Operators Cauchy Dual to 2-hyperexpansive Operators: The Multivariable Case

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Abstract

We introduce an abstract framework to study generating m-tuples, and use it to analyze hypercontractivity and hyperexpansivity in several variables. These two notions encompass (joint) hyponormality and subnormality, as well as toral and spherical isometric-ness; for instance, the Drury-Arveson 2-shift is a spherical complete hyperexpansion. Our approach produces a unified theory that simultaneously covers toral and spherical hypercontractions (and hyperexpansions). As a byproduct, we arrive at a dilation theory for completely hypercontractive and completely hyperexpansive generating tuples. We study in detail the Cauchy duals of toral and spherical 2-hyperexpansive tuples.

Outline of the Talk

- The 1-variable Case
- Generating m-tuples: Abstract Framework
- Hypercontractive Generating Tuples
- Hyperexpansive Generating Tuples
- Toral Cauchy Dual Tuples
- Spherical Cauchy Dual Tuples

The 1-variable Case

(Chavan, 2007)

- Assume T is 2-hyperexpansive, i.e.,

$$I - 2T^*T + T^{*2}T^2 \leq 0,$$

  and let $T' := T(T^*T)^{-1}$ be the Cauchy dual. Then T' is hyponormal.
- Structure of the C^*-algebra generated by a 2-hyperexpansion.
- Berger-Shaw for 2-hyperexpansions.
- Every *analytic* (i.e., $\bigcap_{n\geq 0} T^n\mathcal{H} = 0$) 2-hyperexpansive operator with finite-dimensional cokernel is unitarily equivalent to $U_+ + K$.

A Key Analogy

subnormal	$\longleftrightarrow$	completely hyperexpansive
hyponormal	$\longleftrightarrow$	2-hyperexpansive

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$$S \text{ subnormal}, \ \|S\| \leq 1 \iff B_n(S) := \sum_{p=0}^{n} (-1)^p \binom{n}{p} S^{*p} S^p \geq 0$$

$$(\text{all } n \geq 0)$$

$$\iff S \text{ is completely hypercontractive}$$

T hyponormal if $T^*T - TT^* \geq 0$

T is completely hyperexpansive if $B_n(T) \leq 0$ (all $n \geq 1$)

T is m-hyperexpansive if $B_n(T) \leq 0$ (all n, $1 \leq n \leq m$)

T is m-isometric if $B_m(T) = 0$

1-expansive $\equiv$ expansive

1-isometric $\equiv$ isometric

T is expansive if $I - T^*T \leq 0$

T expansive $\Longrightarrow$ T left invertible

Assume that T is left invertible. Then $T = VP$, with $P \geq 0$ and P invertible.

(Cauchy dual) $T' := T(T^*T)^{-1}$. Observe that $T' = VP^{-1}$.

$$T'^*T' = (T^*T)^{-1}$$

Then T' is left invertible, and $(T')' = T$. Moreover, $C^*(T) = C^*(T')$. (Observe that $V, P \in C^*(T)$.)

If T is a unilateral weighted shift, $T : \{\alpha_n\}$, then

$$\begin{aligned} T^*T &= diag(\alpha_n^2) \\ (T^*T)^{-1} &= diag(\frac{1}{\alpha_n^2}) \end{aligned}$$

so $T' : \{\frac{1}{\alpha_n}\}$.

Shimorin (2001) studied Cauchy duals:

T 2-hyperexpansive $\Longrightarrow$ T' 2-hypercontractive

Athavale (1996): T unilateral weighted shift, β_n moments.

Then T is completely hyperexpansive if and only if

$S_t : \{\sqrt{\frac{t(\beta_n - 1) + 1}{t(\beta_{n+1} - 1) + 1}}\}$ is a contractive subnormal weighted shift for all $t > 0$. (1-parameter family of subnormal shifts.)

In general,

If T is expansive, then T' is a contraction.

If T is a completely hyperexpansive weighted shift, then T' is a subnormal weighted shift.

Recall that an operator S is hyponormal if $S^*S - SS^* \geq 0$.

Equivalently, if there exists a contraction C such that $S^* = CS$.

Theorem (Chavan, 2007)

Let T be 2-hyperexpansive. Then T' is hyponormal.

Proof Observe that

$$T^{*2}T^2 - (T^*T)^2 \le (I - T^*T)^2 + T^{*2}T^2 - (T^*T)^2 = I - 2T^*T + T^{*2}T^2 \le 0,$$

so

$$T^{*2}T^2 \le (T^*T)^2.$$

Thus,

$$\left\|T^2x\right\| \le \|T^*Tx\| \quad (x \in \mathcal{H}).$$

Since

$$T^*T = (T'^*T')^{-1}$$

and

$$T^2 = TT'(T'^*T')^{-1},$$

we get

$$\left\| TT'(T'^{*}T')^{-1}x \right\| \leq \left\| (T'^{*}T')^{-1}x \right\|,$$

or equivalently,

$$\left\| TT'y \right\| \leq \|y\|,$$

which shows that TT' is a contraction. Now,

$$\begin{aligned}(TT')^{*}T' &= (P^{-1}V^{*}PV^{*})VP^{-1} \\ &= P^{-1}V^{*}P(V^{*}V)P^{-1} \\ &= P^{-1}V^{*} \\ &= T'^{*}.\end{aligned}$$

Thus,

$$(TT')^*T' = T'^*,$$

and therefore T' is hyponormal. $\square$

Example

Consider the weighted Bergman space on $\mathbb{D}$, $L_a^2(\mathbb{D}, w)$, where $\log w$ is subharmonic. Then M_z is left invertible and M_z' is 2-hyperexpansive.

Generating m-tuples: Abstract Framework

$\mathbb{N}$: non-negative integers

$\mathbb{N}^m$: cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times).

For $p \equiv (p_1, \cdots, p_m)$ and $n \equiv (n_1, \cdots, n_m)$ in $\mathbb{N}^m$, we write

$$|p| := \sum_{i=1}^{m} p_i$$

and

$$p \leq n \ \text{ if } \ p_i \leq n_i \ \text{ for } \ i = 1, \cdots, m.$$

For $p \leq n$, we let

$$\binom{n}{p} := \prod_{i=1}^{m} \binom{n_i}{p_i}.$$

$\mathcal{H}$: Hilbert space

$B(\mathcal{H})$: the C^*-algebra of bounded linear operators on $\mathcal{H}$

For $T \equiv (T_1, \cdots, T_m)$ a tuple of commuting bounded linear operators acting on $\mathcal{H}$ (or on $B(\mathcal{H})$), we let $T^* := (T_1^*, \cdots, T_m^*)$ and $T^p := T_1^{p_1} \cdots T_m^{p_m}$.

Definition

Let $Q \equiv (Q_1, \cdots, Q_m)$ be a commuting m-tuple of positive, bounded, linear operators acting on $B(\mathcal{H})$. We refer to Q as a *generating m-tuple* on $\mathcal{H}$.

Example

Toral Generating m-tuple: $Q_t := (Q_1, \cdots, Q_m)$, where

$$Q_i(X) := T_i^* X T_i \quad (X \in B(\mathcal{H})).$$

Spherical Generating 1-tuple:

$$Q_s(X) := \sum_{i=1}^{m} T_i^* X T_i \quad (X \in B(\mathcal{H})).$$

$\{Q^n(I)\}_{n \in \mathbb{N}^m}$:= multisequence associated with Q.

$$\text{For } Q_t : \{T^{*n} T^n\}_{n \in \mathbb{N}^m} = \{(y_1 x_1, \cdots, y_m x_m)^n (T^*, T)\}_{n \in \mathbb{N}^m}$$

$$\text{For } Q_s : \left\{ \sum_{p \in \mathbb{N}^m, |p| = n} \frac{n!}{p!} T^{*p} T^p \right\}_{n \in \mathbb{N}} = \{(\sum_{i=1}^{m} y_i x_i)^n (T^*, T)\}_{n \in \mathbb{N}}$$

Why generating m-tuples?

- Unified treatment of operator tuples satisfying certain “toral" and “spherical" inequalities.
- Framework allows one to look at the operator tuples “related" to different domains in $\mathbb{C}^m$, for example, operator 2-tuples (T_1, T_2) related to the Reinhardt domain

$$\left\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\right\},$$

where the generating 1-tuple associated with (T_1, T_2) is given by

$$Q(X) := T_1^* X T_1 + T_2^{*2} X T_2^2 \quad (X \in B(\mathcal{H})).$$

Example

For a 4-tuple (T_1, T_2, T_3, T_4), consider

$$Q_1(X) := T_1^* X T_1 + T_2^* X T_2,\; Q_2(X) := T_3^* X T_3 + T_4^* X T_4 \quad (X \in B(\mathcal{H})).$$

(Q_1, Q_2) is related to the bi-ball

$$\left\{(z_1, z_2, w_1, w_2) \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 < 1, |w_1|^2 + |w_2|^2 < 1\right\}.$$

- We believe that the theory of generating tuples provides, to a certain extent, a *c*oordinate-free approach to multivariable operator theory.
- Gleason-Richter consider operator tuples satisfying certain “spherical" equalities; for instance, spherical generating tuples Q_s satisfying $B_n(Q_s) = 0$, where (in general)

$$B_n(Q) := \sum_{p \in \mathbb{N}^m, 0 \le p \le n} (-1)^{|p|} \binom{n}{p} Q^p(I),$$

- Generating tuples admit a *polynomial functional calculus*. For a finite set $\{c_\alpha\} \subseteq \mathbb{C}$ and a generating m-tuple R, let

$$p(R) := \sum_\alpha c_\alpha R^\alpha,$$

where $p(z) \equiv \sum_\alpha c_\alpha z^\alpha$. The polynomial functional calculus for R is given by $\Phi : p \longmapsto p(R)(I)$. We can actually show that R actually admits a *continuous functional calculus*, under some additional positivity hypotheses.

Hypercontractive Generating Tuples

Recall that

$$B_n(Q) := \sum_{p \in \mathbb{N}^m, 0 \le p \le n} (-1)^{|p|} \binom{n}{p} Q^p(I),$$

where $Q^0(I) := I$.

(a) Q is *contractive* if $B_n(Q) \geq 0$ for all $n \in \mathbb{N}^m$ with $|n| = 1$;

(b) Q is *completely hypercontractive* if

$$B_n(Q) \geq 0 \quad \text{for all } n \in \mathbb{N}^m.$$

Remark

Q on $\mathcal{H}$ is completely hypercontractive if and only if for every $h \in \mathcal{H}$ there is a positive Radon measure μ_h on $[0,1]^m$ such that

$$(Q^n(I)h, h) = \int_{[0,1]^m} t^n d\mu_h(t) \quad (n \in \mathbb{N}^m).$$

It follows that μ_h is a finite Borel measure.

Recall: a *semispectral measure* is a positive operator-valued measure defined on a Borel σ-algebra, which is countably additive in the weak operator topology.

Lemma

Q on $\mathcal{H}$ is completely hypercontractive if and only if there exists a unique semispectral measure E on $[0,1]^m$ such that

$$Q^n(I) = \int_{[0,1]^m} t^n dE(t) \text{ (for all } n \in \mathbb{N}^m).$$

In this case, there exist a Hilbert space $\mathcal{K}$, an operator $R \in B(\mathcal{H}, \mathcal{K})$, and a commuting m-tuple A of positive contractions on $\mathcal{K}$ such that

$$Q^n(I) = R^* A^n R \quad \text{(for all } n \in \mathbb{N}^m).$$

Motivation for Structure Theorem

Proposition

(Toral Case) S subnormal m-tuple on $\mathcal{H}$ with a normal extension N; Q_t, P_t the toral generating m-tuples associated with S and N, respectively. Then:

(i) For all $n \in \mathbb{N}$, we have

$$Q_t^n(I)h = P_{\mathcal{H}} P_t^n(I)h \quad (h \in \mathcal{H}).$$

(ii) If the Taylor joint spectrum $\sigma(N)$ of N is contained in the closed unit polydisc $\overline{\mathbb{D}}^m$ in $\mathbb{C}^m$, then Q_t is completely hypercontractive.

Proposition

(Spherical Case) S subnormal m-tuple on $\mathcal{H}$ with a normal extension N. Let Q_s, P_s denote the spherical generating 1-tuples associated with S and N respectively. Then:

(i) For all $n \in \mathbb{N}$, one has

$$Q_s^n(I)h = P_{\mathcal{H}} P_s^n(I)h \quad (h \in \mathcal{H}).$$

(ii) If the Taylor joint spectrum $\sigma(N)$ of N is contained in the closed unit ball $\overline{\mathbb{B}_1^m}$ in $\mathbb{C}^m$, then Q_s is completely hypercontractive.

Definition

$Q \equiv (Q_1, \cdots, Q_m)$ on $\mathcal{H}$ is *multiplicative* if $Q^n(I) = Q(I)^n$ for every $n \in \mathbb{N}^m$, where

$$Q(I) := (Q_1(I), \cdots, Q_m(I)).$$

Given two generating tuples P and Q on $\mathcal{K}$ and $\mathcal{H}$ respectively, we say that P is a *multiplicative dilation* of Q if P is multiplicative, $\mathcal{H} \subseteq \mathcal{K}$, and

$$Q^n(I)h = P_{\mathcal{H}} P^n(I)h \quad (n \in \mathbb{N}^m, h \in \mathcal{H}).$$

A multiplicative dilation P of Q is said to be *minimal* if

$$\mathcal{K} = \bigvee \{P^n(I)h : n \in \mathbb{N}^m, h \in \mathcal{H}\}.$$

Theorem (Structure)

Every completely hypercontractive generating m-tuple Q on $\mathcal{H}$ admits a minimal multiplicative dilation P on $\mathcal{K}$. Moreover, the semispectral measure E occurring in the representation of Q is related to the spectral measure F of $P(I) := (P_1(I), \cdots, P_m(I))$ by

$$E(\sigma) = P_{\mathcal{H}} F(\sigma)|_{\mathcal{H}} \text{ (for } \sigma \text{ a Borel subset of } [0,1]^m).$$

Furthermore, if P and S acting on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, are two minimal multiplicative dilations of Q then there exists a Hilbert space isomorphism $U : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ such that $Uh = h$ $(h \in \mathcal{H})$ and

$$UP^n(I) = S^n(I)U \ (n \in \mathbb{N}^m).$$

Sketch of Proof

Let $\mathcal{S}$ be the operator system

$$\mathcal{S} := \{p \in C([0,1]^m) : p \in \mathbb{C}[t_1, \cdots, t_m]\},$$

Given $p \in \mathcal{S}$, write $p(z) \equiv \sum_{k \in \mathbb{N}^m, |k| \leq n} \alpha_k t^k$ and consider the mapping $\phi : \mathcal{S} \longrightarrow B(\mathcal{H})$

$$\phi(p) \equiv \phi \left(\sum_{k \in \mathbb{N}^m, |k| \leq n} \alpha_k t^k \right) := \sum_{k \in \mathbb{N}^m, |k| \leq n} \alpha_k Q^k(I) \; (n \in \mathbb{N}, \alpha_k \in \mathbb{C}).$$

We then have

$$(\phi(p)h, h) = \int_{[0,1]^m} p(t) d\mu_h(t)$$

for every $p \in \mathcal{S}$ and every $h \in \mathcal{H}$, where μ_h is a positive Borel measure given by

$$\mu_h(\sigma) = (E(\sigma)h, h) \ \ (\sigma \text{ a Borel subset of } [0,1]^m)$$

with E the semispectral measure considered before. Thus ϕ is positive on $\mathbb{C}[t_1, \cdots, t_m]$. By the Stone-Weierstrass Theorem we may extend ϕ to a positive map $\tilde{\phi}$ on the whole of $C([0,1]^m)$. But any positive map from $C([0,1]^m)$ into a C^*-algebra is completely positive, so we can apply Stinespring's Dilation Theorem.

Corollary

Let Q be a generating m-tuple on $\mathcal{H}$. TFAE:

(i) Q is completely hypercontractive.

(ii) Q admits a minimal multiplicative and contractive dilation P.

In this case, one has

$$\|f(Q)(I)\| \leq \sup\{|f(t)| : t \in [0,1]^m\} \text{ (for every } f \in C([0,1]^m)).$$

Corollary

A completely hypercontractive generating m-tuple Q on $\mathcal{H}$ satisfies

$$Q_i^2(I) \geq Q_i(I)^2 \quad (i = 1, \cdots, m).$$

Corollary (Spectral Result)

Let S be a commuting m-tuple of operators on $\mathcal{H}$ such that Q_s is completely hypercontractive. Then

$$\sigma_T(S) \subseteq \overline{\mathbb{B}_r^m},$$

where $r := \sqrt{\|P(I)\|} \leq 1$ and P is the minimal multiplicative dilation.

Proof uses

Lemma (V. Müller and A. Soltysiak)

Let T be a commuting m-tuple of bounded linear operators on a Hilbert space. Then

$$\sup_{(z_1,\cdots,z_n)\in\sigma(T)} \left(|z_1|^2 + \cdots + |z_n|^2\right)^{\frac{1}{2}} = \lim_{n\to\infty} \|Q_s^n(I)\|^{\frac{1}{2n}},$$

Proposition

Let $p > 0$. If Q is completely hypercontractive and p-isometric, then Q is isometric.

Multivariable Weighted Shifts

For $m \geq 1$, let $\{e_n\}_{n \in \mathbb{N}^m}$ be an orthonormal basis for $\mathcal{H}$, and let $\left\{w_n^{(i)} : 1 \leq i \leq m, n \in \mathbb{N}^m\right\}$ be a bounded subset of the positive real line. An m-variable weighted shift
$T \equiv (T_1, \cdots, T_m) : \{w_n^{(i)}\}$

$$T_i e_n := w_n^{(i)} e_{n+\epsilon(i)} \; (1 \leq i \leq m),$$

where $\epsilon(i)$ is the m-tuple with 1 in the i-th entry and zeros elsewhere.

Example

$m, a > 0$, $m \geq a$, $\mathcal{H}_{a,m}$: reproducing kernel Hilbert space of analytic functions on $\mathbb{B}_1^m$ with kernel

$$\frac{1}{(1 - \langle z, w \rangle)^a},$$

where $(z, w) := \sum_{i=1}^m z_i \overline{w}_i$ $(z, w \in \mathbb{B}_1^m)$. Then $M_{z,a}$ acting on $\mathcal{H}_{a,m}$ is a spherical $(m - a + 1)$-isometry. $M_{z,a}$ is also the multivariable weighted shift with weights

$$\left\{ \sqrt{\frac{n_i + 1}{|n| + a}} : 1 \leq i \leq m, \; n \in \mathbb{N}^m \right\}.$$

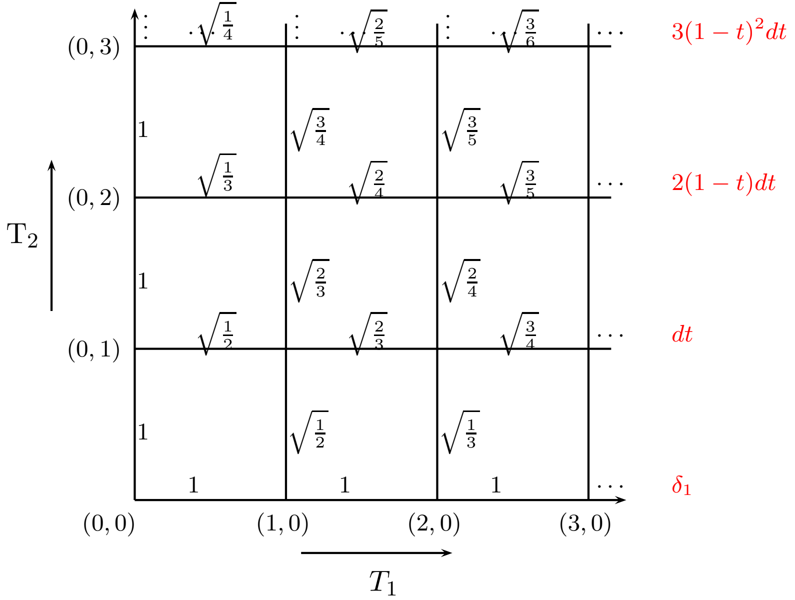
If $a \neq m$, then $M_{z,a}$ is not a spherical isometry, and therefore $M_{z,a}$ is not completely hypercontractive, and a fortiori not subnormal.

Remark

If we choose $a = 1$ and $m = 2$, that is, the space is $\mathcal{H}_{1,2}$, we obtain the Drury-Arveson space. By the preceding example, the Drury-Arveson m-shift $M_{z,1}$ is not subnormal. Of course, one can also use the Six-Point Test to prove that the Drury-Arveson 2-shift is not even jointly hyponormal.

Or even the fact that the Lebesgue measure on $[0, 1]$ is not absolutely continuous w.r.t. δ_1.

Weight Diagram of the Drury-Arveson Shift



Hyperexpansive Generating Tuples

A generating tuple Q is *completely hyperexpansive* if

$$B_n(Q) \leq 0 \text{ for all } n \in \mathbb{N}^m \setminus \{0\}.$$

p-hyperexpansive if

$$B_n(Q) \leq 0 \text{ for all } n \in \mathbb{N}^m \setminus \{0\} \text{ with } |n| \leq p.$$

p-expansive if

$$B_n(Q) \leq 0 \text{ for all } n \in \mathbb{N}^m \setminus \{0\} \text{ with } |n| = p.$$

T is a *toral* (resp. spherical) complete hyperexpansion if the toral generating m-tuple (resp. the spherical generating 1-tuple) associated with T is completely hyperexpansive. Similarly for p-expansions, p-hyperexpansions, p-isometries.

The Drury-Arveson 2-shift $M_{z,1}$ acting on the Hilbert space $\mathcal{H}_{1,2}$ is spherically 2-isometric. By the next result, $M_{z,1}$ is a spherical complete hyperexpansion.

Proposition

Let Q be a 2-expansive generating m-tuple on $\mathcal{H}$. Then:

(i) Q is 2-hyperexpansive. (ii) For each i and $0 \neq h \in \mathcal{H}$, the sequence

$$\left\{ \frac{\left\| \sqrt{Q_i^{n+1}(I)}h \right\|}{\left\| \sqrt{Q_i^{n}(I)}h \right\|} \right\}_{n \in \mathbb{N}} \searrow 1.$$

(iii) Q is 2-isometric if and only if

$$Q^n(I) = I + \sum_{i=1}^{m} n_i(Q_i(I) - I) \; (n \in \mathbb{N}^m).$$

In this case, Q is completely hyperexpansive.

(iv) $Q_i^2(I) \leq Q_i(I)^2 \; (1 \leq i \leq m)$.

Corollary (Spectral Result)

Let Q be a 2-hyperexpansive generating 1-tuple. Then

$$\lim_{n\to\infty} \|Q^n(I)\|^{\frac{1}{n}} = 1.$$

In particular, the Taylor spectrum of any spherical 2-hyperexpansion is contained in the closed unit ball.

Given a generating 1-tuple Q on $\mathcal{H}$, we can associate a unilateral weighted shift by

$$W_{Q,h}\, e_n := \frac{\left\|\sqrt{Q^{n+1}(I)}h\right\|}{\left\|\sqrt{Q^n(I)}h\right\|}\, e_{n+1}\ (n \in \mathbb{N}).$$

It is easy to see that Q is completely hyperexpansive (resp. 2-hyperexpansive, resp. 2-isometric) if and only if so is $W_{Q,h}$, for each $h \in \mathcal{H}$.

If Q is 2-hyperexpansive, $W_{Q,h}$ is a compact perturbation of the unilateral shift, and

$$\sum_{k=1}^{\infty} \left| \frac{\left\| \sqrt{Q_s^{k+1}(I)}h \right\|^2}{\left\| \sqrt{Q_s^{k}(I)}h \right\|^2} - \frac{\left\| \sqrt{Q_s^{k}(I)}h \right\|^2}{\left\| \sqrt{Q_s^{k-1}(I)}h \right\|^2} \right| < \infty.$$

If Q_s is the spherical generating 1-tuple associated with the Drury-Arveson 2-shift, then the associated unilateral weighted shift W_{Q_s,f_0} is nothing but the classical *Dirichlet* shift!

Lemma (Lévy-Khinchin Representation)

A generating m-tuple Q on $\mathcal{H}$ is completely hyperexpansive if and only if there exist semispectral measures $E_1, \cdots, E_m$ on $[0,1]^m$ such that

$$\begin{aligned} Q^n(I) &= I + \sum_{i=1}^{m} n_i E_i(\{\mathbf{1}\}) \\ &\quad + \int_{[0,\,1]^m \setminus \{\mathbf{1}\}} (1 - t^n) \frac{d(E_1 + \cdots + E_m)(t)}{m - t_1 - \cdots - t_m} \quad (n \in \mathbb{N}^m). \end{aligned}$$

Theorem (Structure)

Q compl. hyperexp. on $\mathcal{H}$. For σ Borel, assume

$$F(\sigma) := \int_{\sigma} \frac{d(E_1 + \cdots + E_m)(t)}{m - t_1 - \cdots - t_m}$$

is a $B(\mathcal{H})$-valued Borel measure. Then there exist a multipl. gen. P on $\mathcal{K} \supseteq \mathcal{H}$ and bounded V such that

$$p(Q)(I) = -V^*(p(P)(I))V \text{ for any } p \in \mathcal{S},$$

where $\mathcal{S}$ is the self-adjoint subspace given by

$$\mathcal{S} \ := \left\{ p \in C([0,1]^m) : p \text{ poly } p(\mathbf{1}) = 0, \frac{\partial p}{\partial t_i}(\mathbf{1}) = 0 \ (1 \leq i \leq m) \right\}.$$

Moreover, P is minimal:

$$\mathcal{K} = \bigvee \{P^n(I)h : \ n \in \mathbb{N}^m, h \in \mathcal{H}\}.$$

Remark

(i) Any 2-isometric generating tuple satisfies the hypotheses of the Structure Theorem. (ii) In general, we cannot expect multiplicative dilations for completely hyperexpansive generating tuples. Indeed, any 2-hyperexpansive generating tuple with a multiplicative dilation is necessarily isometric.

Recall:

Proposition

Let Q be a completely hypercontractive generating m-tuple on $\mathcal{H}$, let p be a positive integer, and assume that Q is p-isometric. Then Q is isometric.

In the hyperexpansive case we have:

Proposition

Let Q be a 2-hyperexpansive generating m-tuple on $\mathcal{H}$, and assume that Q is p-isometric for some $p \geq 2$. Then Q is 2-isometric.

Toral Cauchy Dual Tuples

Recall that $T \in B(\mathcal{H})$ is left invertible if and only if T^*T is invertible.

Definition

Let $T \equiv (T_1, \cdots, T_m)$, with each T_i left invertible, and let $Q_t \equiv (Q_1, \cdots, Q_m)$ denote the toral generating m-tuple associated with T. The toral Cauchy dual of T is $T^{\mathfrak{t}} := (T_1^{\mathfrak{t}}, \cdots, T_m^{\mathfrak{t}})$, where $T_i^{\mathfrak{t}} := T_i(Q_i(I))^{-1} \quad (i = 1, \cdots, m)$.

Example

If $D_\lambda : \left\{\sqrt{\frac{\lambda+|n|}{1+|n|}}\right\}$ then $D_\lambda^{\mathfrak{t}} : \left\{\sqrt{\frac{1+|n|}{\lambda+|n|}}\right\}$. For $1 \leq \lambda \leq 2$,
$D_\lambda : \left\{\sqrt{\frac{\lambda+|n|}{1+|n|}}\right\}$ is a toral complete hyperexpansion, so
$D_\lambda^{\mathfrak{t}} : \left\{\sqrt{\frac{1+|n|}{\lambda+|n|}}\right\}$ is a torally contractive subnormal tuple.

Theorem

Let T be a toral 2-hyperexpansive m-tuple of commuting bounded linear operators on $\mathcal{H}$ and let $T^{\mathfrak{t}}$ denote the operator tuple torally Cauchy dual to T. Then for $1 \le i, j \le m$, one has

$$2(T_i^{\mathfrak{t}})^*((T_j^{\mathfrak{t}})^* T_j^{\mathfrak{t}})^{-1} T_i^{\mathfrak{t}} \le I + (T_i^{\mathfrak{t}})^* T_i^{\mathfrak{t}}((T_j^{\mathfrak{t}})^* T_j^{\mathfrak{t}})^{-2}(T_i^{\mathfrak{t}})^* T_i^{\mathfrak{t}}.$$

In particular, the operator tuple $T^{\mathfrak{t}}$ torally Cauchy dual to T consists of hyponormal contractions.

Conjecture: If T is 2-hyperexpansive, then $T^{\mathfrak{t}}$ is (jointly) hyponormal.

Corollary

Let T be a torally 2-hyperexpansive commuting m-variable weighted shift on $\mathcal{H}$. Let $C^*(T)$ denote the C^*-algebra generated by T and let J_T denote the commutator ideal of $C^*(T)$. Then there exists an isometric $*$-isomorphism $\phi : C^*(T)/J_T \longrightarrow C(\mathbb{T}^m)$ such that $\phi(T_i + J_T) = z_i$.

Spherical Cauchy Dual Tuples

Recall: T is *jointly left-invertible* if there exists $k > 0$ such that

$$T_1^* T_1 + \cdots + T_m^* T_m \geq kI.$$

Definition

Let T be a jointly left-invertible m-tuple, and let Q_s be the spherical generating tuple associated with T. The spherical Cauchy dual of T is $T^{\mathfrak{s}} := (T_1^{\mathfrak{s}}, \cdots, T_m^{\mathfrak{s}})$, where $T_i^{\mathfrak{s}} := T_i(Q_s(I))^{-1} \quad (i = 1, \cdots, m)$.

Summary

Although the theories of hypercontractions and hyperexpansions are not mutually dual, they can still be developed in tandem, and results in one can be generally adapted to results in the other.

Cauchy duality (toral, spherical, etc) can lead to a better understanding of both theories, and can also produce spectral results for multiplication operators on functional Hilbert spaces over Reinhardt domains.