# $N$-TUPLES OF OPERATORS SATISFYING $\sigma_T(AB) = \sigma_T(BA)$

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ABSTRACT. Let $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ be $n$-tuples of operators on a Banach space satisfying criss-cross commutativity, i.e., $A_iB_kA_j = A_jB_kA_i$ and $B_iA_kB_j = B_jA_kB_i$ (all $i, j, k$), and let $\mathbf{AB} := (A_1B_1, ..., A_nB_n)$ and $\mathbf{BA} := (B_1A_1, ..., B_nA_n)$ be the associated commuting $n$-tuples. We show that $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$ whenever (i) $\mathbf{0} \in \sigma_\pi(A) \cap \sigma_\pi(A^*)$, or (ii) $\mathbf{a} \circ \mathbf{A} := a_1A_1 + ... + a_nA_n$ is invertible for some $\mathbf{a} \in \mathbb{C}^n$. We also give some applications of this result.

## 1. Introduction

Given two operators $A$ and $B$ acting on a Banach space, it is well known that

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\} \tag{1.1}$$

[2, Prop. 6, p.16], [1]. On Hilbert space, more can be said: if $A$ is normal or invertible, then $\sigma(AB) = \sigma(BA)$ for every $B$. We study the latter identity for $n$-tuples of operators on a Banach space; we specifically ask, when is it true that $\sigma(\mathbf{AB}) = \sigma(\mathbf{BA})$, for $\sigma$ a joint spectral system?

Let $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ be $n$-tuples of operators on a Banach space, and let $\mathbf{AB} := (A_1B_1, ..., A_nB_n)$ and $\mathbf{BA} := (B_1A_1, ..., B_nA_n)$. In [12], V. Wrobel proved that if $A_1 = ... = A_n = A$ and $AB_j = B_jA$ (all $j = 1, ..., n$), then $\sigma_T(\mathbf{AB}) \setminus \{\mathbf{0}\} = \sigma_T(\mathbf{BA}) \setminus \{\mathbf{0}\}$. To deal with more general tuples, we need the following

**Definition 1.1.** $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ are said to *criss-cross commute* if

$$A_iB_kA_j = A_jB_kA_i \text{ and } B_iA_kB_j = B_jA_kB_i \text{ (all } i, j, k).$$

It is clear that if $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ criss-cross commute, then $\mathbf{AB}$ and $\mathbf{BA}$ are commuting $n$-tuples. Several authors have obtained analogues of (1.1) for various joint spectra, under special conditions. In [6], S. Li proved that if $\mathbf{A}$ and $\mathbf{B}$ are criss-cross commuting then $\sigma_T(\mathbf{AB}) \setminus \{\mathbf{0}\} = \sigma_T(\mathbf{BA}) \setminus \{\mathbf{0}\}$, and in [7] he showed that $ind(\mathbf{AB} - \mathbf{z}) = ind(\mathbf{BA} - \mathbf{z})$ for $\mathbf{z} \neq \mathbf{0}$. In [5], R. Harte extended this result to different kinds of joint spectra for criss-cross commuting pairs of $n$-tuples.

For a commuting $n$-tuple $\mathbf{T} = (T_1, ..., T_n)$ acting on a Banach space $\mathcal{X}$, let $K(\mathbf{T} - \mathbf{z})$ denote the Koszul complex associated with $\mathbf{T} - \mathbf{z}$ (cf. Section 3 below, [4], [10], [11]). We define the Taylor spectrum $\sigma_T(\mathbf{T})$ and approximate point

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spectrum $\sigma_\pi(\mathbf{T})$ as follows:

$$\sigma_T(\mathbf{T}) := \{\mathbf{z} \in \mathbf{C}^n : K(\mathbf{T} - \mathbf{z}) \text{ is not exact}\}$$

and

$$\sigma_\pi(\mathbf{T}) := \{\mathbf{z} \in \mathbf{C}^n : \inf_{||x||=1} \sum_{i=1}^{n} ||(T_i - z_i)x|| = 0\}.$$

Given a bounded linear operator $T$ on a Banach space $\mathcal{X}$, we let $T^*$ denote the adjoint of $T$, acting on $\mathcal{X}^*$, the dual space of $\mathcal{X}$. Also, $\sigma(T)$ and $\sigma_\pi(T)$ denote the spectrum and the approximate point spectrum of $T$, respectively. In this paper we show that for a pair $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ of criss-cross commuting $n$-tuples on a Banach space, if (i) $\mathbf{0} \in \sigma_\pi(\mathbf{A}) \cap \sigma_\pi(\mathbf{A}^*)$ or (ii) there exists an invertible operator which is a linear combination of $\{A_1, \cdots, A_n\}$, then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$, where $\mathbf{A}^* \equiv (A_1^*, ..., A_n^*)$.

## 2. The Singular Case

First we prove

**Theorem 2.1.** *Let $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ be criss-cross commuting. If $\mathbf{0} \in \sigma_\pi(\mathbf{A}) \cap \sigma_\pi(\mathbf{A}^*)$, then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.*

For the proof of Theorem 2.1 we need the following result.

**Theorem 2.2.** [10, Theorem 3.6]. *Let $\mathbf{A} \equiv (A_1, ..., A_n)$ be a commuting $n$-tuple of operators. Then $\sigma_T(\mathbf{A}) = \sigma_T(\mathbf{A}^*)$.*

*Proof of Theorem 2.1.* By Li's Theorem [6], it suffices to show that $\mathbf{0} \in \sigma_T(\mathbf{AB}) \cap \sigma_T(\mathbf{BA})$. By assumption, there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{X}$ such that $A_j x_n \to 0$ as $n \to \infty$ $(j = 1, ..., n)$. Then $B_j A_j x_n \to 0$ as $n \to \infty$ (all $j = 1, ..., n$), so we have $\mathbf{0} \in \sigma_T(\mathbf{BA})$. Now, since $\mathbf{0} \in \sigma_a(\mathbf{A}^*)$, there exists a sequence $\{f_n\}$ of unit vectors in $\mathcal{X}^*$ such that $A_j^* f_n \to 0$ as $n \to \infty$. Then $B_j^* A_j^* f_n \to 0$ as $n \to \infty$ (all $j = 1, ..., n$), so we have $\mathbf{0} \in \sigma_T(\mathbf{B}^*\mathbf{A}^*)$. Since $\sigma_T(\mathbf{B}^*\mathbf{A}^*) = \sigma_T((\mathbf{AB})^*)$, we must have $\mathbf{0} \in \sigma_T(\mathbf{AB})$ by Theorem 2.2. Hence $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$, as desired. □

Next we study the condition:

$$(*) \qquad \mathbf{0} \in \sigma_T(\mathbf{A}) \Rightarrow \mathbf{0} \in \sigma_a(\mathbf{A}) \cap \sigma_a(\mathbf{A}^*).$$

**Definition 2.3.** An $n$-tuple $\mathbf{A} = (A_1, ..., A_n)$ is called *strongly commuting* if, for each $1 \leq j \leq n$, there exist operators $H_j$ and $K_j$, each with real spectrum, such that $A_j = H_j + iK_j$ and $\mathbf{S} = (H_1, K_1, ..., H_n, K_n)$ is a commuting $2n$-tuple (cf. [8]).

**Theorem 2.4.** *If $\mathbf{A} = (A_1, ..., A_n)$ is a strongly commuting $n$-tuple, then $\mathbf{A}$ has condition $(*)$.*

For the proof of Theorem 2.4 we need the following result.

**Theorem 2.5.** [3, Theorem 2.1]. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a strongly commuting $n$-tuple of operators. Then $\sigma_T(\mathbf{T}) = \sigma_\pi(\mathbf{T})$.*

*Proof of Theorem 2.4.* By Theorem 2.5 we have $\mathbf{0} \in \sigma_\pi(\mathbf{A})$. Since by Theorem 2.2, $\sigma_T(\mathbf{A}^*) = \sigma_T(\mathbf{A})$ and since $\mathbf{A}^* = (A_1^*, ..., A_n^*)$ is also strongly commuting, by Theorem 2.5 we have $\mathbf{0} \in \sigma_\pi(\mathbf{A}^*)$. □

For the rest of this section, we consider the single operator case.

**Corollary 2.6.** *If an operator $A$ satisfies condition $(*)$, then $\sigma(AB) = \sigma(BA)$ for every operator $B \in B(\mathcal{X})$.*

*Proof.* If $0 \notin \sigma(A)$, then it is clear $\sigma(AB) = \sigma(BA)$. If $0 \in \sigma(A)$, then by Theorem 2.1 we have $\sigma(AB) = \sigma(BA)$. □

We now let

$$\Pi := \{(x, f) \in \mathcal{X} \times \mathcal{X}^* \ : \ \|f\| = f(x) = \|x\| = 1\}.$$

**Definition 2.7.** For an operator $T \in B(\mathcal{X})$, *the numerical range* $V(T)$ of $T$ is defined by

$$V(T) := \{f(Tx) \ : \ (x, f) \in \Pi\}.$$

An operator $T$ is said to be hermitian if $V(T) \subseteq \mathbf{R}$; $T$ is said to be normal if there exist hermitian operators $H$ and $K$ such that $HK = KH$ and $T = H + iK$.

It is well known that $\sigma(T) \subseteq \overline{V(T)}$, where $\overline{V(T)}$ is the closure of $V(T)$. Hence normal operators satisfy condition $(*)$. We thus have

**Corollary 2.8.** *If $A$ is normal, then $\sigma(AB) = \sigma(BA)$ for every operator $B \in B(\mathcal{X})$.*

## 3. The Nonsingular Case

For a *commuting* $n$-tuple of operators $\mathbf{A} = (A_1, ..., A_n)$, we consider the following properties $(P_1)$ and $(P_2)$:

$(P_1)$ $\quad \exists \mathbf{a} = (a_1, ..., a_n) \in \mathbf{C}^n$ and $\mathbf{a} \circ \mathbf{A} := a_1A_1 + \cdots + a_nA_n$ is invertible

$(P_2)$ $$\mathbf{0} = (0, ..., 0) \notin \sigma_T(\mathbf{A}).$$

**Proposition 3.1.** *For a commuting $n$-tuple of operators $\mathbf{A} = (A_1, ..., A_n)$, $(P_1)$ implies $(P_2)$.*

*Proof.* If $\mathbf{a} \circ \mathbf{A}$ is invertible, then by the spectral mapping theorem we have

$$\sigma(\mathbf{a} \circ \mathbf{A}) = \mathbf{a} \circ \sigma_T(\mathbf{A}).$$

Hence

$$0 \notin \sigma(\mathbf{a} \circ \mathbf{A}) \iff 0 \notin \mathbf{a} \circ \sigma_T(\mathbf{A})$$

$$\iff \mathbf{a} \circ \mathbf{z} : = a_1z_1 + \cdots + a_nz_n \neq 0 \text{ (all } \mathbf{z} = (z_1, ..., z_n) \in \sigma_T(\mathbf{A})).$$

Therefore, we have $\mathbf{0} \notin \sigma_T(\mathbf{A})$, so $(P_2)$ holds. □

*Remark* 3.2. In general, $(P_2) \not\Rightarrow (P_1)$. To see this, we shall need the following result ([10, Theorem 4.1]) : There exists a 5-tuple $\mathbf{A} = (A_1, ..., A_5)$ such that $\mathbf{A}$ is nonsingular but the equation $A_1B_1 + \cdots + A_5B_5 = I$ cannot be solved for $B_1, ..., B_5 \in (\mathbf{A})'$, where $(\mathbf{A})' := \{T : A_iT = TA_i \text{ for all } A_i \ (i = 1, ..., 5)\}$. Assume now that $(P_2) \implies (P_1)$. Then, for $\mathbf{A} = (A_1, ..., A_5)$ as above, there exists $\mathbf{a} = (a_1, ..., a_5)$ such that $\mathbf{a} \circ \mathbf{A}$ is invertible. Since it is clear that $\sum_{i=1}^{5} a_i(\mathbf{a} \circ \mathbf{A})^{-1}A_i = I$ and $a_i(\mathbf{a} \circ \mathbf{A})^{-1} \in (\mathbf{A})'$ $(i = 1, ..., 5)$, we get a contradiction.

**Theorem 3.3.** *Let $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ be criss-cross commuting $n$-tuples. If there exists an invertible operator $T$ which is a linear combination of $\{A_1, ..., A_n\}$, then $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.*

*Proof.* We need to recall the construction of the Koszul complex. Let $E$ be the exterior algebra on $n$ generators, that is, $E$ is the complex algebra with identity $e_0$ generated by indeterminates $e_1, ..., e_n$ such that $e_i \wedge e_j = -e_j \wedge e_i$ for all $i, j$, where $\wedge$ denotes multiplication. The elements $e_{j_1} \wedge \cdots \wedge e_{j_k},\ 1 \le j_1 < \cdots < j_k \le n$ form a basis for the subspace of $k$-forms, $E_k$ $(k = 1, ..., n)$, while $E_0 = \mathbf{C}e$. Thus, $E$ is a graded algebra, with $E = \bigoplus_{k=0}^{n} E_k$. For $\mathcal{X}$ a Banach space, let $E_k(\mathcal{X}) := E_k \otimes_{\mathbf{C}} \mathcal{X}$. For $\mathbf{z} = (z_1, ..., z_n) \in \mathbf{C}^n$, we define $D_k$ and $D^k : E_k(\mathcal{X}) \to E_{k-1}(\mathcal{X})$ by

$$D_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1} (A_{j_i} B_{j_i} - z_{j_i}) x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k}$$

and

$$D^k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^{k} (-1)^{i+1} (B_{j_i} A_{j_i} - z_{j_i}) x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

respectively (here $\check{e}_{j_i}$ means deletion). We thus have two chain complexes (Koszul complexes)

$$K(\mathbf{AB} - \mathbf{z}) : 0 \to E_n(\mathcal{X}) \xrightarrow{D_n} E_{n-1}(\mathcal{X}) \xrightarrow{D_{n-1}} \cdots \xrightarrow{D_2} E_1(\mathcal{X}) \xrightarrow{D_1} E_0(\mathcal{X}) \to 0$$

and

$$K(\mathbf{BA} - \mathbf{z}) : 0 \to E_n(\mathcal{X}) \xrightarrow{D^n} E_{n-1}(\mathcal{X}) \xrightarrow{D^{n-1}} \cdots \xrightarrow{D^2} E_1(\mathcal{X}) \xrightarrow{D^1} E_0(\mathcal{X}) \to 0.$$

By hypothesis, there exist complex numbers $a_i$ $(i = 1, ..., n)$ such that $T := a_1 A_1 + \cdots + a_n A_n$ is invertible. For $i = 1, ..., n,\ x \in \mathcal{X}$, and a complex number $z$,

$$(A_i B_i - z)x = (A_i B_i - z) T T^{-1} x$$

$$= (A_i B_i - z)(\alpha_1 A_1 + \cdots + \alpha_n A_n) T^{-1} x = T(B_i A_i - z) T^{-1} x,$$

because $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ criss-cross commute. It follows that

$$D_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) = T D^k((T^{-1}x) \otimes e_{j_1} \wedge \cdots \wedge e_{j_k})\ (k = 0, ..., n).$$

This identity readily implies that $K(\mathbf{AB} - \mathbf{z})$ is exact if and only if so is $K(\mathbf{BA} - \mathbf{z})$. Therefore, $\sigma(\mathbf{AB}) = \sigma(\mathbf{BA})$. □

We proved in Section 2 that a normal operator on a Banach space satisfies condition $(*)$. Applying this result and Theorems 2.1 and 3.3, we obtain the following

**Corollary 3.4.** *Let* $\mathbf{A} = (A, ..., A)$ *and* $\mathbf{B} = (B_1, ..., B_n)$ *be criss-cross commuting. If* $A$ *is normal, then* $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.

We conclude this section with an application of Theorem 3.3 to commuting $n$-tuples of operators in a somewhat restricted form.

**Corollary 3.5.** *Let* $\mathbf{A} = (A_1, ..., A_n)$ *be a commuting* $n$*-tuple operators which is non-singular, that is,* $\mathbf{0} = (0, ..., 0) \notin \sigma(\mathbf{A})$*. Suppose that for* $i = 1, ..., n$*, if* $0 \in \sigma(A_i)$ *then* $0$ *is an isolated point of* $\sigma(A_i)$*. Let* $\mathbf{B} = (B_1, ..., B_n)$ *be an* $n$*-tuple of operators such that* $\mathbf{A}$ *and* $\mathbf{B}$ *criss-cross commute. Then* $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$.

For the proof of Corollary 3.5, we need the following lemma. For $i = 1, ..., n$, let $P_i : \mathbb{C}^n \to \mathbb{C}$ denote the orthogonal projection onto the $i$-th coordinate.

**Lemma 3.6.** *Let $M \subseteq \mathbb{C}^n$ be a compact subset and assume that $\mathbf{0} \notin M$. Assume that $0 \in P_i(M)$ and that $0$ is an isolated point of $P_i(M)$ (all $i = 1, ..., n$). Then there exist numbers $a_1, ..., a_n$ such that $a_1x_1 + \cdots + a_nx_n \neq 0$ for every $(x_1, ..., x_n) \in M$.*

*Proof.* Since 0 is an isolated point of $P_i(M)$, there exist positive numbers $b_i, c_i$ such that if $0 \neq z \in P_i(M)$ then $b_i \leq |z| \leq c_i$ $(i = 1, ..., n)$. Let $a_1 := 1$ and select $a_2 > 0$ such that $a_2b_2 > c_1$. Next select $a_3 > 0$ such that $a_3b_3 > c_1 + a_2c_2$. Inductively, select $a_i > 0$ such that $a_ib_i > c_1 + a_2c_2 + \cdots + a_{i-1}c_{i-1}$ $(i = 3, ..., n)$. Then for all $(x_1, ..., x_n) \in M$ we must have

$$x_1 + a_2x_2 + \cdots + a_nx_n \neq 0.$$

For, assume that $x_1 + a_2x_2 + \cdots + a_nx_n = 0$. If $x_n \neq 0$, then

$$\begin{aligned} a_nb_n &\leq a_n|x_n| = |x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1}| \\ &\leq c_1 + a_2c_2 + ... + a_{n-1}c_{n-1} < a_nb_n, \end{aligned}$$

a contradiction. Thus, we must have $x_n = 0$. Therefore, $x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} = 0$, and a repeated application of the above argument shows that $x_1 = x_2 = \cdots = x_n = 0$, contradicting the hypothesis on $M$. □

*Proof of Corollary 3.5.* If there exists $i$ such that $0 \notin \sigma(A_i)$, then let $T := 0 \cdot A_1 + \cdots + 0 \cdot A_{i-1} + A_i + 0 \cdot A_{i+1} + \cdots + 0 \cdot A_n = A_i$. Since $T$ is invertible, by Theorem 3.3 we have $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$. Thus, we may assume $0 \in \sigma(A_i)$ (all $i = 1, ..., n$). By the hypothesis and Lemma 3.6 (applied to $M := \sigma_T(\mathbf{A})$), there exist numbers $a_1, ..., a_n$ such that $a_1x_1 + \cdots + a_nx_n \neq 0$ for all $(x_1, ..., x_n) \in \sigma_T(\mathbf{A})$. Let $f : \mathbb{C}^n \to \mathbb{C}$ be given by $f(z_1, ..., z_n) := a_1z_1 + \cdots + a_nz_n$, and let $T := f(A_1, ..., A_n)$. By the Spectral Mapping Theorem for the Taylor spectrum we have $\sigma(T) = \{a_1x_1 + \cdots + a_nx_n : (x_1, ..., x_n) \in \sigma_T(\mathbf{A})\}$, and thus $0 \notin \sigma(T)$. Therefore, $T$ is invertible, so by Theorem 3.3 we have $\sigma_T(\mathbf{AB}) = \sigma_T(\mathbf{BA})$. □

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