 2-hyponormality is generally far from subnormality. However, there are special classes of operators for which these two notions are equivalent. For example, in [CL1, Theorem 3.2], it was shown that 2-hyponormality and subnormality coincide for Toeplitz operators T_φ with trigonometric polynomial symbols $\varphi \in L^\infty$. Also 2-hyponormality and subnormality enjoy some common properties. One of them is the following fact [CL2]: If $T \in \mathcal{B}(\mathcal{H})$ is 2-hyponormal then $\ker\,[T^*, T]$ is invariant for T. Thus Theorem 1.1 can be rephrased as:

Corollary 3.8. Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Assume Φ has a matrix pole, or equivalently, if we write

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization)},$$

then Θ has a nonconstant diagonal-constant inner divisor. Then the following are equivalent:

1. T_Φ is 2-hyponormal;
2. T_Φ is subnormal;
3. T_Φ is normal.

Proof. Immediate from Theorem 1.1. □

In particular, [CL1, Theorem 3.2] can be generalized to the matrix-valued case, as follows.

Corollary 3.9. Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial whose co-analytic outer coefficient is invertible. Then the 2-hyponormality and the normality of T_Φ coincide.

Proof. Using the notation of Corollary 3.8, write $\Phi_- := \sum_{j=1}^m B_{-j}z^j$, where B_{-m} is invertible. We have

$$\Theta := z^m I_n \quad \text{and} \quad B = B_{-m}^* + B_{-m+1}^* z + \cdots + B_{-1}^* z^{m-1},$$

and hence by (16) and our assumption, Θ and B are right coprime. The assertion now follows at once from Corollary 3.8. $\square$

Since a nonzero coefficient in $\mathbb{C}$ is trivially *invertible*, Corollary 3.9 reduces to [CL1, Theorem 3.2] if $n = 1$.

Example 3.10. Consider the following matrix-valued trigonometric polynomial

$$\Phi := \begin{bmatrix} 2z^3 + \overline{z} & -2z^3 - \overline{z} \\ 2z^2 + \overline{z}^2 & 2z^2 + \overline{z}^2 \end{bmatrix}. \tag{35}$$

Then

$$\Phi_- = \begin{bmatrix} z & z^2 \\ -z & z^2 \end{bmatrix} \quad \text{and} \quad \Phi_+ = 2\begin{bmatrix} z^3 & -z^3 \\ z^2 & z^2 \end{bmatrix}.$$

A straightforward calculation shows that $\Phi^*\Phi = \Phi\Phi^*$. If $K := \frac{1}{4}\begin{bmatrix} -z + z^2 & -z - z^2 \\ 1 + z & 1 - z \end{bmatrix}$, then

$$||K||_\infty \leq 1 \quad \text{and} \quad \Phi_-^* = K\Phi_+^*.$$

Thus by Lemma 2.1, T_Φ is hyponormal. But a direct calculation shows that T_Φ is not normal. We note that

$$\Phi_- \equiv \begin{bmatrix} z & z^2 \\ -z & z^2 \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} z & z^2 \\ -z & z^2 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^*, \tag{36}$$

where $\Theta \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} z & z^2 \\ -z & z^2 \end{bmatrix}$ and $B \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are right coprime by Corollary 3.2. However, $\Theta \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} z & z^2 \\ -z & z^2 \end{bmatrix}$ has a nonconstant diagonal inner divisor of the form zI_2, so that Φ has a matrix pole. But since T_Φ is not normal, it follows from Theorem 1.1 that T_Φ is not subnormal.

Remark 3.11. Theorem 1.1 may fail if we drop the assumption "Φ has a matrix pole", or equivalently, "Θ has a nonconstant diagonal-constant inner divisor" in the right coprime factorization $\Phi_- = \Theta B^*$. To see this we again consider the function (7):

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z} + 2z \\ \overline{z} + 2z & \overline{z} \end{bmatrix}.$$

We then have

$$\Phi_- = \begin{bmatrix} z & z \\ z & z \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\right)^*,$$

where

$$\Theta \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \text{ and } B \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \text{ are right coprime (by Corollary 3.2).}$$

As we saw in (8), T_Φ is quasinormal, and hence subnormal. But clearly, T_Φ is neither normal nor analytic. Here we note that Θ does not have any nonconstant diagonal-constant inner divisor of the form θI_n with a Blaschke factor θ.

4. Proof of Theorem 1.3

Since the self-commutator measures a form of deviation from normality, one might expect that subnormal (or hyponormal) operators with finite rank self-commutators are well behaved. Particular attention has been paid to the case of rank-one self-commutators. For example, B. Morrel [Mo] showed that every pure subnormal operator with rank-one self-commutator is unitarily equivalent to a linear function of the unilateral shift. Subnormal operators with finite rank self-commutators have been much investigated by many authors. Recently, D. Yakubovich [Ya] gave a nice characterization of subnormal operators with finite rank self-commutators under an assumption on their normal extensions. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to *have no point masses* if it has a normal extension N that has no nonzero eigenvectors.

Yakubovich's Theorem ([Ya, Theorem 2]). If $T \in \mathcal{B}(\mathcal{H})$ is a pure subnormal operator with finite rank self-commutator and without point masses then it is unitarily equivalent to a Toeplitz operator T_Φ with a matrix-valued analytic rational symbol Φ.

By using Yakubovich's Theorem, we first prove the following:

Theorem 4.1. Every pure quasinormal operator with finite rank self-commutator is unitarily equivalent to a Toeplitz operator with a matrix-valued analytic rational symbol.

Proof. Suppose $T \in \mathcal{B}(\mathcal{H})$ is a pure quasinormal operator with finite rank self-commutator. We want to show that T has no point masses. Since $\ker T$ reduces T, it follows that T is a direct sum of 0 and an operator with trivial kernel. But since T is pure we have that T is one-one. Let $T = U|T|$ be the polar decomposition of T. Then since T is one-one and quasinormal, and hence U is isometric and $U|T| = |T|U$, we can see that (cf. [Hal3, Problem 195])

$$N = \begin{bmatrix} U|T| & (I-UU^*)|T| \\ 0 & U^*|T| \end{bmatrix}$$

is a normal extension of T. We first show that N is one-one. To see this we let

$$\begin{bmatrix} U|T| & (I-UU^*)|T| \\ 0 & U^*|T| \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which gives

$$U|T|x + (I-UU^*)|T|y = 0 \quad \text{and} \quad U^*|T|y = 0. \tag{37}$$

Since U is isometric it follows from the first equation of (37) that

$$U^*U|T|x + U^*(I-UU^*)|T|y = 0,$$

which implies that $|T|x = 0$ and hence, again by (37), $|T|y = UU^*|T|y = 0$. Thus $x = y = 0$ since T is one-one. This shows that N is one-one. On the other hand, it was known (cf. [Br], [Con, Theorem II.3.2]) that every pure quasinormal operator on a Hilbert space $\mathcal{H}$ is unitarily equivalent to $U_+ \otimes A$, where U_+ is the unilateral shift on l^2 and A is a one-one positive operator on a Hilbert space $\mathcal{L}$, so that we can write

$$T' := W^*TW = \begin{bmatrix} 0 & 0 & 0 & \dots \\ A & 0 & 0 & \dots \\ 0 & A & 0 & \dots \\ \cdot & \cdot & A & \dots \end{bmatrix} \quad \text{on } \mathcal{L}^{(\infty)},$$

where $W : \mathcal{L}^{(\infty)} \to \mathcal{H}$ is a unitary operator. Note that $[T'^*, T'] = A^2 \oplus 0_\infty$, so that $\text{rank}\,[T^*, T] = \text{rank}\, A = \dim \mathcal{L}$. Suppose $\text{rank}\,[T^*, T] = n < \infty$. Then A is a one-one positive operator on $\mathbb{C}^n$, so that A is diagonalizable, i.e., $A = \text{diag}\,(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_i > 0$ for $i = 1, \cdots, n$. Since $T = U|T|$ is the polar decomposition of T, we have that $W^*UW = U_+ \otimes I_n$ and $W^*|T|W = I \otimes A$. We thus have

$$W^*U^*|T|W = U_+^* \otimes A \quad \text{and} \quad W^*(I-UU^*)|T|W = A \oplus 0_\infty.$$

Thus we can write, on $(\mathbb{C}^n)^{(\infty)} \oplus (\mathbb{C}^n)^{(\infty)} \cong l^2 \oplus l^2$,

$$N \equiv \begin{bmatrix} U|T| & (I-UU^*)|T| \\ 0 & U^*|T| \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} U_+ \otimes A & A \oplus 0_\infty \\ 0 & U_+^* \otimes A \end{bmatrix} \begin{bmatrix} W^* & 0 \\ 0 & W^* \end{bmatrix}.$$

We claim that $N - \lambda$ is one-one for each $\lambda \neq 0$. To see this we first assume $n = 2$ and write

$$A \equiv \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \quad (\alpha > 0,\ \beta > 0).$$

Let $x, y \in \mathcal{H}$ be such that $\begin{bmatrix} x \\ y \end{bmatrix} \in \ker(N - \lambda)$. Write

$$W^* x := (x_0, x_1, x_2, \cdots)^T \in l^2 \quad \text{and} \quad W^* y := (y_0, y_1, y_2, \cdots)^T \in l^2.$$

Then we have

$$\begin{bmatrix} (U_+ \otimes A) - \lambda & A \oplus 0_\infty \\ 0 & (U_+^* \otimes A) - \lambda \end{bmatrix} \begin{bmatrix} W^* x \\ W^* y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{38}$$

By s straightforward calculation, (38) gives

$$\begin{cases} -\lambda x_0 + \alpha y_0 = 0 \\ -\lambda x_1 + \beta y_1 = 0 \\ \alpha x_0 - \lambda x_2 = 0 \\ \beta x_1 - \lambda x_3 = 0 \\ \alpha x_2 - \lambda x_4 = 0 \\ \beta x_3 - \lambda x_5 = 0 \\ \quad \vdots \end{cases} \quad \text{and} \quad \begin{cases} -\lambda y_0 + \alpha y_2 = 0 \\ -\lambda y_1 + \beta y_3 = 0 \\ -\lambda y_2 + \alpha y_4 = 0 \\ \quad \vdots \end{cases}. \tag{39}$$

If $x_0 \neq 0$, say $x_0 = 1$, then by (39),

$$\begin{cases} x_2 = \frac{\alpha}{\lambda},\ \ x_4 = \left(\frac{\alpha}{\lambda}\right)^2,\ \ x_6 = \left(\frac{\alpha}{\lambda}\right)^3, \cdots, x_{2n} = \left(\frac{\alpha}{\lambda}\right)^n, \cdots \\ y_0 = \frac{\lambda}{\alpha},\ \ y_2 = \left(\frac{\lambda}{\alpha}\right)^2,\ \ y_4 = \left(\frac{\lambda}{\alpha}\right)^3, \cdots, y_{2n-2} = \left(\frac{\lambda}{\alpha}\right)^n, \cdots, \end{cases}$$

which implies that either $x \notin l^2$ or $y \notin l^2$, a contradiction and therefore x_0 should be zero. Thus by (39),

$$x_{2n} = 0 \text{ (since } \lambda \neq 0) \quad \text{and} \quad y_{2n} = 0 \text{ (since } \alpha \neq 0) \quad \text{for each } n = 0, 1, \cdots. \tag{40}$$

If instead $x_1 \neq 0$, then the same argument gives that either $x \notin l^2$ or $y \notin l^2$, a contradiction and therefore x_1 should be zero. Thus again by (39),

$$x_{2n-1} = 0 \quad \text{and} \quad y_{2n-1} = 0 \quad \text{for each } n = 1, 2, \cdots. \tag{41}$$

Therefore by (40) and (41), we must have $W^* x = W^* y = 0$, and hence $x = y = 0$, which implies that $N - \lambda$ is one-one for each $\lambda \neq 0$. More generally, if $n > 2$, then the same argument shows that $N - \lambda$ is also one-one for each $\lambda \neq 0$. Consequently, N has no nonzero eigenvectors, i.e., T has no point masses. Thus by Yakubovich's Theorem, we can conclude that T is unitarily equivalent to a Toeplitz operator T_Φ with a matrix-valued analytic rational symbol Φ. □

We are ready for:

Proof of Theorem 1.3. Let $\Phi \in L^\infty_{M_n}$ be a rational function and suppose T_Φ is quasinormal (and hence hyponormal). Thus Φ is normal (cf. [GHR]), and hence we have

$$[T_\Phi^*, T_\Phi] = H_{\Phi^*}^* H_{\Phi^*} - H_\Phi^* H_\Phi. \tag{42}$$

On the other hand, since Φ is rational, it follows from the Kronecker's lemma that H_{Φ^*} and H_Φ are of finite rank. Thus by (42), T_Φ has finite rank self-commutator. Now the theorem follows at once from Theorem 4.1. □

Remark 4.2. For special classes of operators, quasinormality in the presence of a finite rank self-commutator may force normality. And it is even possible that the condition "finite rank self-commutator" without any additional assumptions may force zero self-commutator, i.e., normality. For example, consider the case of composition operators. Let φ be a conformal automorphism of the unit disk (i.e., φ is a Blaschke factor of the form $\varphi(z) := \xi\frac{z-\lambda}{1-\overline{\lambda}z}, \lambda \in \mathbb{D}, |\xi| = 1$), and let C_φ be the composition operator with symbol φ (defined by $C_\varphi f := f \circ \varphi$) acting on the Hardy space $H^2(\mathbb{D})$ or the Bergman space $A^2(\mathbb{D})$ of the open unit disk $\mathbb{D}$. Then C_φ is normal whenever the self-commutator $[C_\varphi^*, C_\varphi]$ is compact (cf. [Zo], [MP]). However if C_φ acts on the Dirichlet space $\mathcal{D}$, then this is not the case. Indeed, if C_φ is a composition operator with a conformal automorphism symbol φ, then it was shown in [Abd, Theorem 2.1] that

$$(C_\varphi^* C_\varphi f)(z) = f(\varphi(0))K_{\varphi(0)}(z) - f(0) + f(z) \tag{43}$$

and

$$(C_\varphi C_\varphi^* f)(z) = f(0)K_{\varphi(0)}(\varphi(z)) - f(\varphi^*(0)) + f(z), \tag{44}$$

where $z \in \mathbb{D}$, $f \in \mathcal{D}$, $\varphi^*(z) := \frac{\lambda+\overline{\alpha}z}{1+\overline{\alpha}\overline{\lambda}z}$, and $K_w(\cdot)$ is the reproducing kernel for the Dirichlet space $\mathcal{D}$ defined by

$$K_w(z) := 1 + \log\frac{1}{1-\overline{w}z} \quad (w \in \mathbb{D}).$$

Hence evidently, $[C_\varphi^*, C_\varphi]$ is of finite rank, but is not zero in general, i.e., C_φ is not normal. However, if C_φ is quasinormal then a straightforward calculation together with (43) and (44) shows that for any $f \in \mathcal{D}$,

$$\begin{aligned} 0 &= [C_\varphi^* C_\varphi,\, C_\varphi]f(z) \\ &= (f\circ\varphi\circ\varphi)(0)K_{\varphi(0)}(z) - (f\circ\varphi)(0) - (f\circ\varphi)(0)\big(K_{\varphi(0)}\circ\varphi\big)(z) + f(0). \end{aligned} \tag{45}$$

If we take $f \equiv K_0 = 1$ in (45) then we have $K_{\varphi(0)}(z) = (K_{\varphi(0)} \circ \varphi)(z)$, so that

$$\log\frac{1}{1-\overline{\varphi(0)}\,z} = \log\frac{1}{1-\overline{\varphi(0)}\,\varphi(z)},$$

which implies

$$\overline{\varphi(0)}\,\big(z-\varphi(z)\big) = 0.$$

If $\varphi(0) \neq 0$ then $z = \varphi(z)$, a contradiction. Therefore we should have $\varphi(0) = 0$, so that $\varphi(z) = \xi z$ ($|\xi| = 1$), which implies that C_φ is normal. $\square$

In Remark 3.11 we have noticed that Theorem 1.1 may fail if the assumption "Φ has a matrix pole" is dropped via the Toeplitz operator T_Φ with symbol Φ given by (7). However, as we have also observed in (9), this Toeplitz operator T_Φ is unitarily equivalent to a direct sum of a normal operator and an analytic Toeplitz operator. From this viewpoint, we might expect that this is not coincidental for subnormal rational Toeplitz operators. Thus we propose:

Conjecture 4.3. Every subnormal rational Toeplitz operator is unitarily equivalent to a direct sum of a normal operator and an analytic Toeplitz operator.

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