2-variable Weighted Shifts Built From Elementary Tensors

Raúl E. Curto, IEOp 2009
(joint work with S.H. Lee and J. Yoon)

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Notation and Preliminaries

$\mathcal{L}(\mathcal{H})$: algebra of bounded operators on a Hilbert space $\mathcal{H}$
$T \in \mathcal{L}(\mathcal{H})$ is

- normal if $T^*T = TT^*$
- subnormal if $T = N|_{\mathcal{H}}$, where N is normal and $N\mathcal{H} \subseteq \mathcal{H}$
- hyponormal if $T^*T \geq TT^*$
- For $S, T \in \mathcal{B}(\mathcal{H})$, $[S, T] := ST - TS$.
- An n-tuple $\mathbf{T} \equiv (T_1, ..., T_n)$ is (jointly) hyponormal if

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \cdots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.$$

- For $k \geq 1$, an operator T is k-hyponormal if $(T, ..., T^k)$ is (jointly) hyponormal, i.e.,

$$\begin{pmatrix} [T^*, T] & \cdots & [T^{*k}, T] \\ \vdots & \ddots & \vdots \\ [T^*, T^k] & \cdots & [T^{*k}, T^k] \end{pmatrix} \geq 0$$

- An n-tuple $\mathbf{T}$ is normal if $\mathbf{T}$ is commuting and each T_i is normal
- $\mathbf{T}$ is subnormal if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace
- normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.
- (Bram-Halmos, $n = 1$)

$$\begin{aligned} T \text{ subnormal } &\Leftrightarrow T \text{ is } k\text{-hyponormal for all } k \geq 1 \\ &\Leftrightarrow (T, T^2, ..., T^k) \text{ is hyponormal for all } k \geq 1. \end{aligned}$$

Problem

(Lifting Problem for Commuting Subnormals (LPCS)) Given a commuting pair $\mathbf{T} \equiv (T_1, T_2)$*, find necessary and sufficient conditions on* T_1 *and* T_2 *to guarantee the subnormality of* $\mathbf{T}$*.*

It is well known that the subnormality of each of T_1 and T_2 is necessary but not sufficient.

Conjecture

(RC-Muhly-Xia, 1988) Let $\mathbf{T} \equiv (T_1, T_2)$ *be a pair of commuting subnormal operators. Then* $\mathbf{T}$ *is subnormal if and only if* $\mathbf{T}$ *is hyponormal.*

Definition

$\mathbf{T} \equiv (T_1, T_2)$ is *k*-hyponormal if $(T_1, T_2, T_1^2, T_1T_2, T_2^2, \cdots, T_1^k, T_1^{k-1}T_2, \cdots, T_2^k)$ is hyponormal.

Theorem

There exist multiple examples of nonsubnormal hyponormal $\mathbf{T}$ *consisting of commuting subnormals. One of the constructions even allows for* $T_1 \cong T_2$*. We also know that for every* $k \geq 1$ *one can build a pair* $\mathbf{T}$ *which is k-hyponormal but not* $(k+1)$*-hyponormal, therefore not subnormal.*

Theorem

(E. Franks, 1994) A commuting pair $\mathbf{T}$ *is subnormal if and only if* $p(\mathbf{T})$ *is subnormal for all poly's p with degree at most* 5.

Definition

$\mathfrak{H}_0 := \{\mathbf{T} \equiv (T_1, T_2) : T_1 T_2 = T_2 T_1 \text{ and } T_1 \text{ subnormal } (i = 1, 2)\}$
$\mathfrak{H}_1 := \{\mathbf{T} \in \mathfrak{H}_0 : \mathbf{T} \text{ is hyponormal}\}$
$\mathfrak{H}_k := \{\mathbf{T} \in \mathfrak{H}_0 : \mathbf{T} \text{ is } k\text{-hyponormal}\}$
$\mathfrak{H}_\infty := \{\mathbf{T} \in \mathfrak{H}_0 : \mathbf{T} \text{ is subnormal}\}$.

Remark

Clearly,

$$\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0;$$

Our recent results show that these inclusions are all proper. Suitable examples have been found in the class $\mathcal{TC}$, a large class of 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2)$ for which hyponormality and subnormality are far apart.
Briefly described, $\mathcal{TC}$ consists of hyponormal pairs $\mathbf{T}$ of subnormal operators such that

$$c(\mathbf{T}) := \mathbf{T}|_{\mathcal{L}} \cong (I \bigotimes W_\alpha, W_\beta \bigotimes I),$$

where $\mathcal{L}$ is a large invariant subspace.

Unilateral Weighted Shifts

- $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty} \in \ell^\infty(\mathbb{Z}_+)$, $\alpha_k > 0$ (all $k \geq 0$)
- $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$

$$W_\alpha e_k := \alpha_k e_{k+1} \quad (k \geq 0)$$

- When $\alpha_k = 1$ (all $k \geq 0$), $W_\alpha = U_+$, the (unweighted) unilateral shift

$$W_\alpha = U_+ D_\alpha \quad \text{(polar decomposition)}$$

- $\|W_\alpha\| = \sup_k \alpha_k$
  $W_\alpha^n e_k = \alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1} e_{k+n}$, so

$$W_\alpha^n \cong \bigoplus_{i=0}^{n-1} W_{\beta(i)}.$$

Weighted Shifts and Berger's Theorem

Recall

$$W_\alpha e_k := \alpha_k e_{k+1} \ \ (k \geq 0).$$

The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

- W_α is never normal
- W_α is hyponormal $\Leftrightarrow \alpha_k \leq \alpha_{k+1}$ (all $k \geq 0$)
- (Berger; Gellar-Wallen) W_α is subnormal iff there exists a positive Borel measure ξ on $[0, \|W_\alpha\|^2]$ such that

$$\gamma_k = \int t^k \, d\xi(t) \ \ (\text{all } k \geq 0).$$

ξ is the Berger measure of W_α.

- The Berger measures of U_+ is δ_1.
- For $0 < a < 1$ we let $S_a := shift\{a, 1, 1, ...)$.
- The Berger measure of S_a is $(1 - a^2)\delta_0 + a^2\delta_1$.
- The Berger measure of B_+ (the Bergman shift) is Lebesgue measure on the interval $[0, 1]$; the weights of B_+ are $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 0)$.

Definition

W_α is flat (or briefly, α is flat) if $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.

Theorem

(Propagation)
(i) (Stampfli, 66) Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. In this case, the Berger measure is $\xi \equiv (1 - a_0^2)\delta_0 + a_0^2 \delta_{a_1^2}$.
(ii) (RC, 88) Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.
(iii) (Choi, 2000) Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.

$$\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2), \quad \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$$

$$\ell^2(\mathbb{Z}_+^2) \cong \ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+).$$

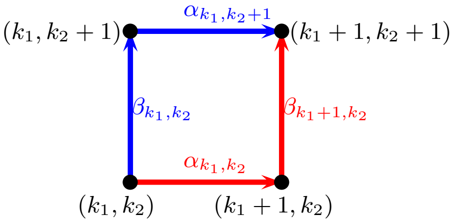
We define the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ by

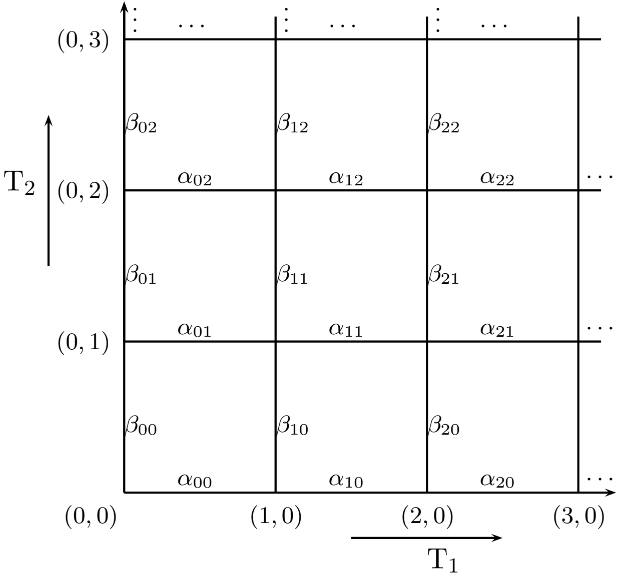
$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$$

$$T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1, 0)$ and $\varepsilon_2 := (0, 1)$. Clearly,

$$T_1 T_2 = T_2 T_1 \iff \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \quad \text{(all } \mathbf{k}\text{)}.$$




(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T2
T1
β02
β12
β22
α02
α12
α22
β01
β11
β21
α01
α11
α21
β00
β10
β20
α00
α10
α20

- Trivially, a pair of unilateral weighted shifts W_α and W_β gives rise to a 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $k_1, k_2 \in \mathbb{Z}_+^2$). In this case, $\mathbf{T}$ is subnormal (resp. hyponormal) if and only if so are T_1 and T_2; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ and $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$, one has

$$T_1 \cong I \bigotimes W_\alpha$$

and

$$T_2 \cong W_\beta \bigotimes I,$$

and $\mathbf{T}$ is also doubly commuting. These are the elementary tensors we use to build appropriate classes on which to study LPCS.

To detect hyponormality, there is a simple criterion:

Theorem

(RC, 1988) (Six-point Test) Let $\mathbf{T} \equiv (T_1, T_2)$ *be a 2-variable weighted shift, with weight sequences* α *and* β*. Then*

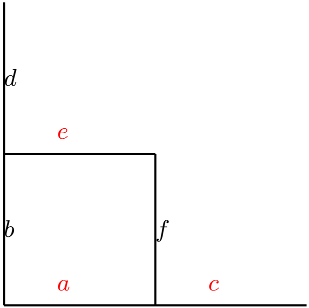
$$\mathbf{T} \text{ is hyponormal } \Leftrightarrow \begin{pmatrix} \alpha^2_{\mathbf{k}+\varepsilon_1} - \alpha^2_{\mathbf{k}} & \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} \\ \alpha_{\mathbf{k}+\varepsilon_2}\beta_{\mathbf{k}+\varepsilon_1} - \alpha_{\mathbf{k}}\beta_{\mathbf{k}} & \beta^2_{\mathbf{k}+\varepsilon_2} - \beta^2_{\mathbf{k}} \end{pmatrix} \geq 0$$

(all $\mathbf{k} \in \mathbb{Z}^2_+$*).*

$$\mathbf{T} \text{ is hyponormal } \Leftrightarrow \begin{pmatrix} c^2 - a^2 & ef - ab \\ ef - ab & d^2 - b^2 \end{pmatrix} \geq 0.$$

Application

$a = c \Rightarrow ef = ab \Rightarrow aef = a^2 b \Rightarrow be^2 = a^2 b \Rightarrow e = a$



We now recall the notion of moment of order $\mathbf{k}$ for a commuting pair (α, β). Given $\mathbf{k} \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is $\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta)$

$$:= \begin{cases} 1 & \text{if } \mathbf{k} = 0 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdot \ldots \cdot \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdot \ldots \cdot \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdot \ldots \cdot \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

By commutativity, $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to (k_1, k_2).

- (Jewell-Lubin)

$$W_\alpha \text{ is subnormal} \quad \Leftrightarrow \quad \gamma_{\mathbf{k}} := \prod_{i=0}^{k_1-1} \alpha_{(i,0)}^2 \cdot \prod_{j=0}^{k_2-1} \beta_{(k_1-1,j)}^2$$

$$= \quad \int t_1^{k_1} t_2^{k_2} \, d\mu(t_1, t_2) \ \ (\text{all } \mathbf{k} \geq \mathbf{0}).$$

Thus, the study of subnormality for multivariable weighted shifts is intimately connected to multivariable real moment problems.

A Measure-Theoretic Formulation of LPCS

- $\mathbf{T} \equiv (T_1, T_2)$ subnormal $\Rightarrow T_i$ subnormal for $i = 1, 2$. For instance,

$$T_1 \cong \bigoplus_{j=0}^{\infty} W_{\alpha^{(j)}},$$

where $\alpha_i^{(j)} := \alpha_{(i,j)}$ is the j-th row.
If μ is the Berger measure of $\mathbf{T}$, and if

$$d\mu(t_1, t_2) \equiv d\Phi_{t_1}(t_2) d\eta(t_1)$$

is the canonical disintegration of μ by vertical slices, we prove that the Berger measure of $W_{\alpha^{(j)}}$ is

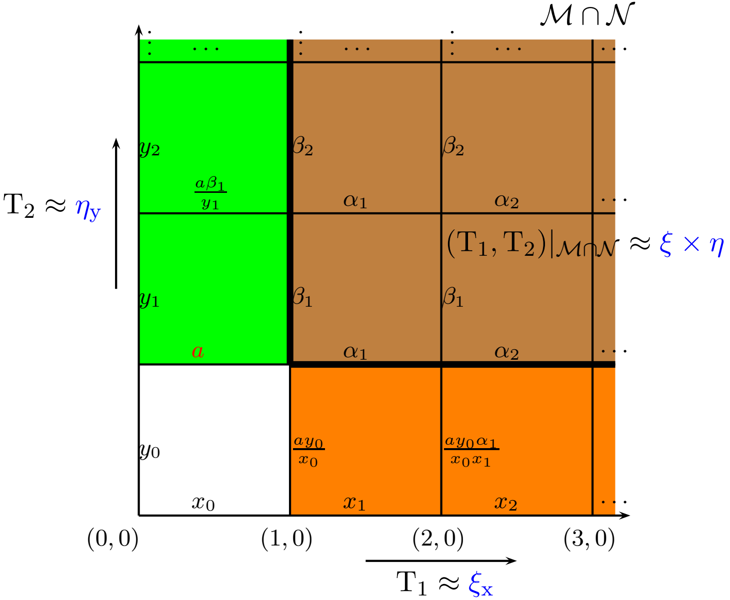
$$d\nu_j(t_1) := \frac{1}{\gamma_{(0,j)}} \int_{[0,a_2]} t_2^j d\Phi_{t_1}(t_2).$$

In terms of the marginal measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$, LPCS can be phrased as a

Problem (Reconstruction-of-Measure Problem (ROMP))

Under what conditions on the measures $\{\nu_j\}_{j=0}^{\infty}$ and $\{\omega_i\}_{i=0}^{\infty}$ does there exist a 2-variable measure μ correctly interpolating all the powers $t_1^{k_1} t_2^{k_2}$ $(k_1, k_2 \geq 0)$.

Special Case (tensor core): Given $\xi, \eta, \xi_x, \eta_y, a$, find μ.



Definition

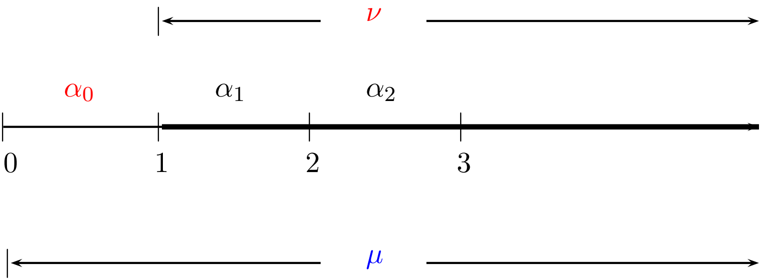
(i) The *core* of a 2-variable weighted shift $\mathbf{T}$ is $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M} \cap \mathcal{N}}$;
(ii) $\mathbf{T}$ is said to be *of tensor form* if $\mathbf{T} \cong (I \otimes W_\alpha, W_\beta \otimes I)$. (When $\mathbf{T}$ is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product $\xi \times \eta$);
(iii) $\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$.

Problem

Let $\mathbf{T} \in \mathcal{TC}$ and assume $\mathbf{T}$ is hyponormal. Additionally, assume that $c(\mathbf{T})$ is subnormal, with Berger measure $\xi \times \eta$. Find necessary and sufficient conditions on ξ, η, ξ_x, η_y and a to guarantee the subnormality of $\mathbf{T}$.

Problem (Baby Version of ROMP, one variable)

Given α_0 and ν, find μ



Proposition

(Subnormal backward extension for 1-variable weighted shifts) (cf. RC, 1988) Let T be a weighted shift whose restriction $T_{\mathcal{M}}$ to $\mathcal{M} := \vee\{e_1, e_2, \cdots\}$ is subnormal, with associated measure ν. Then T is subnormal (with associated measure μ) if and only if
(i) $\frac{1}{t} \in L^1(\nu)$
(ii) $\alpha_0^2 \leq (\|\frac{1}{t}\|_{L^1(\nu)})^{-1}$
In this case,

$$d\mu(t) = \frac{\alpha_0^2}{t} d\nu(t) + (1 - \alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\nu)}) d\delta_0(t),$$

where δ_0 denotes Dirac measure at 0. In particular, T is never subnormal when $\nu(\{0\}) > 0$.

Extremal and Marginal Measures

Definition

μ probability measure on $X \times Y$, with $\frac{1}{t} \in L^1(\mu)$. The extremal measure μ_{ext} (also a probability measure) on $X \times Y$ is given by

$$d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\mu)}} d\mu(s,t).$$

Definition

For μ on $X \times Y$, the marginal measure μ^X is given by

$$\mu^X := \mu \circ \pi_X^{-1},$$

where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus,

$$\mu^X(E) = \mu(E \times Y),$$

for every $E \subseteq X$. If μ is a probability measure, then so is μ^X.

Lemma

μ : *Berger measure of 2-variable weighted shift* $\mathbf{T}$
ν : *Berger measure of shift* $(\alpha_{00}, \alpha_{10}, \ldots)$.
Then $\nu = \mu^X$. *As a consequence,*

$$\iint f(s)\, d\mu(s,t) = \int f(s)\, d\mu^X(s) \quad (\text{all } f \in C(X)).$$

Corollary

μ : *Berger measure of 2-variable weighted shift* $\mathbf{T}$.
For $j \geq 1$, *let* $d\mu_j(s,t) := \frac{1}{\gamma_{0j}} t^j d\mu(s,t)$.
Then the Berger measure of shift $(\alpha_{0j}, \alpha_{1j}, \ldots)$ *is* $\nu_j \equiv \mu_j^X$.

Example

$(\xi \times \eta)^X = \xi$.

Lemma

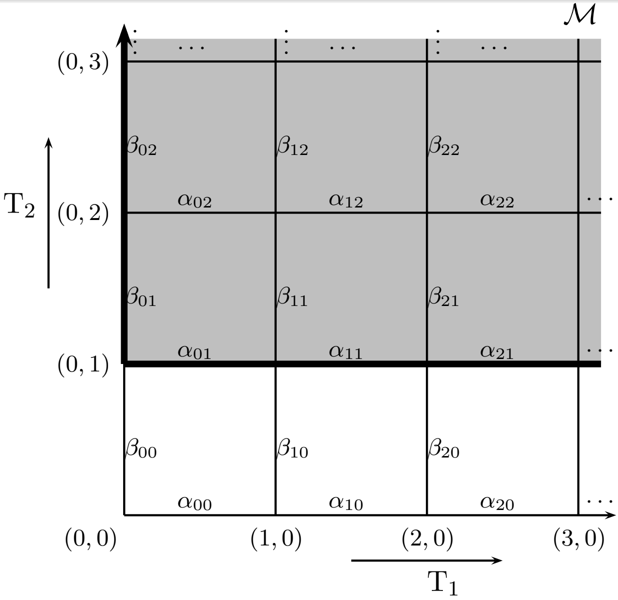
Let μ and ω be two measures on $X \times Y$, and assume that $\mu \leq \omega$. Then $\mu^X \leq \omega^X$.

Proposition (Special Case of ROMP)

(Subnormal backward extension for 2-variable weighted shifts)
$\mathcal{M}$: *subspace of* $\ell^2(Z_+^2)$ *associated to indices* $\mathbf{k}$ *with* $k_2 \geq 1$.
Assume $\mathbf{T}_{\mathcal{M}}$ *subnormal with measure* $\mu_{\mathcal{M}}$ *and*
$W_0 := shift(\alpha_{00}, \alpha_{10}, \cdots)$ *subnormal with measure* ν.
Then $\mathbf{T}$ *is subnormal if and only if*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;

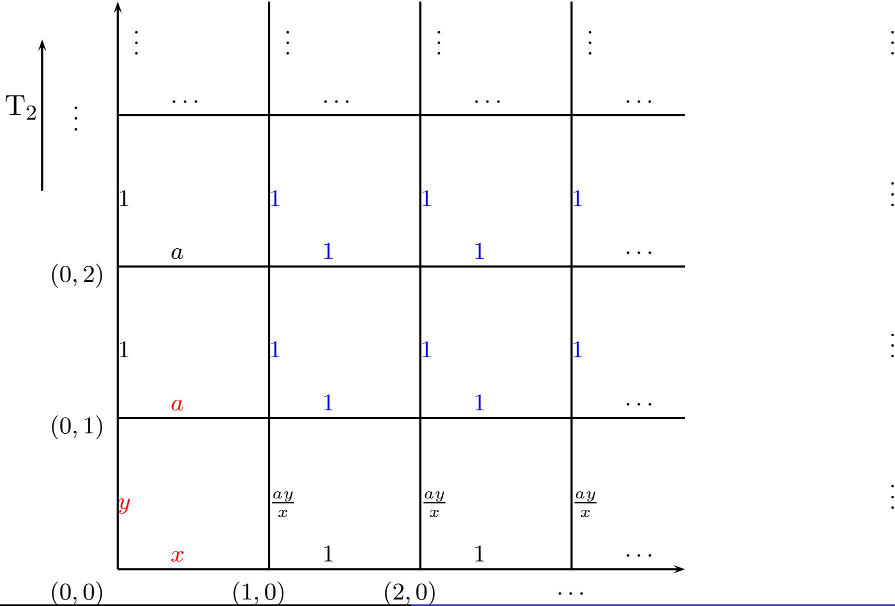
(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \nu$.
Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, *then* $(\mu_{\mathcal{M}})_{ext}^X = \nu$.
When $\mathbf{T}$ *is subnormal, its Berger measure is*

$$\begin{aligned} d\mu(s,t) &= \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &\quad + (d\nu(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s)) d\delta_0(t). \end{aligned}$$


$\mathcal{M}$
T_2
$(0,3)$
$(0,2)$
$(0,1)$
$(0,0)$
$(1,0)$
$(2,0)$
$(3,0)$
T_1
β_{02} β_{12} β_{22}
α_{02} α_{12} α_{22}
β_{01} β_{11} β_{21}
α_{01} α_{11} α_{21}
β_{00} β_{10} β_{20}
α_{00} α_{10} α_{20}

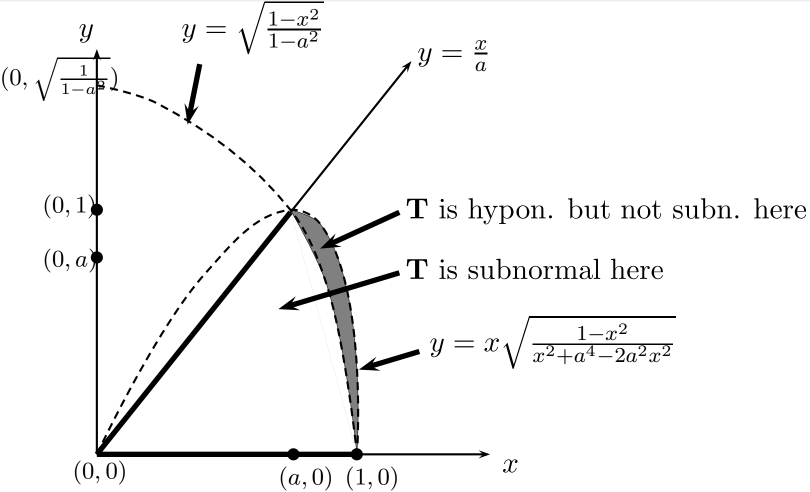
(i)

Application. Consider now the 2-variable weighted shift, with $\max\{y, x, \frac{ay}{x}\} < 1$.



Theorem

T *is hyponormal and not subnormal if and only if* $x > a$ *and* $\sqrt{\frac{1-x^2}{1-a^2}} < y \leq x\sqrt{\frac{(1-x^2)}{x^2+a^4-2a^2x^2}}$.


y
$y = \sqrt{\frac{1-x^2}{1-a^2}}$
$y = \frac{x}{a}$
$(0, \sqrt{\frac{1}{1-a^2}})$
$(0,1)$
$(0,a)$
$\mathbf{T}$ is hypon. but not subn. here
$\mathbf{T}$ is subnormal here
$y = x\sqrt{\frac{1-x^2}{x^2+a^4-2a^2x^2}}$
$(0,0)$
$(a,0)$
$(1,0)$
x

A Necessary Condition for the Existence of a Lifting

Theorem

Let μ be the Berger measure of a subnormal 2-variable weighted shift, and for $j \geq 0$ let ξ_j be the Berger measure of the associated j-th horizontal 1-variable weighted shift $W_{\alpha^{(j)}}$. Then

$$d\xi_j(s) = \{\frac{1}{\gamma_{0j}} \int_Y t^j \, d\Phi_s(t)\} \, d\mu^X(s),$$

where $d\mu(s,t) = d\Phi_s(t) \, d\mu^X(s)$. A similar result holds for the Berger measure η_i of the associated i-th vertical 1-variable weighted shifts $W_{\beta^{(i)}}$ $(i \geq 0)$.

Theorem

Let μ, ξ_j *and* η_i *be as before. For every* $i, j \geq 0$ *we have*

$$\xi_{j+1} \ll \xi_j$$

and

$$\eta_{i+1} \ll \eta_i.$$

Theorem

For $i, j \geq 1$,

$$\xi_{j+1} \approx \xi_j$$

and

$$\eta_{i+1} \approx \eta_i.$$

Corollary

$$(i) \quad \cdots \quad = \quad \text{supp } \xi_3 = \text{supp } \xi_2 = \text{supp } \xi_1 \subseteq \text{supp } \xi_0$$
$$(ii) \quad \cdots \quad = \quad \text{supp } \eta_3 = \text{supp } \eta_2 = \text{supp } \eta_1 \subseteq \text{supp } \eta_0$$

Corollary

$$(i) \quad \cdots \quad = \quad \|W_{\alpha^{(3)}}\| = \|W_{\alpha^{(2)}}\| = \|W_{\alpha^{(1)}}\| \leq \|W_{\alpha^{(0)}}\|$$
$$(ii) \quad \cdots \quad = \quad \left\|W_{\beta^{(3)}}\right\| = \left\|W_{\beta^{(2)}}\right\| = \left\|W_{\beta^{(1)}}\right\| \leq \left\|W_{\beta^{(0)}}\right\|$$

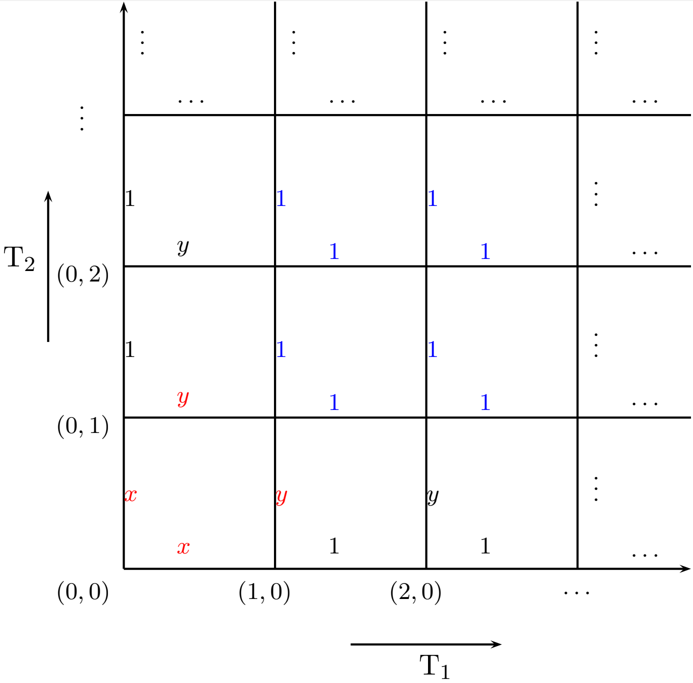
The Necessary Condition is Not Sufficient

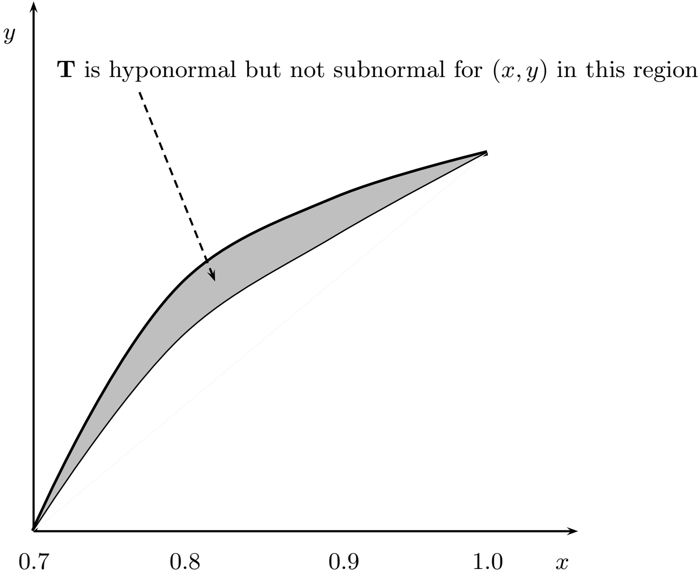
Proposition

Let $\mathbf{T} \equiv (T_1, T_2)$ *be the following 2-variable weighted shift. Then*
(i) $\mathbf{T}$ *is hyponormal* $\Leftrightarrow 1 - 2x^2 + y^2 \geq 0$
(ii) $\mathbf{T}$ *is subnormal* $\Leftrightarrow 1 - 2x^2 + x^2y^2 \geq 0.$
As a consequence, for $(x, y) \in \mathbb{R}_+^2$ *such that*

$$1 - 2x^2 + x^2y^2 < 0 \leq 1 - 2x^2 + y^2,$$

$\mathbf{T}$ *is hyponormal but not subnormal.*


T_2
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
T_1
x
y
1


y
T is hyponormal but not subnormal for (x, y) in this region
0.7
0.8
0.9
1.0
x

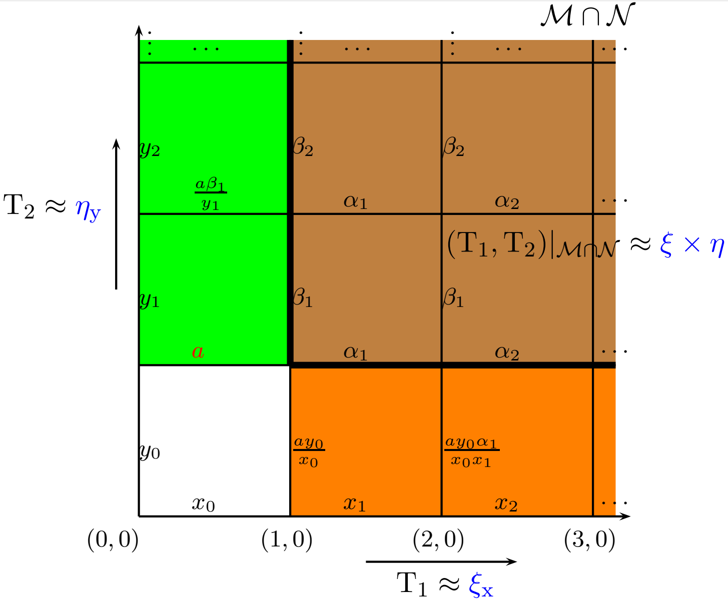
2-variable Shifts Whose Cores are of Tensor Form

Our main result provides a complete solution to Problem 14: $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$ is subnormal if and only if two associated 1-variable Borel measures are positive. As an application, we give a concrete condition for the subnormality of flat 2-variable weighted shifts.

Recall the definition of the class $\mathcal{TC}$. First, we need some notation: $\mathcal{M} := \vee\{e_{k_1,k_2} : k_2 \geq 1\}$ and $\mathcal{N} := \vee\{e_{k_1,k_2} : k_1 \geq 1\}$.

Definition

(i) The *core* of a 2-variable weighted shift $\mathbf{T}$ is $c(\mathbf{T}) := \mathbf{T}|_{\mathcal{M}\cap\mathcal{N}}$;
(ii) $\mathbf{T}$ is said to be *of tensor form* if $\mathbf{T} \cong (I \otimes W_\alpha, W_\beta \otimes I)$. (When $\mathbf{T}$ is subnormal, this is equivalent to requiring that the Berger measure be a Cartesian product $\xi \times \eta$);
(iii) $\mathcal{TC} := \{\mathbf{T} \in \mathfrak{H}_0 : c(\mathbf{T}) \text{ is of tensor form}\}$.


$\mathcal{M} \cap \mathcal{N}$
$T_2 \approx \eta_y$
y_2
$\frac{a\beta_1}{y_1}$
y_1
a
y_0
x_0
β_2
β_2
α_1
α_2
$(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}} \approx \xi \times \eta$
β_1
β_1
α_1
α_2
$\frac{ay_0}{x_0}$
$\frac{ay_0\alpha_1}{x_0x_1}$
x_1
x_2
$(0, 0)$
$(1, 0)$
$(2, 0)$
$(3, 0)$
$T_1 \approx \xi_x$

WLOG, we always assume that $\frac{1}{s} \in L^1(\xi)$ and $\frac{1}{t} \in L^1(\eta)$. We write

$$d\widetilde{\eta}(t) := \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\eta)}} d\eta(t).$$

Thus,

$$d(\xi \times \eta)_{ext}(s,t) = \frac{1}{t \left\| \frac{1}{t} \right\|_{L^1(\eta)}} d\xi(s) d\eta(t) = d\xi(s) d\widetilde{\eta}(t)$$

and $(\xi \times \eta)^X = \xi$. Assume that $c(\mathbf{T})$ is subnormal, with Berger measure $\xi \times \eta$. We let

$$\psi := (\eta_y)_1 - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \eta$$

where $(\eta_y)_1$ is the Berger measure of the subnormal shift $shift(y_1, y_2, \cdots)$.

We also let

$$\varphi := \xi_x - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \widetilde{\xi},$$

Trivially, ψ and φ are measures, but they may or may not be *positive* measures.

Proposition

Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ *be a shift whose core is of tensor form. Then* $\mathbf{T}|_{\mathcal{M}} \in \mathfrak{H}_\infty \iff \psi \geq 0$. *In this case, the Berger measure of* $\mathbf{T}|_{\mathcal{M}}$ *is*

$$\mu_{\mathcal{M}} = a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi.$$

Sketch of Proof.

$(\Rightarrow)$ If $\mathbf{T}|_{\mathcal{M}} \in \mathfrak{H}_\infty$ then $\mathbf{T}|_{\mathcal{M}\cap\mathcal{N}} \in \mathfrak{H}_\infty$ with Berger measure $\mu_{\mathcal{M}\cap\mathcal{N}} = \xi \times \eta$. Note that $\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} = \left\|\frac{1}{s}\right\|_{L^1(\xi)}$. If we think of $\mathbf{T}|_{\mathcal{M}}$ as the backward extension of $\mathbf{T}|_{\mathcal{M}\cap\mathcal{N}}$ (in the s direction), we then have

$$(\eta_y)_1 \geq a^2 \left\|\tfrac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} (\mu_{\mathcal{M}\cap\mathcal{N}})^Y_{ext}$$

$$\Leftrightarrow (\eta_y)_1 - a^2 \left\|\tfrac{1}{s}\right\|_{L^1(\xi)} \xi \geq 0 \Leftrightarrow \psi \geq 0.$$

Thus, ψ is a positive measure.

Sketch of Proof cont.

($\Leftarrow$) If ψ is a positive measure then $\phi := a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi$ is a well defined and positive measure. By a direct calculation, we can see that

$$\begin{cases} \iint d\phi(s,t) = 1, & \text{if } k_1 = 0 \text{ and } k_2 = 0 \\ \iint t^{k_2} d\phi(s,t) = y_1^2 \cdots y_{k_2}^2, & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \iint s^{k_1} d\phi(s,t) = a^2 \alpha_1^2 \cdots \alpha_{k_1-1}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \iint s^{k_1} t^{k_2} d\phi(s,t) = a^2 \alpha_1^2 \cdots \alpha_{k_1-1}^2 \beta_1^2 \cdots \beta_{k_2}^2, & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1 \end{cases}$$

Therefore, ϕ interpolates all moments of $\mathbf{T}|_{\mathcal{M}}$, so $\mathbf{T}|_{\mathcal{M}} \in \mathfrak{H}_\infty$ and $\mu_{\mathcal{M}} = \phi \equiv a^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \widetilde{\xi} \times \eta + \delta_0 \times \psi$. □

We now have:

Theorem

Let $\mathbf{T} \equiv (T_1, T_2) \in \mathfrak{H}_0$ *be a 2-variable weighted shift whose core is of tensor form. Then* $\mathbf{T} \in \mathfrak{H}_\infty \iff \psi \geq 0$ *and* $\varphi \geq 0$.

Sketch of Proof.

$(\Leftarrow)$ It suffices to find a probability measure μ satisfying

$$\gamma_{\mathbf{k}}(\mathbf{T}) = \int t^{\mathbf{k}} d\mu(\mathbf{t}) := \int t_1^{k_1} t_2^{k_2} d\mu(\mathbf{t}), \text{ for all } \mathbf{k} \geq 0.$$

Let

$$\mu := \varphi \times (\delta_0 - \widetilde{\eta}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \widetilde{\eta} \right) + \xi_x \times \widetilde{\eta}.$$

□

Sketch of Proof cont.

Observe that

$$
\begin{aligned}
\mu &= \varphi \times (\delta_0 - \widetilde{\eta}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \widetilde{\eta} \right) + \xi_x \times \widetilde{\eta} \\
&= (\xi_x - \varphi - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0) \times \widetilde{\eta} + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \widetilde{\psi} + \varphi \times \delta_0 \\
&= a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)} \widetilde{\xi} \times \widetilde{\eta} + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \widetilde{\psi} + \varphi \times \delta_0 .
\end{aligned}
$$

□

Since we are assuming that ψ and φ are positive measures, it follows that μ is also positive. Furthermore,

$$\begin{aligned}\iint d\mu(s,t) &= y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \int d[\widetilde{\psi}(t) - \widetilde{\psi}(t)] + 1 \\ &= 1 \ \ (\text{since } \widetilde{\psi} \text{ is a probability measure}).\end{aligned}$$

Therefore, μ is a probability measure. Next, observe that

$$\varphi([0,+\infty)) = \int d\varphi(s) = 1 - y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} - a^2 y_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \left\| \frac{1}{t} \right\|_{L^1(\eta)}$$

and, for $k_2 \geq 1$,

$$\left\| \frac{1}{t} \right\|_{L^1(\psi)} \int t^{k_2} d\widetilde{\psi}(t) = \int t^{k_2-1} d\,(\eta_y)_1\,(t) - a^2 \left\| \frac{1}{s} \right\|_{L^1(\xi)} \int t^{k_2-1} d\eta(t).$$

Now compute all the moments.

Remark

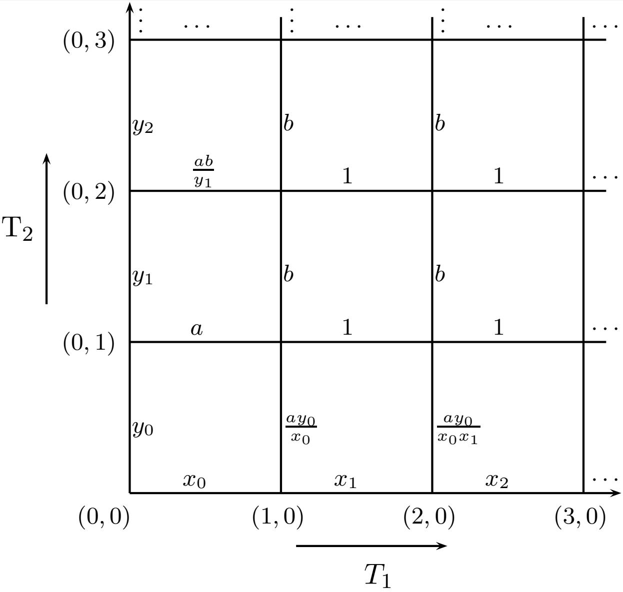
The proof gives a concrete formula for the Berger measure of **T**, namely

$$\mu := \varphi \times (\delta_0 - \widetilde{\eta}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \widetilde{\eta} \right) + \xi_x \times \widetilde{\eta}.$$

Application to Flat 2-variable Weighted Shifts

$\mathbf{T} \equiv (T_1, T_2)$ is

- *horizontally flat* if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $k_1, k_2 \geq 1$
- *vertically flat* if $\beta_{(k_1,k_2)} = \beta_{(1,1)}$ for all $k_1, k_2 \geq 1$
- *flat* if $\mathbf{T}$ is horizontally flat and vertically flat.
- flat if and only if $\mathbf{T} \in \mathcal{TC}$, with ξ and η 1-atomic.
  WLOG, can assume that $\xi = \delta_1$.


(0, 3)
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
(3, 0)
T_2
T_1
y_2
b
b
$\frac{ab}{y_1}$
1
1
y_1
b
b
a
1
1
y_0
$\frac{ay_0}{x_0}$
$\frac{ay_0}{x_0 x_1}$
x_0
x_1
x_2

Recall:

- for $0 < \alpha < \beta$, *shift* $(\alpha, \beta, \beta, \cdots)$ is subnormal with Berger measure $(1 - \frac{\alpha^2}{\beta^2})\delta_0 + \frac{\alpha^2}{\beta^2}\delta_{\beta^2}$
- if **T** is flat and subnormal then ξ_x and η_y have the form

$$\begin{aligned} \xi_x &= p\delta_0 + q\delta_1 + [1 - (p + q)]\rho \\ \eta_y &= \ell\delta_0 + m\delta_{b^2} + [1 - (\ell + m)]\sigma, \end{aligned}$$

where $0 < p, q, \ell, m < 1$, $p + q \leq 1$, $\ell + m \leq 1$, and ρ and σ are probability measures with $\rho(\{0\} \cup \{1\}) = 0$, $\sigma(\{0\} \cup \{b^2\}) = 0$.

Example

Consider the above mentioned 2-variable weighted shift $\mathbf{T} \in \mathfrak{H}_0$. TFAE:

(i) $\mathbf{T} \in \mathfrak{H}_\infty$;

(ii) ψ and φ are positive measures;

(iii) $\frac{b}{a}\sqrt{m} \geq y_0$ and

$$\xi_x \geq y_0^2 \left\{ \left(\left\| \frac{1}{t} \right\|_{L^1((\eta_y)_1)} - \frac{a^2}{b^2} \right) \delta_0 + \frac{a^2}{b^2} \delta_1 \right\}.$$

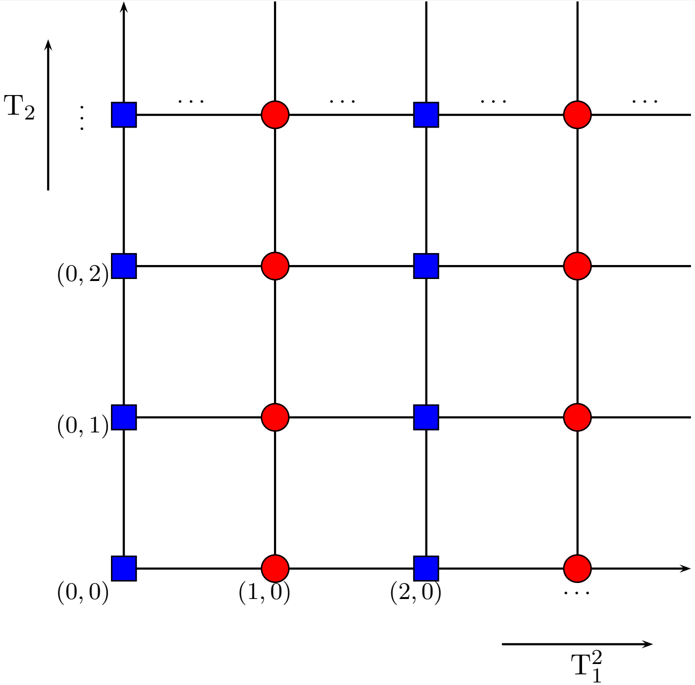
Moreover, when $\mathbf{T}$ is subnormal, its Berger measure is given as

$$\mu = \varphi \times (\delta_0 - \delta_{b^2}) + y_0^2 \left\| \frac{1}{t} \right\|_{L^1(\psi)} \delta_0 \times \left(\widetilde{\psi} - \delta_{b^2} \right) + \delta_1 \times \delta_{b^2}$$

Can Powers Detect Lifting?

J. Stampfli (1962)
(i) If T is hyponormal and T^n is normal for some $n \geq 1$, then T is normal.
(ii) There exists T hyponormal, non-subnormal, such that T^n is subnormal for all $n \geq 1$; e.g., a shift with weights $a, b, 1, 1, \cdots$, where $0 < a < b < 1$.
If W_α is a weighted shift, then W_α^2 is a direct sum of two shifts, with weights $\alpha_0\alpha_1, \alpha_2\alpha_3, \cdots$ and $\alpha_1\alpha_2, \alpha_3\alpha_4, \cdots$, resp.
In two variables, we consider (T_1^2, T_2) and (T_1, T_2^2), which can be split as orthogonal direct sums. For instance,

$$(T_1^2, T_2) \cong (W_{\alpha(2:0)} \oplus (I \otimes S_a), T_2|_{\mathcal{H}_0}) \bigoplus (W_{\alpha(2:1)} \oplus (I \otimes U_+), T_2|_{\mathcal{H}_1}).$$


T2
(0, 2)
(0, 1)
(0, 0)
(1, 0)
(2, 0)
T21

Theorem

Let $\mathbf{T} \equiv (T_1, T_2) \in \mathcal{TC}$. *Then*

$$(T_1, T_2^2) \in \mathfrak{H}_\infty \iff (T_1^2, T_2) \in \mathfrak{H}_\infty \iff (T_1, T_2) \in \mathfrak{H}_\infty.$$

Summary

- LPCS has abstract solution (Multivariable Bram-Halmos), but we seek concrete solution for multivariable weighted shifts.
- ROMP is a measure-theoretic formulation of LPCS for Berger measures.
- For a large collection of 2-variable weighted shifts (those whose core is of tensor form), we find a complete solution.
- Solution entails checking the positivity of two 1-variable measures, naturally associated to the initial data.
- We briefly addressed the question: Can the powers (T_1^2, T_2) and (T_1, T_2^2) detect the existence of a lifting?

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