Truncated Moment Problems, Positive Linear Functionals, and Finite Algebraic Varieties Arising From Cubic Column Relations

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(joint work with L.A. Fialkow;

with L.A. Fialkow and H.M. Möller; and with S. Yoo)

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Outline of the Talk

- Brief Review of Full Moment Problem
- Truncated Moment Problems (Basic Positivity, Functional Calculus, Algebraic Variety)
- Moment Matrix Extension Approach
- Positive Linear Functional Approach
- TMP Version of the Riesz-Haviland Theorem
- Structure of Positive Polynomials
- Cubic Column Relations

General Idea to Study TMP

- TMP is more general than FMP:
  fewer moments $\Longrightarrow$ less data
- Stochel: link between TMP and FMP
- Existing approaches are directed at enlarging the data by acquiring new moments, and eventually making the problem into one of flat data type (i.e., with intrinsic recursiveness).
- This naturally leads to a full MP.
- If such a flat extension of the initial data cannot be accomplished, then TMP has no representing measure.
- **H**elpful tool: Smul'jan's Theorem on positivity of 2×2 matrices

The Classical (Full) Moment Problem

Let $\beta \equiv \beta^{(\infty)} = \{\beta_i\}_{i \in \mathbb{Z}_+^d}$ denote a d-dimensional real multisequence, and let K (closed) $\subseteq \mathbb{R}^d$. The (full) K-moment problem asks for necessary and sufficient conditions on β to guarantee the existence of a positive Borel measure μ supported in K such that

$$\beta_i = \int x^i d\mu \qquad (i \in \mathbb{Z}_+^d);$$

μ is called a **rep. meas.** for β.

Associated with β is a moment matrix $M \equiv M(\infty)$, defined by

$$M_{ij} := \beta_{i+j} \qquad (i, j \in \mathbb{Z}_+^d).$$

Basic Positivity Condition

$\mathcal{P}_n$: polynomials p over $\mathbb{R}$ with $\deg p \leq n$

- Given $p \in \mathcal{P}_n, \; p(x) \equiv \sum_{0 \leq i+j \leq n} a_i x^i$,

$$0 \leq \int p(x)^2 d\mu(x)$$

$$= \sum_{ij} a_i a_j \int x^{i+j} d\mu(x) = \sum_{ij} a_i a_j \beta_{i+j}.$$

- Now recall that we're working in d real variables. To understand this **"matricial" positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, X_1, \ldots, X_d, X_1^2, X_2 X_1, \ldots, X_d^2, \ldots$$

Also recall that

$$M(n)_{i,j} := \beta_{i+j}.$$

Then

$$\textbf{(“matricial” positivity)} \quad \sum_{ij} a_i a_j \beta_{i+j} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\beta) \geq 0.$$

For example, for moment problems in $\mathbb{R}^2$,

$$M(1) = \begin{pmatrix} \beta_{00} & \beta_{01} & \beta_{10} \\ \beta_{01} & \beta_{02} & \beta_{11} \\ \beta_{10} & \beta_{11} & \beta_{20} \end{pmatrix},$$

$$M(2) = \begin{pmatrix} \beta_{00} & \beta_{01} & \beta_{10} & \beta_{02} & \beta_{11} & \beta_{20} \\ \beta_{01} & \beta_{02} & \beta_{11} & \beta_{03} & \beta_{12} & \beta_{21} \\ \beta_{10} & \beta_{11} & \beta_{20} & \beta_{12} & \beta_{21} & \beta_{30} \\ \beta_{02} & \beta_{03} & \beta_{12} & \beta_{04} & \beta_{13} & \beta_{22} \\ \beta_{11} & \beta_{12} & \beta_{21} & \beta_{13} & \beta_{22} & \beta_{31} \\ \beta_{20} & \beta_{21} & \beta_{30} & \beta_{22} & \beta_{31} & \beta_{40} \end{pmatrix}.$$

- In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Similarly, one can build $M(\infty) \equiv M(\infty)(\beta) \equiv M(\beta)$.

- The link between TMP and FMP is provided by a result of Stochel (2001):

Theorem (Stochel's Theorem)

$\beta^{(\infty)}$ has a rep. meas. supported in a closed set $K \subseteq \mathbb{R}^d$ if and only if, for each n, $\beta^{(2n)}$ has a rep. meas. supported in K.

Moment Problems and Nonnegative Polynomials (FULL MP Case)

- $\mathcal{M} := \{\beta \equiv \beta^{(\infty)} : \beta$ admits a rep. meas. $\mu\}$
- $\mathcal{B}_+ := \{\beta \equiv \beta^{(\infty)} : M(\infty)(\beta) \geq 0\}$

  Clearly, $\mathcal{M} \subseteq \mathcal{B}_+$

- (Berg, Christensen and Ressel) $\beta \in \mathcal{B}_+$, β bounded $\Rightarrow \beta \in \mathcal{M}$
- (Berg and Maserick) $\beta \in \mathcal{B}_+$, β exponentially bounded $\Rightarrow \beta \in \mathcal{M}$
- (RC and L. Fialkow) $\beta \in \mathcal{B}_+$, $M(\beta)$ finite rank $\Rightarrow \beta \in \mathcal{M}$

- $\mathcal{P}_+$: nonnegative poly's
  Σ^2 : sums of squares of poly's
  Clearly, $\Sigma^2 \subseteq \mathcal{P}_+$

Duality

For C a cone in $\mathbb{R}^{\mathbb{Z}_+^d}$, we let

$$C^* := \{\xi \in \mathbb{R}^{\mathbb{Z}_+^d} : \text{supp}(\xi) \text{ is finite and } \langle p, \xi \rangle \geq 0 \text{ for all } p \in C\}.$$

- (Riesz-Haviland) $\mathcal{P}_+^* = \mathcal{M}$
  For, consider the Riesz functional $\Lambda_\beta(p) := p(\beta) \equiv \langle p, \beta \rangle$, which induces a map $\mathcal{M} \to \mathcal{P}_+^*$ ($\beta \mapsto \Lambda_\beta$); Haviland's Theorem says that this maps is onto, that is,

$\exists\ \mu$ rep. meas. for $\beta \Leftrightarrow \Lambda_\beta \geq 0$ on $\mathcal{P}_+$.

- $\mathcal{P}_+ = \mathcal{M}^*$ (straightforward once we have a r.m.)
- $\mathcal{B}_+ = (\Sigma^2)^*$ (straightforward)
- (Berg, Christensen and Jensen) $(\mathcal{B}_+)^* = \Sigma^2$
- $(n = 1)$ $\mathcal{P}_+ = \Sigma^2 \Rightarrow \mathcal{P}_+^* = (\Sigma^2)^* \Rightarrow \mathcal{M} = \mathcal{B}_+$ (Hamburger)
- Generally, SOS implies the existence of a representing measure.

Localizing Matrices

Consider the **full, complex** MP

$$\int \bar{z}^i z^j \, d\mu = \gamma_{ij} \quad (i, j \geq 0),$$

where supp $\mu \subseteq K$, for K a closed subset of $\mathbb{C}$.

- The **Riesz functional** is given by

$$\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij} \quad (i, j \geq 0).$$

- Riesz-Haviland:

  There exists μ with supp $\mu \subseteq K \Leftrightarrow \Lambda_\gamma(p) \geq 0$ for all p such that $p|_K \geq 0$.

If q is a polynomial in z and $\bar{z}$, and

$$K \equiv K_q := \{z \in \mathbb{C} : q(z, \bar{z}) \geq 0\},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for μ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p} \, d\mu \geq 0 \quad (\text{all } p).$$

- K. Schmüdgen (1991): If K_q is compact, $\Lambda_\gamma(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all p, then there exists μ with supp $\mu \subseteq K_q$.

The Truncated (Complex) Moment Problem

- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \dots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \le i + j \le 2n);$$

  μ is called a **rep. meas.** for γ.
- In earlier joint work with L. Fialkow,
- We have introduced an approach based on matrix positivity and extension, combined with a new “functional calculus” for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0, \infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a, b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We obtain applications to quadrature problems in numerical analysis.
- We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on $\mathbb{R}^d$.

Our results have been applied by

- S. McCullough to obtain a dilation-type structure theorem in Fejér-Riesz factorization theory;
- J. Lasserre to obtain concrete necessary and sufficient conditions on the coefficients of a polynomial so that all of its zeros lie in a prescribed semi-algebraic subset of the plane; and
- J. Lasserre, M. Laurent and others to convert polynomial optimization into an instance of semidefinite programming.

- More recently, we have begun to use our methods to solve FULL moment problems, by first solving truncated MP's, and then applying J. Stochel's limiting argument.
- Our matrix extension approach works equally well to **localize the support** of a rep. meas.
- In the specific case of $K := \text{supp } \mu$, a semi-algebraic set determined by a finite collection of complex polynomials $\mathcal{P} = \{p_i(z, \bar{z})\}_{i=1}^m$, i.e.,

$$K = K_P := \{z \in \mathbb{C} : p_i(z, \bar{z}) \geq 0, 1 \leq i \leq m\},$$

we obtain an existence criterion expressed in terms of positivity and extension properties of the moment matrix $M(n)(\gamma)$ associated to γ and of the localizing matrix M_{p_i} corresponding to each p_i.

Recall that

$$M(n)_{i,j} := \beta_{i+j}.$$

Then

$$\text{(“matricial” positivity)} \quad \sum_{ij} a_i a_j \beta_{i+j} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\beta) \geq 0.$$

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\beta_{00}, \beta_{01}, \beta_{10}, ..., \beta_{06}, ..., \beta_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. A similar phenomenon, with non-invertible moment matrix, also occurs when $n = 2$.

Functional Calculus

For $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$ define

$$p(Z, \bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j,$$

where $\bar{Z}^i Z^j$ denote the columns of $M(n)$, indexed as $1, Z, \bar{Z}, Z^2, \bar{Z}Z, \ldots$.

If there exists a rep. meas. μ, then

$$p(Z, \bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

Also in the presence of a rep.meas., we must have following analogue of recursiveness for the TCMP

(RG) **If** $p, q, pq \in \mathcal{P}_n$, **and** $p(Z, \bar{Z}) = 0$,

then $(pq)(Z, \bar{Z}) = 0$.

Algebraic Variety

Recall that $p(Z, \bar{Z}) = 0$ implies $\text{supp } \mu \subseteq \mathcal{Z}(p)$. We define the algebraic variety of γ as

$$\mathcal{V}(\gamma) := \bigcap_{\substack{p \in \mathcal{P}_n \\ p(Z,\bar{Z})=0}} \mathcal{Z}(p),$$

and observe that

$$r := \text{rank } M(n) \leq \text{card supp } \mu \leq v := \text{card } \mathcal{V}(\gamma),$$

from which it follows that

card $\mathcal{V}(\gamma) <$ rank $M(n) \Rightarrow$ there is **no** rep. meas. μ.

Singular TMP; Real Case

- Given a finite family of moments, build moment matrix
- Identify all column relations
- Build algebraic variety $\mathcal{V}$
- Always true:

$$r := \text{rank}\, \mathcal{M}(n) \leq \text{card supp } \mu \leq v := \text{card}\, \mathcal{V}(\gamma),$$

so if the variety is finite there's a natural candidate for supp μ, i.e., supp $\mu = \mathcal{V}(\gamma)$

Singular TMP; Real Case, cont.

- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
- Strategy: Build positive extension, check for $r = v$, repeat, until we reach $r = v$ (extremal problem) or $r > v$ (no representing measure)
  rank $M(n) \leq$ rank $M(n+1) \leq$ card $\mathcal{V}(M(n+1)) \leq$ card $\mathcal{V}(M(n))$
- General case.

First Existence Criterion

Theorem

(RC-L. Fialkow, 1998) Let γ be a truncated moment sequence. TFAE:

(i) γ has a rep. meas.;

(ii) γ has a rep. meas. with moments of all orders;

(iii) γ has a compactly supported rep. meas.;

(iv) γ has a finitely atomic rep. meas. (with at most $(n+2)(2n+3)$ atoms);

(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n+k+1)$ (here $k \leq 2n^2+6n+6$)).

Case of Flat Data

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

γ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

Theorem

(RC-L. Fialkow, 1996) If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

Theorem

(RC-L. Fialkow, 1996) The truncated moment sequence γ has a rank *$M(n)$-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

To find μ concretely, let r :=rank $M(n)$ and look for the relation

$$Z^r = c_0 1 + c_1 Z + ... + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + ... + c_{r-1} z^{r-1})$$

and solve the Vandermonde equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

Localization of Support: Main Theorem

Theorem

(RC-LF, 2000) Let $M(n) \geq 0$ *and suppose* $\deg(q) = 2k$ *or* $2k-1$ *for some* $k \leq n$. *Then* $\exists\ \mu$ *with* rank $M(n)$ *atoms and* supp $\mu \subseteq K_q$ *if and only if* $\exists$ *a flat extension* $M(n+1)$ *for which* $M_q(n+k) \geq 0$. *In this case,* $\exists\ \mu$ *with exactly* rank $M(n)$ – rank $M_q(n+k)$ *atoms in* $\mathcal{Z}(q)$.

Remark

M. Laurent (2005) has found an alternative proof, using ideas from real algebraic geometry.

A Version of Riesz-Haviland for TMP

Recall the Riesz-Haviland Theorem:

$\exists\ \mu$ rep. meas. for $\beta \Leftrightarrow\ \Lambda_\beta \geq 0$ on $\mathcal{P}_+$.

- For TMP, the natural analogue won't work.
- We say that the Riesz functional L is K*-positive* if

p$\in \mathcal{P}$ and $p|K\ \geq 0 \Rightarrow$ L(p)$\geq$ 0.

- Consider the case

  $d = 1, \ K = \mathbb{R}$, and

$$M(2) := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0.$$

Then Λ_β is $\mathbb{R}$-positive, but no rep.meas. exists. For, in this case,

$$\mathrm{L}(\mathrm{a}_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) := a_0 + a_1 + a_2 + a_3 + 2a_4$$

To see that L is $\mathbb{R}$-positive, recall that if $p \in \mathcal{P}_4$ satisfies $p|_{\mathbb{R}} \geq 0$, then there exist $f, g \in \mathcal{P}_2$ such that $p = f^2 + g^2$.

Now

$$L(p) = L(f^2 + g^2) = \langle M(2)\hat{f}, \hat{f}\rangle + \langle M(2)\hat{g}, \hat{g}\rangle \geq 0;$$

thus, L is $\mathbb{R}$-positive.

Assume that μ is a representing measure for β. Since

$$\int (x-1)^2 \, d\mu = L(x^2 - 2x + 1) = \beta_2 - 2\beta_1 + \beta_0 = 0,$$

it follows that $(x-1)|_{\text{supp } \mu} \equiv 0$. We thus have $(x-1)x^3|_{\text{supp } \mu} \equiv 0$, so

$$0 = \int (x-1)x^3 \, d\mu = L(x^4 - x^3) = \beta_4 - \beta_3 = 1,$$

a contradiction. Thus L is K-positive, but β has no representing measure.

In TMP, K-positivity is a necessary (but not sufficient) condition for a K-representing measure μ.

Theorem (TMP Version of Riesz-Haviland)

(RC-LF, 2007) $\beta \equiv \beta^{(2n)}$ *admits a* K*-representing measure if and only if* L_β *admits a* K*-positive linear extension* $L : \mathcal{P}_{2n+2} \longmapsto \mathbb{R}$.

This Theorem implies the classical Riesz-Haviland, via Stochel's Theorem.

TMP Version of Riesz-Haviland

$$
\begin{array}{ccc}
\mathcal{P} & & \\
\vdots & & \\
\uparrow & & \\
\mathcal{P}_{2n+2} & & \\
 & \color{red}{K\text{-pos.}} & \\
\uparrow & \searrow & \\
\mathcal{P}_{2n} & \xrightarrow{\color{red}{K\text{-pos.}}} & \mathbb{R}
\end{array}
$$

- **Main tool.** Let $\beta \equiv \beta^{(2n)}$ and let $K \subseteq \mathbb{R}^d$ be closed. Assume L_β is K-positive. Then $\beta \equiv \beta^{(2n-1)}$ has a K-representing measure.
- Related notion: $\mathcal{V}$-positivity, where $\mathcal{V} \equiv V(M(n))$ is the algebraic variety associated to $\beta^{(2n)}$. In general, $\mathcal{V}$-positivity is stronger than K-positivity (since we expect $\mathcal{V} \subseteq K$).
- We have been able to prove that $\mathcal{V}$-positivity for $L_{\beta^{(2n)}}$ implies the existence of representing measures for $\beta^{(2n)}$ when $d = 1$ or when $\mathcal{V}$ is compact.

Problem

Let $\beta \equiv \beta^{(2n)}$ be given, and assume that L_β is $\mathcal{V}$-positive. Is TMP soluble?

In general it is quite difficult to directly verify that an extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ is K-positive. One approach to establishing K-positivity or the existence of representing measures is through **extensions of moment matrices**.

Positivity of Block Matrices

Theorem

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases}$$

Moreover, $\text{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \text{rank}\, A \Leftrightarrow C = W^*AW.$

Corollary

$$A \geq 0 \textit{ and } \operatorname{rank} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \operatorname{rank} A \Rightarrow \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$$

Corollary

$$\begin{aligned} \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} &= \begin{pmatrix} A & AW \\ W^*A & W^*AW \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C - W^*AW \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix}^* \begin{pmatrix} \sqrt{A} & \sqrt{A}W \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix}^* \begin{pmatrix} 0 & \sqrt{C - W^*AW} \end{pmatrix} \end{aligned}$$

(sum-of-squares representation).

Let's look at TMP in $\mathbb{R}^2$:

- Given a finite family of moments, build moment matrix $M(n)$
- Identify all column relations
- Get poly's such that $p(X, Y) = 0$
- Build algebraic variety $\mathcal{V} := \bigcap_{p(X,Y)=0} \mathcal{Z}_p$
- If there exists a rep. meas. μ, then supp $\mu \subseteq \mathcal{V}$

Extremal MP; $r = v$

Recall that $r :=$ rank $M(n)$ and $v :=$ card $\mathcal{V}$.

$$\beta_i = \int_{\mathbb{R}^d} x^i \, d\mu, \ |i| \leq 2n;$$

$\mathcal{P} \equiv \mathbb{R}^d[x] = \mathbb{R}[x_1, ..., x_d]$: space of real valued d-variable polynomials

$\mathcal{P}_k \equiv \mathbb{R}^d_k[x]$: the subspace of $\mathcal{P}$ consisting of polynomials p with $\deg p \leq k \ \ (k \geq 1)$.

$\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}$ (Riesz functional): if $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$, then $\Lambda(p) := \sum_{|i| \leq 2n} a_i \beta_i$

- In the presence of a representing measure μ, we have $\Lambda(p) = \int p \, d\mu$.

$\hat{p}$: coefficient vector (a_i) of p.

$M(n) \equiv M(n)(\beta)$: moment matrix, with rows and columns X^i indexed by the monomials of $\mathcal{P}_n$ in degree-lexicographic order

$d = n = 2$: the columns of $M(2)$ are denoted as $1, X_1, X_2, X_1^2, X_2X_1, X_2^2$

- $M(n)$ is a real symmetric matrix characterized by

$$\langle M(n)\hat{p}, \hat{q}\rangle = \Lambda(pq) \ (p, q \in \mathcal{P}_n).$$

- If μ is a representing measure for β, then $\langle M(n)\hat{p}, \hat{p}\rangle = \Lambda(p^2) = \int p^2 d\mu \geq 0$; it follows that $M(n) \geq 0$).

- The *algebraic variety* of β is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker M(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$. (Observe that $\mathcal{V} = \mathbb{R}^d$ when $M(n)$ is invertible.)

- If β admits a representing measure μ, and $p \in \mathcal{P}_n$, then

$$\hat{p} \in \ker M(n) \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p.$$

Thus supp $\mu \subseteq \mathcal{V}$, so $r :=$ rank $M(n)$ and $v :=$ card $\mathcal{V}$ satisfy

$$r \leq \text{card supp } \mu \leq v.$$

- If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$.

Basic necessary conditions for the existence of a representing measure

(Positivity) $M(n) \geq 0$

(Consistency) $p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0$

(Variety Condition) $r \leq v$, i.e., $\text{rank }\ M(n) \leq \text{card }\ \mathcal{V}$.

Consistency implies

(Recursiveness) $p, q, pq \in \mathcal{P}_n,\ \hat{p} \in \ker M(n) \Longrightarrow \widehat{pq} \in \ker M(n)$.

Previous results:

- For $d = 1$ (the T *Hamburger* MP for $\mathbb{R}$), positivity and recursiveness are sufficient
- For $d = 2$, there exists $M(3) > 0$ for which β has no representing measure
- Thus, in general, *Positivity*, *Consistency* and the *Variety Condition* are **not** sufficient.

Question C

Suppose $M(n)(\beta)$ is singular. If $M(n)$ is positive, β is consistent, and $r \leq v$, does β admit a representing measure?

More generally, the following question remained unsolved until very recently.

Question RG

Suppose $M(n)(\beta)$ is singular. If $M(n)$ is positive, recursively generated, and $r \leq v$, does β admit a representing measure?

- RC-LF: If $d = 2$ and $M(n)\hat{p} = 0$ for some p with $\deg p \leq 2$, then Question RG has an affirmative answer. (Quartic MP)
- RC-LF-MM: If $d = 2$, $Y = X^3$ and $r = v \leq 7$, then Question RG has an affirmative answer.
- RC-LF-MM: If $d = 2$, $Y = X^3$ and $r = v = 8$, **then Question RG has a negative answer.**

The next result gives an affirmative answer to Question C in the *extremal* case, i.e., $r = v$.

Theorem EXT

(RC-LF and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ ***extremal****, i.e., $r = v$, the following are equivalent:*

(i) β has a representing measure;

(ii) β has a unique representing measure, which is rank *$M(n)$-atomic (minimal);*

(iii) $M(n) \geq 0$ and β is consistent.

- The extremal case is inherent in TMP: C. Bayer and J. Teichmann (2006) (extending a classical theorem of V. Tchakaloff and its successive generalizations by I.P. Mysovskikh, M. Putinar and RC-LF) recently proved that if $\beta^{(2n)}$ has a representing measure, then it has a **finitely atomic** representing measure;

- RC-LF showed that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $M(n)$ admits an extension to a positive moment matrix $M(n+k)$ (for some $k \geq 0$, which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $M(n+k+1)$;

- In many instances, $M(n+k+1)$ is an extremal moment matrix for which there is a computable rank $M(n+k)$-atomic representing measure μ. Clearly, μ is also a finitely atomic representing measure for $\beta^{(2n)}$, and every finitely atomic representing measure for $\beta^{(2n)}$ arises in this way.
- there exist extremal TCMP of arbitrarily large degree
- Moment theory can be used to estimate the number and location of the zeros of a prescribed polynomial; e.g., using TMP techniques we have shown that

$$p(z) \equiv z^{2n} + az^{2n-1} - az - 1 \ \ (0 < a < 1)$$

has $2n$ distinct zeros, all in the unit circle.

A Second Approach to Positivity

A second approach to positivity for an extension $\tilde{L} : \mathcal{P}_{2n+2} \longrightarrow \mathbb{R}$ concerns the **structure of positive polynomials**. Recall that the main difficulty associated with the Riesz-Haviland Theorem is that for a general closed set $K \subseteq \mathbb{R}^d$ there is no concrete representation theorem for polynomials that are nonnegative on K.

- In this sense, the Riesz-Haviland Theorem is an “abstract” solution to the moment problem, and, similarly, our analogue is an abstract solution to the truncated moment problem.
- When K is a compact semialgebraic set, Schmüdgen’s Theorem provides a concrete test for the K-positivity of $L_{\beta(\infty)}$, which we have generalized to the case of TMP, as we described above.
- We will now look at the second approach, based on representations of positive polynomials.

Let $\mathcal{Q} = \{q_0, q_1, \ldots, q_m\} \subseteq \mathcal{P}$ (with $q_0 \equiv 1$) and consider the semialgebraic set

$$K_{\mathcal{Q}} = \{x \in \mathbb{R}^d : q_i(x) \geq 0 \; (1 \leq i \leq m)\}.$$

Moreover, let $\mathcal{Q}^\pi$ denote the set of products of **distinct** polynomials in $\mathcal{Q}$, that is,

$$\mathcal{Q}^\pi := \{q_{i_1} \cdots q_{i_s} : q_{i_j} \in \mathcal{Q}, 0 \leq i_1 < \cdots < i_s \leq m, 1 \leq s \leq m+1\};$$

Observe that $\mathcal{Q} \subseteq \mathcal{Q}^\pi$ and that $K_{\mathcal{Q}} = K_{\mathcal{Q}^\pi}$.

Theorem (FMP Case)

(K. Schmüdgen) Suppose $K_{\mathcal{Q}}$ is compact. The sequence $\beta \equiv \beta^{(\infty)}$ has a representing measure supported in $K_{\mathcal{Q}}$ if and only if $M_r \geq 0$ for each polynomial $r \in \mathcal{Q}^{\pi}$.

What about TMP?

Let $K_{\mathcal{Q}}$ be as above and choose n so that $2n \geq \deg\ q_i$ for $i = 1, \cdots, m$. For $k \geq 0$, consider the following properties for $K_{\mathcal{Q}}$:

$$(\mathbf{S}_{n,k}) \quad \left\{ \begin{array}{c} \beta^{(2n)} \text{ has a } K_{\mathcal{Q}}\text{-representing measure if and only if} \\ M(n) \text{ admits a positive extension } M(n+k) \text{ such that} \\ M_{q_i}(n+k) \geq 0 \text{ for } i = 1, \cdots, m \end{array} \right.$$

and

$$(\mathbf{R}_{n,k}) \quad \left\{ \begin{array}{c} \beta^{(2n)} \text{ has a } K_{\mathcal{Q}}\text{-representing measure if and only if} \\ M(n) \text{ admits a positive, recursively generated extension} \\ M(n+k) \text{ such that } M_{q_i}(n+k) \geq 0 \text{ for } i = 1, \cdots, m. \end{array} \right.$$

Clearly, if $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_{n,k})$, then it also satisfies $(\mathbf{R}_{n,k})$.

Whereas Schmüdgen works with the cone $\Sigma_{\mathcal{Q}^\pi} \bigcap \mathcal{P}_{2n}$, we focus on the sub-cone $\Sigma_{\mathcal{Q},n}$, defined by:

$$\Sigma_{\mathcal{Q},n} := \{p \in \mathcal{P}_{2n} : p = \sum_j f_{0j}^2 + q_1 \sum_j f_{1j}^2 + ... + q_m \sum_j f_{mj}^2,\ q_i f_{ij}^2 \in \mathcal{P}_{2n}\}.$$

Here's our TMP analog of one direction of Schmüdgen's Thm.

Theorem

(i) Assume that $K_{\mathcal{Q}}$ satisfies ($\mathbf{S}_{n,k}$) for some n and k. Then every polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k}$.

(ii) Assume that $K_{\mathcal{Q}}$ satisfies ($\mathbf{R}_{n,k}$) for some n and k. Then each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k+1}$.

The previous theorem can be extended to nonnegative polynomials in those cases where the cone $\Sigma_{\mathcal{Q},n+k}$ is closed in $\mathcal{P}_{2(n+k)}$.

What about converses?

Theorem

(i) If $k \geq 1$ and each polynomial in $\mathcal{P}_{2n+2}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n+k}$, then $K_{\mathcal{Q}}$ satisfies ($\mathbf{S}_{n,k}$).
(ii) If $k = 0$, $K_{\mathcal{Q}}$ is compact, and each polynomial in $\mathcal{P}_{2n}$ that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q},n}$, then $K_{\mathcal{Q}}$ satisfies ($\mathbf{S}_{n,0}$).

Proposition ($\overline{\mathbb{D}}$ satisfies $(\mathbf{S}_{1,0})$)

Each polynomial $p \in \mathcal{P}_2$ *satisfying* $p|_{\overline{\mathbb{D}}} \geq 0$ *admits a representation* $p = \Sigma_{i=1}^{5} f_i^2 + \alpha(1 - x^2 - y^2)$, *where* deg $f_i \leq 1$ $(1 \leq i \leq 6)$ *and* $\alpha \geq 0$.

We know that $\overline{\mathbb{D}}$ fails to satisfy $(\mathbf{S}_{3,k})$ for all k, and it appears to be open whether the disk satisfies $(\mathbf{S}_{2,k})$ for some k.

Cubic Column Relations

Since we know how to solve the singular Quartic MP, WLOG we will assume $M(2) > 0$.

Recall

(RC-L. Fialkow) If $M(n)$ admits a column relation of the form $Z^k = p_{k-1}(Z, \bar{Z})$ $(1 \leq k \leq \left[\frac{n}{2}\right] + 1$ and deg $p_{k-1} \leq k-1)$, then $M(n)$ admits a flat extension $M(n+1)$, and therefore a representing measure.

Now, if $k = 3$, Theorem A can be used only if $n \geq 4$. Thus, one strategy is to somehow extend $M(3)$ to $M(4)$ and preserve the column relation $Z^3 = p_2(Z, \bar{Z})$. **However**, this requires checking that the C block in the extension satisfies the Toeplitz condition, something highly nontrivial.

Here's a different approach: We'd like to study the case of harmonic poly's: $q(z,\bar{z}) := f(z) - \overline{g(z)}$, with deg $q = 3$. Recall that

$$\text{rank } M(n) \leq \text{card } \mathcal{Z}(q)$$

so of special interest is the case when card $\mathcal{Z}(q) \geq 7$, since otherwise the TMP admits a flat extension, or has no representing measure. In the case when $g(z) \equiv z$, we have

Lemma

(Wilmshurst '98, Sarason-Crofoot, '99, Khavinson-Swiatek, '03)

$$\text{card } \mathcal{Z}(f(z) - \bar{z}) \leq 7.$$

- To get 7 points is not easy, as most complex cubic harmonic poly's tend to have 5 or fewer zeros. One way to maximize the number of zeros is to impose symmetry conditions on the zero set K. Also, the substitution $w = z + b/3$ (which produces an equivalent TMP) transforms a cubic of the form $z^3 + bz^2 + cz + d$ into $w^3 + \tilde{c}w + \tilde{d}$; thus, WLOG we always assume that there's no quadratic term in the analytic piece.
- Now, for a poly of the form $z^3 + \alpha z + \beta \bar{z}$, it is clear that $0 \in K$ and that $z \in K \Rightarrow -z \in K$. Another natural condition is to require that K be symmetric with respect to the line $y = x$, which in complex notation is $z = i\bar{z}$. When this is required, we obtain $\alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}$. Thus, the column relation becomes $Z^3 = itZ + u\bar{Z}$, with $t, u \in \mathbb{R}$.

Under these conditions, one needs to find only two points, one on the line $y = x$, the other outside that line.

We thus consider the harmonic polynomial

$q_7(z, \bar{z}) := z^3 - itz - u\bar{z}$.

Proposition

(RC-S. Yoo, '09) card $\mathcal{Z}(q_7) = 7$. *In fact, for* $0 < |u| < t < 2|u|$,

$$\mathcal{Z}(q_7) = \{0, p + iq, q + ip, -p - iq, -q - ip, r + ir, -r - ir\},$$

where $p, q, r > 0$, $p^2 + q^2 = u$ *and* $r^2 = \frac{t-u}{2}$.

To prove this result, we first identify the two real poly's $\text{Re } q_7 = x^3 - 3xy^2 + ty - ux$ and $\text{Im } q_7 = -y^3 + 3x^2y - tx + uy$ and calculate $\mathit{Resultant}(Req_7, Imq_7, y)$, which is the determinant of the Sylvester matrix, i.e.,

$$\det \begin{pmatrix} -3x & t & x^3 - ux & 0 & 0 \\ 0 & -3x & t & x^3 - ux & 0 \\ 0 & 0 & -3x & t & x^3 - ux \\ -1 & 0 & 3x^2 + u & -tx & 0 \\ 0 & -1 & 0 & 3x^2 + u & -tx \end{pmatrix}$$

$$= \; x\left(u - t + 2x^2\right)\left(u + t + 2x^2\right)\left(16x^4 - 16x^2u + t^2\right).$$

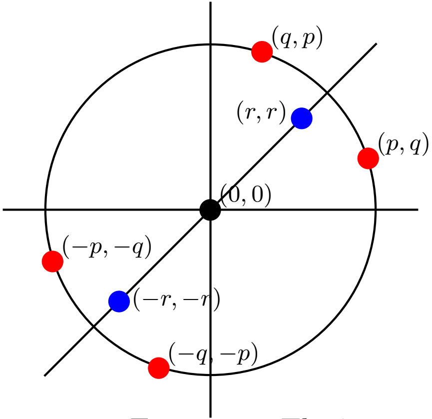

FIGURE 1. The 7-point set $\mathcal{Z}(q_7)$, where $r = \sqrt{\frac{t-u}{2}}$, $p = \frac{1}{2}(2u + \sqrt{4u^2 - t^2})$ and $p^2 + q^2 = u$

The fact that q_7 has the maximum number of zeros predicted by the Lemma is significant to us, in that each sextic TMP with invertible $M(2)$ and a column relation of the form $q_7(Z, \bar{Z}) = 0$ either does not admit a representing measure or is necessarily extremal.

As a consequence, the existence of a representing measure will be established once we prove that such a TMP is consistent. This means that for each poly p of degree at most 6 that vanishes on $\mathcal{Z}(q_7)$ we must verify that $\Lambda(p) = 0$.

Since rank $M(3) = 7$, there must be another column relation besides $q_7(Z, \bar{Z}) = 0$. Clearly the columns

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \bar{Z}Z^2$$

must be linearly independent (otherwise $M(3)$ would be a flat extension of $M(2)$), so the new column relation must involve $\bar{Z}Z^2$ and $\bar{Z}^2Z$. An analysis using the properties of the functional calculus shows that, **in the presence of a representing measure**, the new column relation must be

$$\bar{Z}^2Z + i\bar{Z}Z^2 - iuZ - u\bar{Z} = 0.$$

Notation

In what follows, $\mathbb{C}_6[z,\bar{z}]$ will denote the space of complex polynomials in z and $\bar{z}$ of degree at most 6, and let

$$\begin{aligned} q_{LC}(z,\bar{z}) &:= \bar{z}^2 z + i\bar{z}z^2 - iuz - u\bar{z} \\ &= i(z - i\bar{z})(\bar{z}z - u). \end{aligned}$$

Observe that the zero set of q_{LC} is the union of a line and a circle, and that $\mathcal{Z}(q_7) \subset \mathcal{Z}(q_{LC})$.

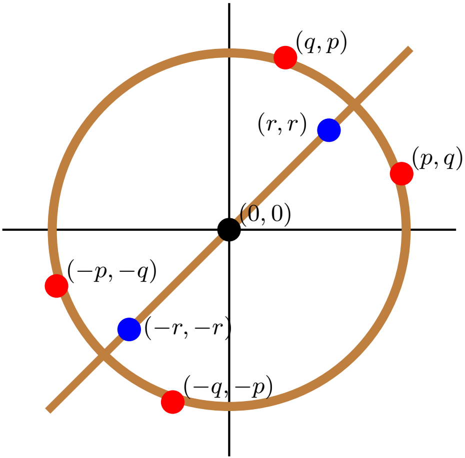

FIGURE 2. The sets $\mathcal{Z}(q_7)$ and $\mathcal{Z}(q_{LC})$

Theorem

(RC - Yoo, '09) Let $M(3) \geq 0$, *with* $M(2) > 0$ *and* $q_7(Z, \bar{Z}) = 0$. *There exists a representing measure for* $M(3)$ *if and only if*

$$\begin{cases} \Lambda(q_{LC}) &= 0 \\ \Lambda(zq_{LC}) &= 0. \end{cases} \tag{0.1}$$

Equivalently,

$$\begin{cases} Re\ \gamma_{12} - Im\ \gamma_{12} = u(Re\ \gamma_{01} - Im\ \gamma_{01}) &= 0 \\ \gamma_{22} = (t+u)\gamma_{11} - 2u\ Im\ \gamma_{02} &= 0. \end{cases}$$

Equivalently,

$$q_{LC}(Z, \bar{Z}) = 0 \tag{0.2}$$

Proof. ($\Longrightarrow$) Let μ be a representing measure. We know that $7 \leq \text{rank } M(3) \leq \text{card supp } \mu \leq \text{card } \mathcal{Z}(q_7) = 7$, so that $\text{supp } \mu = \mathcal{Z}(q_7)$ and $\text{rank } M(3) = 7$. Thus,

$$\Lambda(q_7) = \int q_7 \, d\mu = 0.$$

Similarly, since $\text{supp } \mu \subseteq \mathcal{Z}(q_{LC})$, we also have

$$\Lambda(q_{LC}) = \Lambda(z q_{LC}) = 0,$$

as desired.

$(\Longleftarrow)$ On $\mathcal{Z}(q_7)$ we have $z^3 = itz + u\bar{z}$. Using this relation and (0.1), we can prove that $\Lambda(\bar{z}^i z^j q_{LC}) = 0$ for all $0 \leq i + j \leq 3$. For example,

$$\begin{aligned} \bar{z} q_{LC} - iz q_{LC} &= (\bar{z} - iz)(\bar{z}^2 z + i\bar{z} z^2 - iuz - u\bar{z}) \\ &= -uz^2 + \bar{z} z^3 - u\bar{z}^2 + \bar{z}^3 z \\ &= -uz^2 + \bar{z}(itz + u\bar{z}) - u\bar{z}^2 + (-it\bar{z} + uz)z \\ &= 0, \end{aligned}$$

and therefore $\Lambda(\bar{z} q_{LC}) = i\Lambda(z q_{LC}) = 0$. It follows that for $f, g, h \in \mathbb{C}_3[z, \bar{z}]$ we have $\Lambda(fq_7 + g\bar{q}_7 + hq_{LC}) = 0$.

Consistency will be established once we show that all degree-six polynomials vanishing in $\mathcal{Z}(q_7)$ are of the form $fq_7 + g\bar{q}_7 + hq_{LC}$.

Proposition (Representation of Polynomials)

Let $\mathcal{P}_6 := \{p \in \mathbb{C}_6[z,\bar{z}] : p|_{\mathcal{Z}(q_7)} \equiv 0\}$ *and let* $\mathcal{I} := \{p \in \mathbb{C}_6[z,\bar{z}] : p = fq_7 + g\bar{q}_7 + hq_{LC}$ *for some* $f,g,h \in \mathbb{C}_3[z,\bar{z}]\}$. *Then* $\mathcal{P}_6 = \mathcal{I}$.

Proof. Clearly, $\mathcal{I} \subseteq \mathcal{P}_6$. We shall show that $\dim \mathcal{I} = \dim \mathcal{P}_6$.
Let $T : \mathbb{C}^{30} \longrightarrow \mathbb{C}_6[z,\bar{z}]$ be given by

$$(a_{00}, \cdots, a_{30}, b_{00}, \cdots, b_{30}, c_{00}, \cdots, c_{30}) \longmapsto$$

$$\begin{aligned} &(a_{00} + a_{01}z + a_{10}\bar{z} + \cdots + a_{30}\bar{z}^3)q_7 \\ +&(b_{00} + b_{01}z + b_{10}\bar{z} + \cdots + b_{30}\bar{z}^3)\bar{q}_7 \\ +&(c_{00} + c_{01}z + c_{10}\bar{z} + \cdots + c_{30}\bar{z}^3)q_{LC}. \end{aligned}$$

Recall that $30 = \dim \mathbb{C}^{30} = \dim \ker T + \dim \operatorname{Ran} T$, and observe that $\mathcal{I} = \operatorname{Ran} T$, so that $\dim \mathcal{I} = \operatorname{rank} T$.

To determine $\operatorname{rank} T$, we first determine $\dim \ker T$. Using Gaussian elimination, we prove that $\dim \ker T = 9$ whenever $ut \neq 0$. It follows that $\operatorname{rank} T = 30 - 9 = 21$, that is, $\dim \mathcal{I} = 21$.

Now consider the evaluation map $S : \mathbb{C}_6[z,\bar{z}] \longrightarrow \mathbb{C}^7$ given by

$$\begin{aligned} S(p(z,\bar{z})) \;\; := \;\; & (p(w_0,\bar{w}_0), p(w_1,\bar{w}_1), p(w_2,\bar{w}_2), \\ & p(w_3,\bar{w}_3), p(w_4,\bar{w}_4), p(w_5,\bar{w}_5), p(w_6,\bar{w}_6)). \end{aligned}$$

Again, $\dim \ \ker \ S + \dim \text{Ran} \ S = \dim \ \mathbb{C}_6[z,\bar{z}] = 28$. Using Lagrange Interpolation, it is esay to verify that S is onto, i.e., $\text{rank} \ S = 7$.

Moreover, $\ker \ S = \mathcal{P}_6$. Since $\dim \ \mathbb{C}_6[z,\bar{z}] = 28$, it follows that $\dim \ \ker \ S = 21$, and a fortiori that $\dim \ \mathcal{P}_6 = 21$.

Therefore, $\dim \ \mathcal{I} = 21 = \dim \ \mathcal{P}_6$, and since $\mathcal{I} \subseteq \mathcal{P}_6$, we have established that $\mathcal{I} = \mathcal{P}_6$, as desired.

Summary

- Given a finite family of moments, build moment matrix
- Identify all column relations, and build algebraic variety $\mathcal{V}$
- Consider the ideal generated by poly's arising from column relations
- Always true: $r \leq \text{card } \text{supp } \mu \leq v$
- Finite rank case; flat case
- Quartic Case

- Extremal case (must check Consistency)
- Harmonic cubic poly's in Sextic Case
- The extremal case is inherent in TMP:
  C. Bayer and J. Teichmann (2006) proved that if $\beta^{(2n)}$ has a representing measure, then it has a **finitely atomic** representing measure.
- We have obtained an analogue of the Riesz-Haviland Theorem for TMP, and we have discussed two approaches to K-positivity, including degree-bounded representations of positive polynomials.
- General singular case
- Invertible case still a big mystery...

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