One-step extensions of subnormal 2-variable weighted shifts

Raúl E. Curto, Sang Hoon Lee and Jasang Yoon

Abstract. We study one-step extensions of 2-variable weighted shifts. We provide necessary and sufficient conditions for the subnormality of such extensions, by using backward extensions, disintegration of measures, and k-hyponormality techniques from the theory of 2-variable weighted shifts. We apply our results to solve an interpolation problem for measures on $\mathbb{R}_+^2$.

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Contents

1. Introduction

Consider the following reconstruction-of-the-measure problem: Given two probability measures μ_1 and μ_2 on $\mathbb{R}_+^2$, find necessary and sufficient conditions for the existence of a probability measure μ on $\mathbb{R}_+^2$ with $supp\mu \nsubseteq (\mathbb{R}_+ \times 0) \cup (0 \times \mathbb{R}_+)$ such that

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$$\frac{s\ d\mu(s,t)}{\int s\ d\mu(s,t)} = d\mu_1(s,t) \text{ and } \frac{t\ d\mu(s,t)}{\int t\ d\mu(s,t)} = d\mu_2(s,t). \tag{1.1}$$

Note that (1.1) readily implies that $td\mu_1(s,t) = \lambda s d\mu_2(s,t)$ for some $\lambda > 0$; this condition, while clearly necessary for the existence of μ, is by no means sufficient, as we show in Example 3.

In this paper we solve this interpolation problem using techniques from multivariable operator theory, namely the theory of 2-variable weighted shifts. To describe our results, we first need to introduce some notation and terminology.

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geq TT^*$.

Besides their relevance for the construction of examples and counterexamples in Hilbert space operator theory, weighted shifts can also be used to detect properties such as subnormality, via the Lambert-Lubin Criterion ([15], [16]): A commuting pair (T_1, T_2) of injective operators acting on a Hilbert space $\mathcal{H}$ admits a commuting normal extension if and only if for every nonzero vector $x \in \mathcal{H}$ the 2-variable weighted shift with weights $\alpha_{(i,j)} := \frac{\|T_1^{i+1}T_2^j x\|}{\|T_1^i T_2^j x\|}$ and $\beta_{(i,j)} := \frac{\|T_1^i T_2^{j+1} x\|}{\|T_1^i T_2^j x\|}$ has a normal extension.

For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S,T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive semidefinite on the direct sum of n copies of $\mathcal{H}$ (cf. [1], [2], [4], [7], [9], [13]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty} \in \ell^{\infty}(\mathbb{Z}_+)$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1, & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & \text{if } k > 0. \end{cases}$$

It is easy to see that $W_\alpha \equiv shift(\alpha_0, \alpha_1, \cdots)$ is never normal, and that it is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$.

Similarly, consider double-indexed positive bounded sequences $\alpha \equiv \{\alpha_{(k_1,k_2)}\}, \beta \equiv \{\beta_{(k_1,k_2)}\} \in \ell^{\infty}(\mathbb{Z}_+^2)$, $(k_1,k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$;

in a canonical way, $\ell^2(\mathbb{Z}_+^2)$ is isometrically isomorphic to $\ell^2(\mathbb{Z}_+)\bigotimes\ell^2(\mathbb{Z}_+)$. We define the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ by

$$T_1 e_{(k_1,k_2)} := \alpha_{(k_1,k_2)} e_{(k_1,k_2)+\varepsilon_1}$$

$$T_2 e_{(k_1,k_2)} := \beta_{(k_1,k_2)} e_{(k_1,k_2)+\varepsilon_2},$$

where $\varepsilon_1 := (1,0)$ and $\varepsilon_2 := (0,1)$ (see Figure 1). Clearly, T_1 commutes with T_2 if and only if

$$\beta_{(k_1,k_2)+\varepsilon_1}\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2)+\varepsilon_2}\beta_{(k_1,k_2)} \quad \left(\text{all } (k_1,k_2)\in\mathbb{Z}_+^2\right). \tag{1.2}$$

In an entirely similar way one can define multivariable weighted shifts.

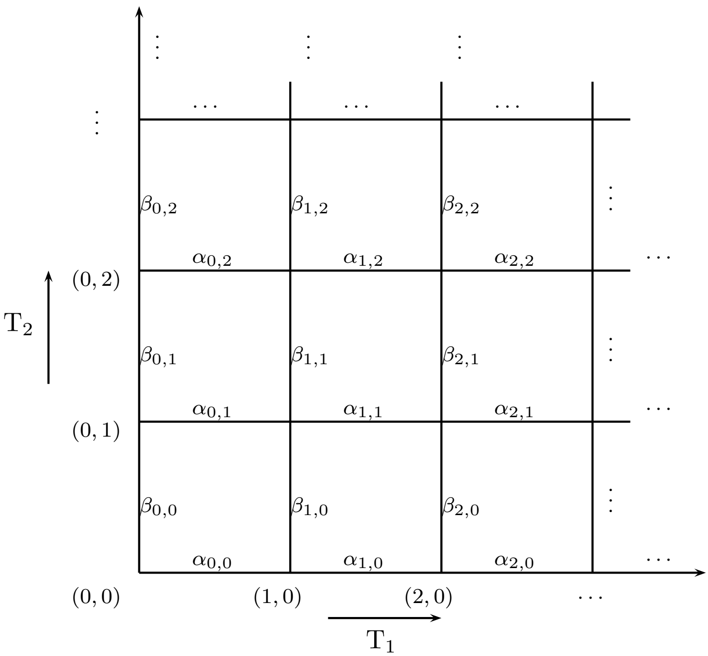

FIGURE 1. Weight diagram of a 2-variable weighted shift

Given $(k_1,k_2)\in\mathbb{Z}_+^2$, the *moment* of (α,β) of order (k_1,k_2) is

$$\gamma_{(k_1,k_2)}(\alpha,\beta) := \begin{cases} 1, & \text{if } (k_1,k_2)=(0,0) \\ \alpha_{(0,0)}^2\cdots\alpha_{(k_1-1,0)}^2, & \text{if } k_1\geq 1 \text{ and } k_2=0 \\ \beta_{(0,0)}^2\cdots\beta_{(0,k_2-1)}^2, & \text{if } k_1=0 \text{ and } k_2\geq 1 \\ \alpha_{(0,0)}^2\cdots\alpha_{(k_1-1,0)}^2\cdot\beta_{(k_1,0)}^2\cdots\beta_{(k_1,k_2-1)}^2, & \text{if } k_1\geq 1 \text{ and } k_2\geq 1. \end{cases} \tag{1.3}$$

We remark that, due to the commutativity condition (1.2), $\gamma_{(k_1,k_2)} \equiv \gamma_{(k_1,k_2)}(\alpha,\beta)$ can be computed using any nondecreasing path from $(0,0)$ to (k_1,k_2).

We also recall a well known characterization of subnormality for multivariable weighted shifts $\mathbf{T} \equiv (T_1,\cdots,T_n)$ [14], due to C. Berger [3, II.6.10] and independently established by R. Gellar and L.J. Wallen [12] in the 1-variable case; for simplicity, we state it in the case $n=2$: $W_{(\alpha,\beta)} \equiv (T_1,T_2)$ is subnormal if and only if there is a probability measure μ defined on the rectangle $R=[0,a_1]\times[0,a_2]$ where $a_i = \|T_i\|^2$ such that

$$\gamma_{(k_1,k_2)} = \int_R s^{k_1} t^{k_2} \, d\mu(s,t),$$

for all $(k_1,k_2) \in \mathbb{Z}_+^2$.

In this article we consider the following problem.

Problem 1.1. Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively. Find necessary and sufficient conditions on $\mu_{\mathcal{M}}$, $\mu_{\mathcal{N}}$ and β_{00} for the subnormality of $W_{(\alpha,\beta)}$.

Our main result (Theorem 1.7) provides a concrete solution to Problem 1.1.

We now recall some auxiliary facts needed for the proof of our main results. In the single variable case, if W_α is subnormal with Berger measure σ, and if we let $\mathcal{L}_j := \bigvee\{e_{k_1} : k_1 \geq j\}$ denote the invariant subspace obtained by removing the first j vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{L}_j}$ is $d\sigma_{\mathcal{L}_j}(s) = \frac{s^j}{\gamma_j} d\sigma(s)$. Similarly, for an arbitrary 2-variable weighted shift $W_{(\alpha,\beta)}$, we let $\mathcal{M}$ (resp. $\mathcal{N}$) be the invariant subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis associated to indices $\mathbf{k}=(k_1,k_2)$ with $k_1 \geq 0$ and $k_2 \geq 1$ (resp. $k_1 \geq 1$ and $k_2 \geq 0$) (see Figure 2).

Lemma 1.2. *(Subnormal backward extension of a* 1*-variable weighted shift) (cf* [5, Proposition 8], [10, Proposition 1.5]*) Let $W_\alpha|_{\mathcal{L}_1}$ be subnormal, with Berger measure $\mu_{\mathcal{L}_1}$. Then W_α is subnormal (with Berger measure μ) if and only if the following conditions hold:*
(i) $\frac{1}{s} \in L^1(\mu_{\mathcal{L}_1})$
(ii) $\alpha_0^2 \leq \left(\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{L}_1})}\right)^{-1}$.
In this case, $d\mu(s) = \frac{\alpha_0^2}{s} d\mu_{\mathcal{L}_1}(s) + \left(1-\alpha_0^2\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{L}_1})}\right) d\delta_0(s)$*, where δ_0 denotes Dirac measure at* 0*. In particular, W_α is never subnormal when* $\mu_{\mathcal{L}_1}(\{0\}) > 0$.

To check the subnormality of 2-variable weighted shifts, we need to introduce some definitions.

Definition 1.3. ([10], [11], [17]) (i) Let μ and ν be two positive measures on $X \equiv \mathbb{R}_+$. We say that $\mu \leq \nu$ on X, if $\mu(E) \leq \nu(E)$ for all Borel subset $E \subseteq X$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$

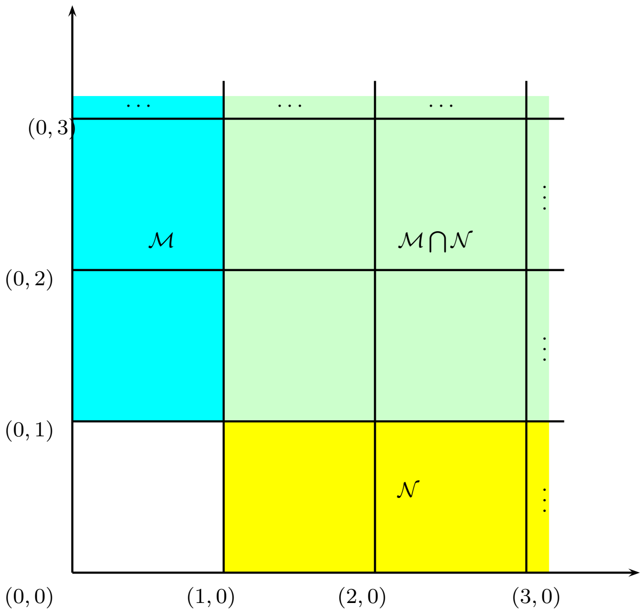

FIGURE 2. The subspaces $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M}\bigcap\mathcal{N}$

such that $f \geq 0$ on X.
(ii) Let μ be a probability measure on $X \times Y$, with $Y \equiv \mathbb{R}_+$, and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $X \times Y$ is given by $d\mu_{ext}(s,t) := \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t)$.
(iii) Given a measure μ on $X \times Y$, the *marginal measure* μ^X is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto X. Thus $\mu^X(E) = \mu(E \times Y)$, for every $E \subseteq X$; equivalently, $d\mu^X(s) = \int_Y d\mu(s,t)$. (Observe that μ^X is a probability measure whenever μ is.)

In what follows, we will have occasion to use the marginal measure of the extremal measure associated with $\mu_{\mathcal{M}}$, which we will denote by $(\mu_{\mathcal{M}})^X_{ext}$.

Lemma 1.4. *(Subnormal backward extension of a 2-variable weighted shift* [10]*) Assume that* $W_{(\alpha,\beta)}|_{\mathcal{M}}$ *is subnormal with Berger measure* $\mu_{\mathcal{M}}$ *and that* $shift(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with Berger measure* σ. *Then* $W_{(\alpha,\beta)}$ *is subnormal if and only if the following conditions hold:*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq \left(\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}\right)^{-1}$;

(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \sigma$.
Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 1$, *then* $(\mu_{\mathcal{M}})_{ext}^X = \sigma$. *In the case when* $W_{(\alpha,\beta)}$ *is subnormal, the Berger measure* μ *of* $W_{(\alpha,\beta)}$ *is given by*

$$\begin{aligned} d\mu(s,t) &= \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) \\ &\quad + \left(d\sigma(s) - \beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s) \right) d\delta_0(t). \end{aligned}$$

Lemma 1.5. *(*[11, Theorem 3.1]*) Let* μ *be the Berger measure of a subnormal 2-variable weighted shift, and let* σ *be the Berger measure of the associated 0-th horizontal 1-variable* $shift(\alpha_{00}, \alpha_{10}, \cdots)$. *Then* $\sigma = \mu^X$.

To detect the k-hyponormality of 2-variable weighted shifts, we use the following result.

Lemma 1.6. *(*[6, Theorem 2.4]*) The following statements are equivalent:*
(i) $W_{(\alpha,\beta)}$ *is k-hyponormal;*
(ii) $M_{(k_1,k_2)}(k)\left(W_{(\alpha,\beta)}\right) := \left(\gamma_{(k_1,k_2)+(m,n)+(p,q)}\right)_{\substack{0\leq m+n\leq k \\ 0\leq p+q\leq k}} \geq 0$ *for all* $(k_1,k_2) \in \mathbb{Z}_+^2$.

Our main result, which follows, provides a concrete solution of Problem 1.1 in terms of $\mu_{\mathcal{M}}$, $\mu_{\mathcal{N}}$ and β_{00}.

Theorem 1.7. *Assume that* $W_{(\alpha,\beta)}|_{\mathcal{M}}$ *and* $W_{(\alpha,\beta)}|_{\mathcal{N}}$ *are subnormal with Berger measures* $\mu_{\mathcal{M}}$ *and* $\mu_{\mathcal{N}}$, *respectively, and let* $c := \frac{\int s \, d\mu_{\mathcal{M}}}{\int t \, d\mu_{\mathcal{N}}} \equiv \frac{\alpha_{01}^2}{\beta_{10}^2}$. *Then* $W_{(\alpha,\beta)}$ *is subnormal if and only if the following four conditions hold:*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\frac{1}{s} \in L^1(\mu_{\mathcal{N}})$;
(iii) $c\beta_{00}^2 \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} \leq 1$;
(iv) $\beta_{00}^2 \left\{ \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X + c \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} \delta_0 - \frac{c}{s}(\mu_{\mathcal{N}})^X \right\} \leq \delta_0$.

We shall give the proof of Theorem 1.7 (and of Theorem 1.8 below) in Section 2. To state the following result, recall from [8] that when the core of a 2-variable weighted shift $W_{(\alpha,\beta)}$ is of tensor form, it follows that the Berger measure of the restriction of $W_{(\alpha,\beta)}$ to $\mathcal{M} \cap \mathcal{N}$ splits as a Cartesian product of two 1-variable measures. As a special case of Theorem 1.7, we now have:

Theorem 1.8. *(The case when* $W_{(\alpha,\beta)}$ *has a core of tensor form; see Figure 3.) Assume that* $W_{(\alpha,\beta)}|_{\mathcal{M}}$ *and* $W_{(\alpha,\beta)}|_{\mathcal{N}}$ *are subnormal with Berger measures* $\mu_{\mathcal{M}}$ *and* $\mu_{\mathcal{N}}$, *respectively, and let* $\rho := \mu_{\mathcal{M}}^X$, *i.e.,* ρ *is the Berger measure of* $shift(\alpha_{01}, \alpha_{11}, \cdots)$. *Also assume that* $\mu_{\mathcal{M}\cap\mathcal{N}} = \xi \times \eta$ *for some 1-variable probability measures* ξ *and* η. *Then* $W_{(\alpha,\beta)}$ *is subnormal if and only if the following conditions hold:*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$;
(iii) $\left(\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\tau_1)}\right) \rho = \left(\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}\right) \rho \leq \sigma$.

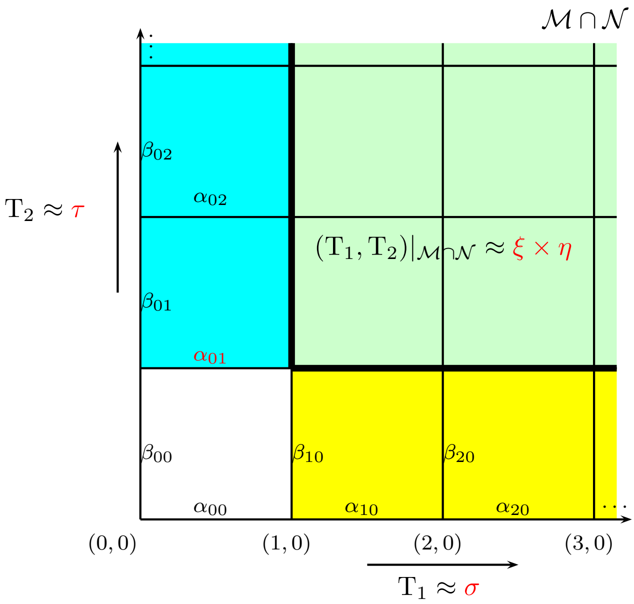

FIGURE 3. A 2-variable weighted shift with core of tensor form

Remark 1.9. Let $W_{(\alpha,\beta)}$ be a 2-variable weighted shift given as in Theorem 1.8, and assume that $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$. By combining Theorems 1.7 and 1.8, we have:

$$\left(\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\tau_1)}\right)\rho \leq \sigma$$

$$\Longleftrightarrow \beta_{00}^2 \left\{\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X + c\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})}\delta_0 - \frac{c}{s}(\mu_{\mathcal{N}})^X\right\} \leq \delta_0.$$

2. Proofs of Theorems 1.7 and 1.8

Proof of Theorem 1.7. ($\Longrightarrow$) Assume $W_{(\alpha,\beta)}$ is subnormal; we need to establish conditions (i) – (iv). By Lemma 1.4, conditions (i) and (ii) are clear. Since $W_{(\alpha,\beta)}|_{\mathcal{N}}$ is subnormal, the 0-th horizontal 1-variable weighted shift for $W_{(\alpha,\beta)}|_{\mathcal{N}}$, that is, $shift(\alpha_{10},\alpha_{20},\cdots)$, is also subnormal. Moreover, $shift(\alpha_{00},\alpha_{10},\cdots)$ is subnormal, and let σ denote its Berger measure. By Lemma 1.2, we have $\alpha_{00}^2 \left\|\frac{1}{s}\right\|_{L^1((\sigma)_{\mathcal{L}_1})} \leq 1$ (recall that $\mathcal{L}_1 := \bigvee\{e_{k_1} : k_1 \geq 1\}$).

By the commutativity of $W_{(\alpha,\beta)}$, we observe that $c\beta_{00}^2 = \alpha_{00}^2$, and therefore

$$\alpha_{00}^2 \left\| \frac{1}{s} \right\|_{L^1\left((\sigma)_{\mathcal{L}_1}\right)} \le 1 \Longleftrightarrow c\beta_{00}^2 \left\| \frac{1}{s} \right\|_{L^1\left((\sigma)_{\mathcal{L}_1}\right)} \le 1,$$

which implies condition (iii), once we note that $(\mu_{\mathcal{N}})^X = (\sigma)_{\mathcal{L}_1}$ (by Lemma 1.5, applied to $\mu_{\mathcal{N}}$ with $j = 0$), and therefore

$$\int_X \int_Y \frac{1}{s}\, d\mu_{\mathcal{N}}(s,t) = \int_X \frac{1}{s} \int_Y d\mu_{\mathcal{N}}(s,t) = \int_X \frac{1}{s}\, d\mu_{\mathcal{N}}^X(s);$$

that is,

$$\left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} = \left\| \frac{1}{s} \right\|_{L^1\left((\sigma)_{\mathcal{L}_1}\right)}.$$

To prove condition (iv), we observe that, by Lemma 1.4(iii),

$$\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \le \sigma. \tag{2.1}$$

On the other hand, by Lemma 1.2,

$$d\sigma(s) = \frac{\alpha_{00}^2}{s} d(\mu_{\mathcal{N}})^X(s) + \left(1 - \alpha_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})}\right) d\delta_0(s),$$

so (2.1) is equivalent to

$$\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d\left(\mu_{\mathcal{M}}(s,t)\right)_{ext}^X \le \frac{\alpha_{00}^2}{s} d(\mu_{\mathcal{N}})^X(s) + \left(1 - \alpha_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})}\right) d\delta_0(s),$$

which in turn is equivalent to

$$\beta_{00}^2 \left\{ \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d\left(\mu_{\mathcal{M}}(s,t)\right)_{ext}^X + c \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} d\delta_0(s) - \frac{c}{s} d(\mu_{\mathcal{N}})^X(s) \right\} \le d\delta_0(s),$$

and finally equivalent to

$$\beta_{00}^2 \left\{ \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X + c \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} \delta_0 - \frac{c}{s} (\mu_{\mathcal{N}})^X \right\} \le \delta_0,$$

which establishes (iv).

($\Longleftarrow$) Suppose that conditions (i), (ii), (iii) and (iv) hold for $W_{(\alpha,\beta)}$, and recall that

$$d\sigma(s) = \frac{\alpha_{00}^2}{s} d(\mu_{\mathcal{N}})^X(s) + \left(1 - \alpha_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})}\right) d\delta_0(s). \tag{2.2}$$

Condition (i) implies condition (i) in Lemma 1.4. Moreover, from (2.1) and the fact that $\int d\left(\mu_{\mathcal{M}}(s,t)\right)_{ext}^X = 1$, we obtain condition (ii) in Lemma 1.4.

To prove condition (iii) in Lemma 1.4, start with condition (iv) above, recall that $c\beta_{00}^2 = \alpha_{00}^2$, that $(\mu_{\mathcal{N}})^X = (\sigma)_{\mathcal{L}_1}$, and that

$$\frac{\alpha_{00}^2}{s} (\sigma)_{\mathcal{L}_1} + \left(1 - \alpha_{00}^2 \left\| \frac{1}{s} \right\|_{L^1\left((\sigma)_{\mathcal{L}_1}\right)}\right) \delta_0 = \sigma.$$

Then

$$\beta_{10}^2\sigma = \frac{\beta_{10}^2\alpha_{00}^2}{s}\mu_{\mathcal{N}}^X + \beta_{10}^2\delta_0 - \beta_{10}^2\alpha_{00}^2\left\|\frac{1}{s}\right\|_{L^1((\mu_{\mathcal{N}}))}\delta_0.$$

From the commutativity of $W_{(\alpha,\beta)}$, it follows that

$$\beta_{10}^2\sigma = \beta_{10}^2\alpha_{00}^2\frac{\mu_{\mathcal{N}}^X}{s} + \beta_{10}^2\delta_0 - \beta_{00}^2\alpha_{01}^2\left\|\frac{1}{s}\right\|_{L^1((\mu_{\mathcal{N}}))}\delta_0,$$

so that

$$\sigma = c\beta_{00}^2\frac{\mu_{\mathcal{N}}^X}{s} + \delta_0 - c\beta_{00}^2\left\|\frac{1}{s}\right\|_{L^1((\mu_{\mathcal{N}}))}\delta_0.$$

Therefore, condition (iv) implies that

$$\beta_{00}^2\left(\left\|\frac{1}{t}\right\|_{L^1((\mu_{\mathcal{M}}))}(\mu_{\mathcal{M}})_{ext}^X + \frac{\delta_0-\sigma}{\beta_{00}^2}\right) \leq \delta_0.$$

It follows that

$$\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1((\mu_{\mathcal{M}}))}(\mu_{\mathcal{M}})_{ext}^X + \delta_0 - \sigma \leq \delta_0,$$

and finally

$$\beta_{00}^2\left\|\frac{1}{t}\right\|_{L^1((\mu_{\mathcal{M}}))}(\mu_{\mathcal{M}})_{ext}^X \leq \sigma.$$

Thus, by (2.2), we have condition (iii) in Lemma 1.4, so our proof is complete. □

We now give the proof of Theorem 1.8.

Proof of Theorem 1.8. ($\Longrightarrow$) Assume $W_{(\alpha,\beta)}$ is subnormal. Clearly, conditions (i) and (ii) in Lemma 1.4 readily imply conditions (i) and (ii) here; thus, we only need to establish condition (iii). We will do this in three steps.
Step 1. We will prove that

$$(\mu_{\mathcal{M}})_{ext}^X = \rho.$$

We do this by using the results in [8]. Let τ denote the Berger measure of $shift(\beta_{00}, \beta_{01}, \cdots)$, and let τ_1 denote the Berger measure of $shift(\beta_{01}, \beta_{02}, \cdots)$. Let $\psi := \tau_1 - \alpha_{01}^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\eta$. By [8, Proposition 2.2] and $\mu_{\mathcal{M}} = \alpha_{01}^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\tilde{\xi}\times\eta + \delta_0\times\psi$, it follows that

$$(\mu_{\mathcal{M}})_{ext} = \alpha_{01}^2\frac{\xi}{s}\times\frac{\eta}{t\left\|\frac{1}{t}\right\|_{L^1(\eta)}} + \delta_0\times((\tau_1)_{ext} - \alpha_{01}^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}\frac{\eta}{t\left\|\frac{1}{t}\right\|_{L^1(\eta)}}),$$

and therefore

$$(\mu_{\mathcal{M}})_{ext}^X = \alpha_{01}^2\frac{\xi}{s} + \delta_0\times(1-\alpha_{01}^2\left\|\frac{1}{s}\right\|_{L^1(\xi)}) = \rho,$$

as desired.

Step 2. We now establish that

$$\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}=\left\|\frac{1}{t}\right\|_{L^1(\tau_1)}.$$

Lemma 1.4(iii) says that $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \sigma$, so that by Step 1 above we have $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \rho \leq \sigma$. Thus,

$$\begin{aligned}\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} &= \alpha_{01}^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \int_X \tilde{\xi} \int_Y \frac{\eta}{t} + \int_Y \frac{\psi}{t} \\ &= \alpha_{01}^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \left\|\frac{1}{t}\right\|_{L^1(\eta)} + \int_Y \frac{\tau_1}{t} - \alpha_{01}^2 \left\|\frac{1}{s}\right\|_{L^1(\xi)} \left\|\frac{1}{t}\right\|_{L^1(\eta)} \\ &= \left\|\frac{1}{t}\right\|_{L^1(\tau_1)},\end{aligned}$$

as desired.

Step 3. We finally prove condition (iii), by combining Steps 1 and 2; we leave the details to the reader.

($\Longleftarrow$) Assume that conditions (i), (ii) and (iii) hold for $W_{(\alpha,\beta)}$. Then clearly, conditions (i) and (ii) in Lemma 1.4 are satisfied. Since $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with Berger measures $\mu_{\mathcal{M}}$, condition (iii) is esaily estabished once we recall that $(\mu_{\mathcal{M}})_{ext}^X = \rho$. This concludes the proof. $\square$

3. An application

We will now see that one-step extensions may not exist, even under very favorable assumptions of subnormality for the restriction of $W_{(\alpha,\beta)}$ to $\mathcal{M} \vee \mathcal{N}$. For instance, both $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ can be unitarily equivalent, and yet for no β_{00} is $W_{(\alpha,\beta)}$ subnormal. To see this, let us assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with the Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively. Assume also that $Y = X$. Let $\mu_{\mathcal{M}} = \mu_{\mathcal{N}}$ be a diagonal measure ϵ on $X \times X$, that is, $supp\ \epsilon \subseteq \{(s,t) \in X \times X : s = t\}$; we loosely describe this by $d\epsilon(s,t) = d\epsilon(s,s) = d\epsilon(t,t)$. Then by the techniques of disintegration of measures ([3], [11, Theorem 3.1], [17, Theorem 2.6]), we can see that

$$\epsilon^X = \epsilon^Y, \quad \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} = \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = \left\|\frac{1}{s}\right\|_{L^1(\epsilon^X)} = \left\|\frac{1}{t}\right\|_{L^1(\epsilon^Y)}$$

and

$$(\mu_{\mathcal{M}})_{ext}^X = (\epsilon)_{ext}^X = \epsilon^X.$$

Thus, in Theorem 1.7 we have $c = 1$ and therefore

$$c\beta_{00}^2 \left\|\tfrac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} \leq 1 \Longleftrightarrow \beta_{00}^2 \left\|\tfrac{1}{s}\right\|_{L^1(\epsilon^X)} \leq 1$$

and

$$\beta_{00}^2 \left\{ \left\| \tfrac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X + \left\| \tfrac{1}{s} \right\|_{L^1(\mu_{\mathcal{N}})} d\delta_0(s) - \tfrac{d(\mu_{\mathcal{N}})^X}{s} \right\} \leq d\delta_0(s)$$

$$\Longleftrightarrow \beta_{00}^2 \left\{ \left\| \tfrac{1}{t} \right\|_{L^1(\epsilon^Y)} \epsilon^X + \left\| \tfrac{1}{t} \right\|_{L^1(\epsilon^Y)} \delta_0 - \tfrac{\epsilon^X}{s} \right\} \leq \delta_0.$$

We can summarize these calculations as follows.

Proposition 3.1. *Let* $W_{(\alpha,\beta)}$ *be the* 2*-variable weighted shift given above. Then* $W_{(\alpha,\beta)}$ *is subnormal if and only if*
(i) $\beta_{00}^2 \left\| \frac{1}{s} \right\|_{L^1(\epsilon^X)} \leq 1$*;*
(ii) $\beta_{00}^2 \left\{ \left\| \frac{1}{t} \right\|_{L^1(\epsilon^Y)} \epsilon^X + \left\| \frac{1}{t} \right\|_{L^1(\epsilon^Y)} \delta_0 - \frac{\epsilon^X}{s} \right\} \leq \delta_0$.

We now present a concrete example.

Example. Let $\mu_{\mathcal{M}} = \mu_{\mathcal{N}}$ be the 2-variable probability measure on $[0,1]^2$ with moments $\gamma_{(k_1,k_2)} := \frac{1}{k_1+k_2+1}$ $(k_1, k_2 \geq 0)$. It is easy to see that $\mu_{\mathcal{M}} = \mu_{\mathcal{N}} = \epsilon$ is a diagonal measure on $[0,1]^2$; specifically, ϵ is normalized Lebesgue measure on the diagonal of $[0,1]^2$. It follows that $\epsilon^X = \epsilon^Y$ is the Lebesgue measure on $[0,1]$. Therefore, we have: $W_{(\alpha,\beta)}$ is never subnormal for any choice of β_{00}. For, $\frac{1}{s} \notin L^1(\epsilon^X)$, which is a necessary condition for subnormality, by Proposition 3.1(i).

Remark 3.2. Although $W_{(\alpha,\beta)}$ given in Example 3 fails to be subnormal for any β_{00}, by Lemma 1.6 and straightforward manipulation of moment matrices, we can show that for any given $k \geq 1$ there exists $m_k > 0$ such that for $\beta_{00} \in (0, m_k]$ the 2-variable weighted shift $W_{(\alpha,\beta)}$ is k-hyponormal. Furthermore, $m_k > m_{k+1}$ $(k \geq 1)$ and $\lim_{k\to\infty} m_k = 0$; since subnormality is equivalent to k-hyponormality for every $k \geq 1$, this provides an alternative proof that $W_{(\alpha,\beta)}$ is never subnormal.

Remark 3.3. The reader may wonder whether Theorem 1.7 can be extended to the case of more general common invariant subspaces for $W_{(\alpha,\beta)}$; for instance, subspaces associated with indices $k \equiv (k_1, k_2)$ such that $k_1 + k_2 \geq n$, where $n \geq 2$. It is indeed possible to slightly generalize Theorem 1.7 to the case of the two invariant subspaces $\mathcal{M}_q := \{(0, k_2) : k_2 \geq q\}$ and $\mathcal{N}_p := \{(k_1, 0) : k_1 \geq p\}$ $(p, q \geq 2)$, by repeated application of Theorem 1.7 and Lemma 1.4. The case of more general invariant subspaces, however, appears inaccessible at present using the tools in this paper.

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References

[1] A. Athavale, On joint hyponormality of operators, *Proc. Amer. Math. Soc.* 103(1988), 417-423.
[2] J. Bram, Subnormal operators, *Duke Math. J.* 22(1955), 75-94.

[3] J. Conway, *The Theory of Subnormal Operators,* Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

[4] R. Curto, Joint hyponormality: A bridge between hyponormality and subnormality, *Proc. Symposia Pure Math.* 51(1990), 69-91.

[5] R. Curto, Quadratically hyponormal weighted shifts, *Integral Equations Operator Theory* 13(1990), 49-66.

[6] R. Curto, S.H. Lee and J. Yoon, k-hyponormality of multivariable weighted shifts, *J. Funct. Anal.* 229(2005), 462-480.

[7] R. Curto, S.H. Lee and J. Yoon, Hyponormality and subnormality for powers of commuting pairs of subnormal operators, *J. Funct. Anal.* 245(2007), 390-412.

[8] R. Curto, S.H. Lee and J. Yoon, Reconstruction of the Berger measure when the core is of tensor form, *Actas del XVI Coloquio Latinoamericano de Álgebra, Bibl. Rev. Mat. Iberoamericana* (2007), 317-331.

[9] R. Curto, P. Muhly and J. Xia, Hyponormal pairs of commuting operators, *Operator Theory: Adv. Appl.* 35(1988), 1-22.

[10] R. Curto and J. Yoon, Jointly hyponormal pairs of subnormal operators need not be jointly subnormal, *Trans. Amer. Math. Soc.* 358(2006), 5139-5159.

[11] R. Curto and J. Yoon, Disintegration-of-measure techniques for commuting multivariable weighted shifts, *Proc. London Math. Soc.* 92(2006), 321-402.

[12] R. Gellar and L.J. Wallen, Subnormal weighted shifts and the Halmos-Bram criterion, *Proc. Japan Acad.* 46(1970), 375-378.

[13] P.R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.* 76(1970), 887-933.

[14] N.P. Jewell and A.R. Lubin, Commuting weighted shifts and analytic function theory in several variables, *J. Operator Theory* 1(1979), 207-223.

[15] A. Lambert, Subnormality of weighted shifts, *J. London Math. Soc.* 14(1976), 476–480.

[16] A.R. Lubin, Weighted shifts and commuting normal extension, *J. Austral. Math. Soc.* Ser. A 27(1979), 17-26.

[17] J. Yoon, Disintegration of measures and contractive 2-variable weighted shifts, *Integral Equations Operator Theory* 59(2007), 281-298.

Raúl E. Curto
Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242
e-mail: `rcurto@math.uiowa.edu`
URL: `http://www.math.uiowa.edu/~rcurto/`

Sang Hoon Lee
Department of Mathematics, Chungnam National University, Daejeon, 305-764, Republic of Korea
e-mail: `slee@cnu.ac.kr`

Jasang Yoon
Department of Mathematics, The University of Texas-Pan American, Edinburg, Texas 78539
e-mail: `yoonj@utpa.edu`
URL: `http://faculty.utpa.edu/yoonj/`