MOMENTS OF THE SURPLUS BEFORE RUIN AND THE DEFICIT AT RUIN IN THE ERLANG(2) RISK PROCESS

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ABSTRACT

This paper investigates the moments of the surplus before ruin and the deficit at ruin in the Erlang(2) risk process. Using the integro-differential equation that we establish, we obtain some explicit expressions for the moments. Furthermore, when the claim size is exponentially and subexponentially distributed, asymptotic relationships for the moments are derived as the initial capital tends to infinity. Also, we show the joint probability density function of the surplus before ruin and the deficit at ruin.

1. INTRODUCTION

In this paper we consider the insurance risk process

\[ U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \]

where \( u \geq 0 \) is the initial capital of the insurance company, \( c > 0 \) is the premium rate, \( \{X_i, i \geq 1\} \) denotes the sequence of i.i.d. non-negative successive claims, and \( \{N(t), t \geq 0\} \) denotes the number of claims up to time \( t \), which is a counting process independent of \( \{X_i, i \geq 1\} \). If \( N(t) \) is the renewal process, that is, the interoccurrence times \( \theta_i, i \geq 1 \), constitute a sequence of i.i.d. non-negative random variables, the model above is the renewal risk model, which was introduced by Sparre Anderson (1957). Two special cases that have received extensive attention in the literature are the compound Poisson model with \( \theta_1 \) exponentially distributed, and the Erlang(2) risk model, where \( \theta_1 \) has an Erlang(2) probability density function (p.d.f.) \( k(t) \) with scale parameter \( \beta > 0 \):

\[ k(t) = \beta^2 t e^{-\beta t} \quad \text{for } t > 0. \quad (1.1) \]

The Erlang(2) risk model has been investigated by Dickson (1998) and Dickson and Hipp (2000, 2001).

In the renewal risk model, we define the time of ruin by

\[ T = \inf\{t \geq 0 | U(t) < 0\}, \]

and the deficit at ruin and the surplus before ruin by \( U(T^-) \) and \( U(T^-) \), respectively. Let

\[ \phi_0(u) = E[e^{-\delta T} \omega(U(T^-), |U(T)| \mathbb{1}_{T < \infty}) | U(0) = u], \]

(1.2)

where \( \mathbb{1}_\cdot \) is an indicator function and \( \omega(x_1, x_2) \) is a non-negative bivariate function of \( x_1 > 0 \) and \( x_2 > 0 \). In the long history of insurance risk theory, a particularly interesting problem is how to calculate the moments of the time of ruin, the deficit at ruin, and the surplus before ruin. Lin and Willmot (2000) derived explicit expressions for \( \phi_0(u) \) in the compound Poisson model. Recently, in the Erlang(2) risk model, Dickson and Hipp (2001) continued to investigate Equation (1.2) but for a special case where \( \omega(x_1, x_2) = 1 \). This paper complements the work of Dickson and Hipp. We address exact and approximate expressions for \( \phi_0(u) \). Hereafter we simply write \( \phi_0(u) \) as \( \phi(u) \).

We remark that choosing different forms of the penalty function \( \omega(x_1, x_2) \) in Equation (1.2) gives rise to different information relating to the deficit at ruin and the surplus before ruin. For example, \( \phi(u) \) will represent the \( \alpha \)-order moment of the
deficit at ruin (or the surplus before ruin) if we specially choose \( \omega(x_1, x_2) = x_2^n \) (or \( x_1^n \)), represent their joint cumulative distribution function (c.d.f.) if \( \omega(x_1, x_2) = \mathbb{1}_{[x_1 \leq x_2 \leq y]} \) represent the c.d.f. of the deficit at ruin if \( \omega(x_1, x_2) = \mathbb{1}_{[x_1 \leq x]} \), and so on. When \( \omega(x_1, x_2) \equiv 1 \), the moment \( \phi_n(u) \) coincides with the well-known probability of ultimate ruin, which is independently defined by

\[
\psi(u) = \mathbb{P}\{T < \infty|U(0) = u\}. \tag{1.3}
\]

The financial explanations on \( \omega(x_1, x_2) \) can be found in Gerber and Shiu (1998).

Throughout this paper we suppose that the claim sizes, \( X_i \), \( i \geq 1 \), are i.i.d. with a common c.d.f. \( F \) supported on \([0, \infty)\), a finite mean \( \mu \), and a probability density function (p.d.f.) \( p(x) \). Its integrated tail c.d.f. is defined by \( F_n(x) = \frac{1}{n} \int_0^x \hat{F}(s) \, ds, \ x \geq 0 \). It is always assumed that the safety loading condition

\[
\rho = \frac{cE\theta_1 - \mu}{\mu} = \frac{2c/\beta - \mu}{\mu} > 0 \tag{1.4}
\]

holds. We denote by \( m_p(s) = \int_0^\infty e^{sp}F(\, dx) \) the moment generating function (m.g.f.) of \( F \) and by \( F^n \) the \( n \)-fold convolution of \( F \), \( n \geq 1 \). All limit relationships, unless otherwise stated, are for \( u \to \infty \) or \( x \to \infty \). The symbol \( \sim \) means that the quotient of the left-hand side and right-hand side of the corresponding relationship tends to 1.

## 2. Establishing Equations for \( \phi \)

### 2.1. An Integro-differential Equation for \( \phi \)

First, we derive an integro-differential equation for \( \phi \) by a device widely used in risk theory; see Dickson and Hipp (2001) and references therein.

**Theorem 2.1**

Consider the Erlang(2) risk model with the safety loading condition (1.4). If

\[
\int_0^\infty \int_0^\infty \omega(x_1, x_2) p(x_1 + x_2) \, dx_1 \, dx_2 < \infty, \tag{2.1}
\]

then it holds that

\[
c^2 \phi''(u) = 2 \beta c \phi'(u) - \beta^2 \phi(u)
+ \beta^2 \int_0^u \phi(u - x) p(x) \, dx
+ \beta^2 \int_u^\infty \omega(u, x - u) p(x) \, dx. \tag{2.2}
\]

**Proof**

By Corollary 3.3 (see below), the condition (2.1) implies that \( \phi(u) < \infty \) holds for all \( u \geq 0 \). Conditioning on the time and the amount of the first claim, we have

\[
\phi(u) = \int_0^\infty k(t) \int_0^{u + ct} \phi(u + ct - x) p(x) \, dx \, dt
+ \int_0^\infty k(t) \int_{u + ct}^\infty \omega(u + ct, x - u - ct)p(x) \, dx \, dt,
\]

where the last term follows from the fact that once the first claim causes the ruin, then the surplus before ruin and the deficit at ruin if \( u + ct \) and \( u + ct - x \), respectively. Substitution of \( s = u + ct \) into the above yields that

\[
c\phi(u) = \int_u^\infty k\left(\frac{s - u}{c}\right) \int_0^s \phi(s - x) p(x) \, dx \, ds
+ \int_u^\infty k\left(\frac{s - u}{c}\right) \int_s^\infty \omega(s, x - s)p(x) \, dx \, ds. \tag{2.3}
\]

Differentiating the two sides of Equation (2.3) with respect to \( u \), we obtain

\[
c\phi'(u) = -\frac{\beta^2}{c} \int_u^\infty e^{-\beta(s - u)/c} \int_0^s \phi(s - x) p(x) \, dx \, ds
- \frac{\beta^2}{c} \int_u^\infty e^{-\beta(s - u)/c} \int_s^\infty \omega(s, x - s)p(x) \, dx \, ds + \beta \phi(u), \tag{2.4}
\]
where we used Equation (2.3) and the fact that 
\[ k'(t) = \beta^2 e^{-\beta t} - \beta k(t) \] (recall Eq. [1.1]). Differentiating the two sides of Equation (2.4) once again, we conclude that
\[
e^u \phi''(u) = -\frac{\beta^3}{c^2} \int_u^\infty e^{-\beta(s-u)c} \phi(s-x)p(x) \, dx \, ds
- \frac{\beta^3}{c^2} \int_u^\infty e^{-\beta(s-u)c} \int_s^\infty \varphi(s,x-s)p(x) \, dx \, ds
+ \frac{\beta^3}{c^2} \left[ \int_0^u \phi(u-x)p(x) \, dx \right.
+ \int_u^\infty \varphi(u,x-u)p(x) \, dx \right]
+ \beta \phi'(u)
= \frac{\beta}{c} (c\phi'(u) - \beta \phi(u)) + \beta \phi'(u)
+ \frac{\beta^3}{c^2} \left[ \int_0^u \phi(u-x)p(x) \, dx \right.
+ \int_u^\infty \varphi(u,x-u)p(x) \, dx \right],
\]
where the last equation results from Equation (2.4). Hence Equation (2.2) holds.

2.2. The Laplace Transform of \( \phi \)

In this subsection we will apply the Laplace transform to solve the second order differential equation (2.2). Henceforth, the Laplace transform of a function \( f \) is, as usual, defined by
\[
f^*(s) = \int_0^\infty e^{-su} f(u) \, du.
\]
Besides the Laplace transform, there is another operation, \( T_r f \), that is often used in the recent literature; see, for example, Dickson and Hipp (2001). For a given integrable function \( f \), this operation is defined by
\[
T_r f(x) = \int_x^\infty e^{-r(u-x)} f(u) \, du
\]
for real \( r \). It is easy to check that
\[
T_r f^*(s) = \frac{f^*(s) - f^*(r)}{r - s}
\]
holds for any real \( r \neq s \); see Dickson and Hipp (2001), p. 337.

Now we aim to derive an expression for the Laplace transform of \( \phi \). Consider the equation
\[
(cs - \beta)^2 = \beta^2 \phi^*(s),
\]
which has at least two roots, \( r_1 = 0 \) and \( r_2 > \beta/c \), as can be easily verified. With the notations
\[
\varphi(u) = \int_u^\infty \varphi(u,x-u)p(x) \, dx,
\]
\[
\eta(u) = T_r T_s \varphi(u)
\]
and
\[
\xi(u) = T_r T_s \varphi(u),
\]
the announced expression for the Laplace transform of \( \phi \) is given in the theorem below.

**Theorem 2.2**

Consider the Erlang(2) risk model with the safety loading condition (1.4). If Equation (2.1) holds, then the Laplace transform of \( \phi \) is
\[
\phi^*(s) = \frac{\beta^2 \eta^*(s)}{c^2 - \beta^2 \xi^*(s)}.
\]

**Proof**

Under the assumption (2.1), Corollary 3.3 below guarantees that \( \phi^*(s) < \infty \) for all \( s > 0 \). We can easily check that the Laplace transforms of \( \phi(u) \), \( \phi'(u) \), and \( \int_0^u \phi(u-x)p(x) \, dx \) are \( 2s \phi^*(s) - \phi(0) \), \( s^2 \phi^*(s) - s \phi(0) - \phi'(0) \), and \( \phi^*(s) \phi^*(s) \), respectively. Taking the Laplace transform on the both sides of Equation (2.2) gives
\[
[(cs - \beta)^2 - \beta^2 \phi^*(s)] \phi^*(s)
= (c^2 s - 2\beta c) \phi(0) + c^2 \phi'(0) + \beta^2 \varphi(s).
\]
(2.8)

Recalling that \( r_2 > \beta/c \) is the root of Equation (2.6), by modeling the proof of Theorem 3.1 of Dickson and Hipp (2001), we can obtain that
\[
(cs - \beta)^2 - \beta^2 \phi^*(s) = s(s - r_2)[c^2 - \beta^2 \xi^*(s)]
\]
and that the right-hand side of Equation (2.8)
equals \((s - r_2)[c^2\phi(0) - \beta^2T_r\varphi(s)]\). From these discussions it follows that

\[
s\Phi^*(s) = \frac{c^2\phi(0) - \beta^2T_r\varphi(s)}{c^2 - \beta^2\varphi(s)}. \tag{2.9}
\]

Letting \(s \to 0^+\), the limit of the right-hand side of Equation (2.9), hence the limit of \(s\Phi^*(s)\), is finite. In addition, from Corollary 3.3, \(\phi(u)\) tends to 0 as \(u \to \infty\). Thus, by virtue of the properties of the Laplace transform, we have that

\[
\lim_{s \to 0^+} s\Phi^*(s) = \lim_{u \to \infty} \phi(u) = 0.
\]

This, together with the continuity (in \(s\)) of the Laplace transform \(T_r\varphi(s)\) and the finiteness of the limit of the denominator of the right-hand side of Equation (2.9), implies that

\[
c^2\phi(0) - \beta^2T_r\varphi(0) = 0. \tag{2.10}
\]

Finally, going along the same line as Dickson and Hipp (2001), we obtain Equation (2.7) immediately. □

Clearly, Equation (2.10) gives the result below.

**Corollary 2.3**

Under the conditions of Theorem 2.2, \(\phi(0)\) can be written as

\[
\phi(0) = \frac{\beta^2(\varphi^*(0) - \varphi^*(r_2))}{c^2r_2},
\]

where \(\varphi()\) and \(r_2\) are given in Theorem 2.2.

**3. Exact Representations for \(\phi\)**

In this section we aim at exact representations for the moment \(\phi(u)\) and the joint p.d.f. of the surplus before ruin and the deficit at ruin. For this purpose we introduce two proper c.d.f.’s \(G(u)\) and \(H(u)\), which are supported on \([0, \infty)\) with tails

\[
\bar{G}(u) = \int_u^\infty \tilde{\zeta}(t) \, dt \equiv \frac{1}{D_1} \int_u^\infty \int_x^\infty e^{-r(t-x)} \bar{F}(t) \, dt \, dx,
\]

\[
\bar{H}(u) = \int_0^u \frac{\eta(u)}{e^{-r\varphi(t)}} \int_x^\infty \varphi(t) \, dt \, dx
\equiv \frac{1}{D_2} \int_u^\infty e^{-r(x-u)} \int_x^\infty \varphi(t) \, dt \, dx, \tag{3.1}
\]

respectively.

Here \(G(u)\) is similar to what Lin and Willmot (1999) used. Since \(r_2\) is a root of Equation (2.6), we have

\[
D_1 = \int_0^\infty \tilde{\zeta}(t) \, dt
= \frac{1}{r_2} \left( \frac{1}{r_2} - 1 - r^2 \right)
= \frac{1}{r_2} \left( \frac{2c^2}{\beta} + \frac{c^2}{\beta^2} \right). \tag{3.3}
\]

It is not difficult to see that Equation (2.7) is equivalent to the equation

\[
\phi(u) = \frac{1}{1 + b} \int_0^u \phi(u - t)G(dt) + \frac{D_2}{(1 + b)D_1} \bar{H}(u), \tag{3.4}
\]

where \(b = D_1^{-1}c^{-2} - 1\). Recalling Equation (3.3) and the safety loading condition (1.4), we have

\[
\frac{1}{1 + b} = \frac{\beta^2}{c^2r_2} \left( \frac{\mu - 2c}{\beta} \right) + 1 < 1. \tag{3.5}
\]

Thus \(b > 0\). Equation (3.4) is immediately recognized as a defective renewal equation in view of Equation (3.5). The renewal equation in defective form like Equation (3.4) has been widely investigated in the literature; see Gerber and Shiu (1998) and Lin and Willmot (1999), among others.

Now we define an associated compound geometric c.d.f. \(K(u)\) by

\[
\bar{K}(u) = \sum_{n=1}^\infty \frac{b}{(1 + b)^n} G^n(u), \quad u \geq 0. \tag{3.6}
\]

According to Lin and Willmot (1999), we obtain the following exact representations for the function \(\phi(u)\).

**Theorem 3.1**

Consider the Erlang(2) risk model with the safety loading condition (1.4). If Equation (2.1)
holds, then the function \( \phi(u) \) can be expressed as
\[
\phi(u) = \frac{D_2}{c^2} \left( \frac{1}{b} \int_0^u \bar{H}(u-x) K(dx) + \bar{H}(u) \right),
\]
(3.7)
or equivalently,
\[
\phi(u) = \frac{D_2}{(1+b)D_1} \times \left( \frac{1}{b} \bar{K} + \bar{H}(u) - \frac{1}{b} \bar{K}(u) + \bar{H}(u) \right),
\]
(3.8)
where the notations involved are given above.

Along the same line as above, we can also derive an explicit expression for the joint p.d.f. of the surplus before ruin and the deficit at ruin. This is shown in the result below.

**Theorem 3.2**

In the Erlang(2) risk model with the safety loading condition (1.4), the joint p.d.f. of the non-negative random pair \((U(T^-, [U(T)])\)) is given by
\[
f(x_1, x_2) = \frac{\beta^2 p(x_1 + x_2)}{c^2 r_2} \times \left( \frac{1}{b} \int_{(u-x_1)^+}^u (1 - e^{-r_2(x_1-u+x)}) \right) K(dx) + (1 - e^{-r_2(x_1-u)}) \right)
\]
where \( u^+ = u I_{(u>0)} \).

**Proof**

For the two arbitrarily fixed positive numbers \( x_1 \) and \( x_2 \) in Equation (3.9), we choose
\[
\varphi(x, y) = I_{(x \leq x_1, y \leq x_2)}.
\]
(3.10)
In this special case, \( \phi(u) = \phi(x_1, x_2; u) \) is just the joint c.d.f. of the random pair \((U(T^-), [U(T)])\) (hence \( |\phi(u)| \leq 1 \)), and, trivially, condition (2.1) holds. In the present case, it is also clear that
\[
\lim_{u \to \infty} \phi(u) = \lim_{u \to \infty} \psi(u) = 0,
\]
where the last equality is implied by the safety loading condition (1.4). So Theorems 2.2 and 3.1 are applicable to the present case.

In order to derive the exact expression for \( \phi(u) \) (or equivalently, for the joint c.d.f. of \((U(T^-), [U(T)])\)), we now start to deal with the two terms on the right-hand side of Equation (3.7). Since Equation (3.10) implies that \( \varphi(t) = [\bar{F}(t) - \bar{F}(t + x_2)] I_{(t \leq x_1)} \), we have
\[
\int_x^\infty \varphi(t) \, dt = \int_x^{x_1} \left[ \bar{F}(t) - \bar{F}(t + x_2) \right] dt \cdot I_{(x \leq x_1)}.
\]
Substituting this into Equation (3.2) leads to
\[
\bar{H}(u) = \frac{1}{D_2} \int_u^\infty e^{-r_2(x-u)} \int_x^{x_1} \left[ \bar{F}(t) - \bar{F}(t + x_2) \right] dt \cdot I_{(x \leq x_1)}
\]
(3.11)
where the notations involved are given above.

It follows that
\[
\frac{\partial^2 \bar{H}(u)}{\partial x_1 \partial x_2} = \frac{1}{r_2 D_2} \left( 1 - e^{-r_2(x_1-u)} \right) p(x_1 + x_2) I_{(u \leq x_1)}.
\]
(3.12)
Now we turn to the first term on the right-hand side of Equation (3.7). For simplicity, we write
\[
\varphi = g(x_1, x_2; u) = \int_0^u \bar{H}(u-x) K(dx).
\]
(3.13)
When \( u \leq x_1 \), substituting Equation (3.11) into Equation (3.13) yields that
\[
\varphi = \frac{1}{r_2 D_2} \int_0^{x_1} \int_{u-x}^u \left( 1 - e^{-r_2(x_1-u+x)} \right)
\times \left[ \bar{F}(t) - \bar{F}(t + x_2) \right] dt K(dx),
\]
which implies that
\[
\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = \frac{1}{r_2 D_2} \left( 1 - e^{-r_2(x_1-u+x)} \right) K(dx).
\]
(3.14)
When \( x_1 < u \), substituting Equation (3.11) into
Equation (3.13) and applying Fubini’s theorem, we obtain
\[
g = \frac{1}{r^2D_2} \int_{u-x_1}^{u} \int_{u-x}^{x_1} (1 - e^{-r_2(t-u+x)}) \\
\times [\tilde{F}(t) - \bar{F}(t + x)] \, dtK(dx)
\]
\[
= \frac{1}{r^2D_2} \int_{u-x_1}^{u} \int_{u-x}^{x_1} (1 - e^{-r_2(t-u+x)}) \\
\times [\tilde{F}(t) - \bar{F}(t + x)] \, dtK(dx)
\]
which implies that
\[
\frac{\partial^2 g}{\partial x_1 \partial x_2} = \frac{p(x_1 + x_2)}{r^2D_2} \\
\times \int_{u-x_1}^{u} (1 - e^{-r_2(x_1-u+x)}) \, K(dx).
\]
Thus in any case we conclude that
\[
\frac{\partial^2 g}{\partial x_1 \partial x_2} = \frac{p(x_1 + x_2)}{r^2D_2} \\
\times \int_{u-x_1}^{u} (1 - e^{-r_2(x_1-u+x)}) \, K(dx). \tag{3.14}
\]
Finally, substituting Equations (3.12) and (3.14) into Equation (3.7), we complete the proof of Equation (3.9). \(\square\)

By virtue of Theorem 3.2 above, we can solve the problems on the finiteness of \(\phi(u)\) and its Laplace transform \(\phi^*(s)\).

**Corollary 3.3**
Consider the Erlang\((2)\) risk model with the safety loading condition (1.4). If Equation (2.1) holds, then \(\phi(u) < \infty\) for all \(u \geq 0\), \(\phi^*(s) < \infty\) for all \(s > 0\), and \(\lim_{u \to \infty} \phi(u) = 0\).

**Proof**
From Equation (3.9), we know that
\[
\phi(u) = \int_{0}^{\infty} \int_{0}^{\infty} \omega(x_1, x_2)f(x_1, x_2) \, dx_1 \, dx_2
\]
\[
\leq \frac{\beta^2}{c^2} \left( \frac{1}{b} + 1 \right) \\
\times \int_{0}^{\infty} \int_{0}^{\infty} \omega(x_1, x_2)p(x_1 + x_2) \, dx_1 \, dx_2 < \infty,
\]
which further implies that \(\phi^*(s) < \infty\) for any \(s > 0\).

We continue to prove the last assertion that \(\lim_{u \to \infty} \phi(u) = 0\). We divide the first double integral of Equation (3.15) into three parts as
\[
\frac{c^2}{\beta^2} \phi(u) = \int_{0}^{u} \int_{0}^{\infty} p(x_1 + x_2) \frac{r_2b}{r_2} \omega(x_1, x_2) \, dx_1 \, dx_2
\]
\[
\times \int_{u-x_1}^{u} (1 - e^{-r_2(x_1-u+x)}) \, K(dx) \, dx_1
\]
\[
+ \int_{u-x_1}^{u} \int_{0}^{\infty} p(x_1 + x_2) \frac{r_2b}{r_2} \omega(x_1, x_2) \, dx_1 \, dx_2
\]
\[
\times \int_{0}^{u-x_1} (1 - e^{-r_2(x_1-u+x)}) \, K(dx) \, dx_1
\]
\[
= I_1 + I_2 + I_3.
\]
As for the first term \(I_1\), we derive its bounds as
\[
I_1 \leq \int_{0}^{u} \int_{0}^{\infty} \frac{p(x_1 + x_2)}{r_2} \omega(x_1, x_2) \\
\times \int_{u-x_1}^{u} K(u - x_1) - \bar{K}(u) \, dx_1 \\
= \frac{1}{r_2} \left[ \left( \int_{0}^{u/2} + \int_{u/2}^{u} \right) \omega(x_1) \bar{K}(u - x_1) \, dx_1 \\
- \bar{K}(u) \int_{0}^{u} \omega(x_1) \, dx_1 \right]
\]
\[
\leq \frac{1}{r_2} \left[ \bar{K}(u/2) \int_{0}^{\infty} \omega(x_1) \, dx_1 + \int_{u/2}^{\infty} \omega(x_1) \, dx_1 \\
- \bar{K}(u) \int_{0}^{u} \omega(x_1) \, dx_1 \right].
\]
Recalling the condition (2.1), this shows that \(I_1 \to 0\) as \(u \to \infty\). The treatment on the last two terms \(I_2\) and \(I_3\) is similar. This proves \(\lim_{u \to \infty} \phi(u) = 0\). \(\square\)
4. Approximations for $\Phi$

We remark that in Theorem 3.1 above, the expression (3.7) for $\phi(u)$, although exact, is rather involved. In fact, $\phi(u)$ can be evaluated only through expression (3.7) for several special choices of claim size distributions such as combinations of exponentials, mixtures of Erlangs, phase-type distributions, etc. However, asymptotics for $\phi(u)$ are available for many very general choices of claim size distributions. This is mainly due to the fact that $\phi(u)$ is just the solution of the defective renewal equation (3.4), which allows for the use of the exiting theoretical technique in this field.

This section is devoted to asymptotic properties of $\phi(u)$ as the initial capital $u$ tends to infinity. Like many references, we roughly classify the present investigation on asymptotics into three cases: the light-tailed, the subexponential, and the intermediate cases, according to the tail behavior of the claim size distribution. It will be seen that these asymptotics for $\phi(u)$ depend heavily on the tail behavior of the claim size distribution.

4.1. Result for the Light-Tailed Case

We say the claim size distribution $F$ is light-tailed if its m.g.f. $m_F(s)$ exists finite for some positive number $s$; otherwise, the c.d.f. $F$ is called heavy-tailed. Typical examples that are light-tailed but do not belong to the class $\mathcal{S}(\gamma)$, $\gamma > 0$, are the exponential, the gamma, etc.; see below for the definition of the class $\mathcal{S}(\gamma)$.

We introduce an auxiliary function

$$f(s) = m_F(s) - \left(1 + \frac{cs}{\beta}\right)^2,$$

and assume that $0 \leq f(s_0) < \infty$ for some $s_0 > 0$. For any $0 < s \leq s_0$, from the definition of the c.d.f. $G$ in Equation (3.1), after simplifications, we obtain that

$$\frac{1}{1 + b}m_G(s) = \frac{s^{-1}(-1 + m_F(s)) + r_2^{-1}(-1 + m_F(-r_2))}{s + r_2} \frac{\beta^2}{c^2}\cdot$$

Substituting $m_F(-r_2) = (cr_2 - \beta)^2/\beta^2$ (recall Eq. [2.6]) into the above leads to

$$f_1(s) = 1 - \frac{1}{1 + b}m_G(s) = -\frac{\beta^2}{c^2} \frac{f(s)}{s(s + r_2)}.$$  (4.1)

It is not difficult to see that there exists a unique constant $0 < \kappa \leq s_0$ such that $f_1(\kappa) = 0$ (hence $f(\kappa) = 0$), which implies that

$$1 = \frac{1}{1 + b} m_G(\kappa).$$

In fact, the constant $\kappa > 0$ is usually identified as the Lundberg adjustment coefficient of the risk model; see, for example, Bowers et al. (1997), chapter 13. If we further assume that the function $e^{\kappa u}H(u)$ is directly Riemann integrable, then, applying the key renewal theorem to the defective renewal equation (3.4), we immediately obtain an asymptotic relation for $\phi(u)$ that

$$\phi(u) \sim \frac{\eta^n(-\kappa)}{D_1 \int_0^\infty xe^{\kappa u}g(dx)} e^{-\kappa u}. \quad (4.2)$$

For more about the key renewal theorem, we refer the reader to Feller (1971). The same approach as above has been employed by Gerber and Shiu (1998).

We summarize the discussion above into the result below.

Theorem 4.1

Consider the Erlang(2) risk model with the safety loading condition (1.4). If Equation (2.1) holds, $0 \leq f(s_0) < \infty$ for some $s_0 > 0$, and $e^{\kappa u}H(u)$ is directly Riemann integrable for the adjustment coefficient $\kappa > 0$, which is determined by $f(\kappa) = 0$, then Equation (4.2) holds.

4.2. Class $\mathcal{S}(\gamma)$ and Its Properties

The following classes of distribution functions are essential for modeling large claims; for their applications to ruin theory, refer to Veraverbeke (1977) and Embrechts and Veraverbeke (1982), among others.

Definition 4.2

A c.d.f. $F$ supported on $[0, \infty)$ belongs to the class $\mathcal{S}(\gamma)$ with $\gamma \geq 0$ if and only if

(I) $\lim_{x \to \infty} F_{\gamma}(x) = 2m_F(\gamma) < \infty.$

(II) $\lim_{x \to \infty} \frac{F(x) - y}{F(y)} = e^y$ for all $y$ real.

A c.d.f. $F$ belongs to the class $\mathcal{L}(\gamma)$ if and only if $F$ satisfies condition (II).

Clearly, $\mathcal{S}(\gamma) \subset \mathcal{L}(\gamma)$. In Definition 4.2 $\gamma$ is the right abscissa of the convergence of $m_F(\cdot)$, and the
convergence in (II) is automatically uniform on $y$-compacta. When $\gamma = 0$, we denote $S(0)$ and $\mathcal{L}(0)$ simply by $S$ and $\mathcal{L}$, respectively, which are two important classes of heavy-tailed cumulative distribution functions. $S$ is the celebrated subexponential class, which includes many popular distributions in statistics like the Pareto, the log-normal, the Weibull, the log-gamma, the Burr, and the Benktander I and II distributions; see Table 1.2.6 in Embrechts, Klüppelberg, and Mikosch (1997). We also mention that among those interesting distributions from the class $\mathcal{S}(\gamma)$ with $\gamma > 0$ is the inverse Gaussian c.d.f., which has a p.d.f.

$$p(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\pi} \left(x - \mu\right)^2\right\}, \quad x > 0,$$

where $\lambda > 0$ and $-\infty < \mu < \infty$ are two parameters. The membership of the inverse Gaussian distribution in the class $\mathcal{S}(\gamma)$ can easily be verified by Theorem 3 in Cline (1986).

The following result can be found in Teugels (1975) and Cline (1987); see also Klüppelberg (1989) for some more general discussions.

**Lemma 4.3**

Let $F_1$ and $F_2$ be two c.d.f.’s satisfying $\overline{F}_1(x) \sim C \overline{F}_2(x)$ for some $C > 0$. Then $F_1 \in \mathcal{S}(\gamma)$ if and only if $F_2 \in \mathcal{S}(\gamma)$.

In the next subsection we shall need the following result for the class $\mathcal{S}(\gamma)$ with $\gamma > 0$.

**Lemma 4.4**

Let $F$ be a c.d.f. supported on $[0, \infty)$ with a finite mean $\mu$, and $F_\varepsilon$ be its integrated tail distribution. For $\gamma > 0$, $F \in \mathcal{S}(\gamma)$ if and only if $F_\varepsilon \in \mathcal{S}(\gamma)$, each of which implies that

$$\overline{F}_\varepsilon(x) \sim \frac{1}{\mu \gamma} \overline{F}(x).$$

**Proof**

The assertion $F_\varepsilon \in \mathcal{S}(\gamma) \Leftrightarrow F \in \mathcal{S}(\gamma)$ can be proved directly from Corollary 2.2 of Klüppelberg (1989), as was mentioned in Cheng, Tang, and Yang (2002). If $F \in \mathcal{S}(\gamma)$, then according to Bingham, Goldie, and Teugels (1987), $\overline{F}$ can be expressed as

$$\overline{F}(x) = C(x) \exp\left\{-\int_0^x b(t) \, dt\right\},$$

where $C(x) > 0$ and $b(x)$ are functions such that $\lim_{x \to \infty} C(x) = C_0 > 0$ for a constant $C_0$ and $\lim_{x \to \infty} b(x) = \gamma > 0$. Hence, for any $0 < \varepsilon < 1$,

$$\frac{\overline{F}(x + s)}{\overline{F}(x)} \leq (1 + \varepsilon)e^{-(1-\varepsilon)\gamma s} \quad (4.3)$$

holds for all sufficiently large $x$ and all $s \geq 0$. Therefore,

$$\lim_{x \to \infty} \frac{\overline{F}_\varepsilon(x)}{\overline{F}(x)} = \frac{1}{\mu} \lim_{s \to \infty} \int_0^s \frac{\overline{F}(x + s)}{\overline{F}(x)} \, ds = \frac{1}{\mu \gamma}$$

follows from the dominated convergence, which is justified by Equation (4.3).

**Lemma 4.5**

Let c.d.f.’s $F_i \in \mathcal{L}(\gamma)$ for $i = 1, 2, 3, 4$ and $m_i(\gamma) = m_i(\gamma)$. If $\overline{F}_1 \sim C_1 \overline{F}_3$ and $\overline{F}_2 \sim C_2 \overline{F}_4$ for $C_i > 0, i = 1, 2$, then

$$\overline{F}_1 \ast \overline{F}_2 \sim C_1 C_2 \overline{F}_3 \ast \overline{F}_4 + C_1 (m_2(\gamma) - C_2 m_4(\gamma)) \overline{F}_3 + C_3 (m_1(\gamma) - C_1 m_3(\gamma)) \overline{F}_4. \quad (4.4)$$

**Proof**

Equation (4.4) follows from the proof of Lemma 2.4 in Cline (1987) with some adjustments.

### 4.3. Unifying Results for the Subexponential and Intermediate Cases

This subsection addresses some asymptotic results for the moments of the deficit at ruin and the surplus before ruin in the case where the claim size distribution belongs to the class $\mathcal{S}(\gamma)$ with $\gamma \geq 0$. The related work, but only on the deficit at ruin, was done by Cheng, Tang, and Yang (2002). For the sake of self-containment, however, we still decide to consider in this subsection the asymptotic results for both the deficit at ruin and the surplus before ruin. In doing so, the penalty function $\omega(x_1, x_2)$ will be chosen as $x_1^\alpha$ or $x_2^\alpha$ for $\alpha > 0$.

The following preliminary is an immediate consequence of the definition of direct Riemann integrability; some closely related materials can be found, for example, in Ross (1983) and Asmussen (1987).
Lemma 4.6
Suppose that \( h(t) \) is a bounded and non-negative continuous function. Then it is directly Riemann integrable if and only if \( \bar{\sigma}(\delta) < \infty \) for some \( \delta > 0 \), where the sum \( \bar{\sigma}(\delta) \) is defined by

\[ \bar{\sigma}(\delta) = \delta \sum_{k=1}^{\infty} c_k \quad \text{with} \quad c_k = \sup_{(k-1)\delta \leq x < k\delta} h(x). \]

Here is the main result of this subsection.

Theorem 4.7
Consider the Erlang(2) risk model with the safety loading condition \((1.4)\). Suppose that the claim size c.d.f. \( F \) supported on \([0, \infty)\) has a p.d.f. \( p(x) \), \( F_c \in \mathcal{S}(\gamma) \) for some \( \gamma \geq 0 \).

A. If \( \gamma = 0 \), \( \mathbb{E}X_1^{\alpha+1} < \infty \) for some constant \( \alpha > 0 \) and \( \varpi(x_1, x_2) = x_1^\alpha \) or \( x_2^\alpha \), then it holds that

\[ \phi(u) = \frac{1}{\rho \mu} \int_0^u \frac{t}{F_c(u-t)} \varpi(t) \, dt \]

\[ + \frac{1}{\rho \mu} \int_u^\infty \varpi(t) \, dt. \]

B. Let \( \gamma > 0 \) and \( f(\gamma) < 0 \).

B1. If \( \varpi(x_1, x_2) = x_1^\alpha \), it holds that

\[ \phi(u) \sim \frac{\gamma}{f'((\gamma)} \left[ \int_0^u t^\alpha \tilde{F}(u-t) \tilde{F}(t) \, dt \right. \]

\[ + \left. (1-f(\gamma) - m_f(\gamma)) \int_u^\infty t^\alpha \tilde{F}(t) \, dt \right]. \]

B2. If \( \varpi(x_1, x_2) = x_2^\alpha \), then

\[ \phi(u) \sim \left( \frac{r_2 + \gamma \gamma D_2(m_f(\gamma) - 1)}{f'(\gamma)} \right) \]

\[ + \frac{\gamma^{-\alpha} \Gamma(\alpha + 1)}{-f'(\gamma)} \tilde{F}(u). \]

C. If \( \gamma > 0, f(\gamma) > 0 \), and \( \varpi(x_1, x_2) = x_1^\alpha \) or \( x_2^\alpha \), then Equation \((4.2)\) holds, where \( \gamma > \kappa > 0 \) is uniquely determined by \( f(\kappa) = 0 \).

Proof
When \( \varpi(x_1, x_2) = x_1^\alpha \) or \( x_2^\alpha \), the function \( \varpi(t) \) equals \( t^\alpha \tilde{F}(t) \) or \( \int_t^\infty (x-t)^\alpha p(x) \, dx \) correspondingly. Then, the condition \((2.1)\), that is, \( \int_0^\infty \varpi(t) \, dt < \infty \), can be justified by \( \mathbb{E}X_1^{\alpha+1} < \infty \). So the result of Theorem \(3.1\) can be used. For further uses, we define a c.d.f. \( W(u) = \tilde{W}(u) = \int_u^\infty \varpi(t) \, dt \), where \( D_3 = \int_0^\infty \varpi(t) \, dt \). Now we divide the proof of Theorem \(4.7\) into five steps to complete.

Step 1. The c.d.f. \( W \) belongs to \( \mathcal{L}(\gamma) \).

To prove \( W \in \mathcal{L}(\gamma) \), it suffices to show that for any \( \gamma > 0 \) it holds that

\[ \lim_{u \to \infty} \int_u^{u+y} \varpi(t) \, dt = e^{-\gamma} \varpi(t). \quad (4.8) \]

When \( \varpi(x_1, x_2) = x_1^\alpha \), hence \( \varpi(t) = t^\alpha \tilde{F}(t) \), then according to integration by parts and \( F_c \in \mathcal{S}(\gamma) \subset \mathcal{L}(\gamma) \), we obtain that

\[ \lim_{u \to \infty} \int_u^{u+y} t^\alpha \tilde{F}(t) \, dt \]

\[ = \lim_{u \to \infty} \frac{u+y)^\alpha \tilde{F}(u+y) + \alpha \int_0^u t^\alpha \tilde{F}(t) \, dt \]

\[ = \lim_{u \to \infty} \frac{u^\alpha \tilde{F}_c(u) + \alpha \int_0^u t^\alpha \tilde{F}(t) \, dt \}

\[ = e^{-\gamma}, \]

where in the last step we used the fact that

\[ (u+y)^\alpha \tilde{F}_c(u+y) \sim e^{-\gamma} u^\alpha \tilde{F}_c(u) \]

and that

\[ \int_0^\infty (t+u+y)^{\alpha-1} \tilde{F}_c(t+u+y) \, dt \]

\[ \sim e^{-\gamma} \int_0^\infty (t+u)^{\alpha-1} \tilde{F}_c(t+u) \, dt. \]

This proves Equation \((4.8)\) in the case of \( \varpi(x_1, x_2) = x_1^\alpha \). For the other case of \( \varpi(x_1, x_2) = x_2^\alpha \), so that \( \varpi(t) = \int_t^\infty (x-t)^\alpha p(x) \, dx \), by reversing the order of the integrals and using the
integration by parts, we can derive analogously that

$$
\lim_{u \to \infty} \int_{u+y}^{\infty} \varphi(t) \, dt \frac{f_u^x (x - t)^\alpha \, dt}{p(x) \, dx} = \lim_{u \to \infty} \int_{u}^{\infty} \left( f_u^x (x - t)^\alpha \, dt \right) p(x) \, dx
$$

Then Equation (4.8) holds for any cases.

**Step 2.** The relationship between $\bar{H}(u)$ and $\bar{F}_c(u)$.

From Equations (4.8) and (3.2) and the dominated convergence theorem, we obtain that

$$
\frac{\bar{H}(u)}{\bar{F}_c(u)} = \frac{1}{D_2} \int_0^\infty \int_0^\infty e^{-r_2 \gamma_1 x} \, dx = \frac{1}{(r_2 + \gamma)D_2}.
$$

Thus, by Equation (4.9), it holds that

$$
\bar{H}(u) \sim \frac{1}{(r_2 + \gamma)D_2} \int_0^\infty \varphi(t) \, dt
$$

**Step 3.** The relationship between $\bar{G}(u)$ and $\bar{F}_c(u)$.

First, we derive a relationship between $\bar{G}(u)$ and $\bar{F}_c(u)$. Recalling Equation (3.1), by the dominated convergence theorem and $F_c(x) \in \mathcal{L}(\gamma) \subseteq \mathcal{L}(\gamma)$, we obtain that

$$
\lim_{u \to \infty} \frac{\bar{G}(u)}{\bar{F}_c(u)} = \lim_{u \to \infty} \int_0^x \int_{x+u}^{\infty} e^{-r_2(t-x-u)} \, dt \, dx
$$

$$
= \frac{\int_0^x (x-u)^{\alpha+1} p(x) \, dx}{(\alpha+1)D_2 \bar{F}_c(u)}
$$

$$
= \frac{\alpha \int_0^x (x-u)^{\alpha+1} \bar{F}_c(x) \, dx}{\bar{F}_c(u)}
$$

$$
= \frac{\alpha \int_0^x x^{\alpha+1} \bar{F}_c(x+u) \, dx}{\bar{F}_c(u)} \to \infty.
$$

Thus Equation (4.10) holds. Or, equivalently, $\bar{F}_c(u) = o(\bar{H}(u))$ from Equation (4.9).

For the other case of $\varphi(x_1, x_2) = x_2^\alpha$ and $\gamma > 0$, recalling Equation (4.3), the dominated convergence gives

$$
\lim_{u \to \infty} \int_0^\infty \varphi(t) \, dt \frac{\int_0^x x^{\alpha} \bar{F}_c(x+u) \, dx}{\bar{F}_c(u)} = \int_0^x x^{\alpha} e^{-t} \, dx = \gamma^{-(\alpha+1)} \Gamma(\alpha+1).
$$

Then Equation (4.10) holds. Or, equivalently, $\bar{F}_c(u) = o(\bar{H}(u))$ from Equation (4.9).

For the other case of $\varphi(x_1, x_2) = x_2^\alpha$ and $\gamma > 0$, recalling Equation (4.3), the dominated convergence gives

$$
\lim_{u \to \infty} \frac{\int_0^\infty \varphi(t) \, dt}{\bar{F}(u)} = \lim_{u \to \infty} \frac{\int_0^x x^{\alpha} \bar{F}_c(x+u) \, dx}{\bar{F}(u)} = \int_0^x x^{\alpha} e^{-t} \, dx = \gamma^{-(\alpha+1)} \Gamma(\alpha+1).
$$

Thus, by Equation (4.9), it holds that

$$
\bar{H}(u) \sim \frac{1}{(r_2 + \gamma)D_2} \int_0^\infty \varphi(t) \, dt
$$

$$
\sim \frac{1}{(r_2 + \gamma)D_2} \gamma^{-(\alpha+1)} \Gamma(\alpha+1) \bar{F}(u) = C_2 \bar{F}(u).
$$

(4.11)
\[ \lim_{n \to \infty} \frac{\mu}{D_1} \int_0^u e^{-\gamma(t-u)}F_c(t) \, dt = \lim_{n \to \infty} \frac{\mu}{D_1} \int_0^u e^{-\gamma(t-u)}F_c(t+u) \, dt = \frac{\mu}{D_1} \int_0^u e^{-(r+\gamma)u} \, dt \]

This results in \( G(u) \in \mathcal{S}(\gamma) \) by Lemma 4.3. The result (4.12) is some similar to Proposition 1 of Willmot (1999), which essentially deals with the regularly varying subclass.

In the case of (A) and (B) of Theorem 4.7, recalling Equation (4.1), we know that \( 1 + 1/b \) \( m_G(\gamma) < 1 \). From this, Equation (4.12) and the standard arguments on the compound geometric c.d.f. \( \tilde{R}(u) \) in Equation (3.6) (see, for example, Veraverbeke 1977 or Embrechts and Veraverbeke 1982), it holds that

\[ \tilde{R}(u) \sim \sum_{n=1}^\infty bn(m_G(\gamma))^{n-1} \frac{1}{(1+b)^n} \bar{G}(u) = \frac{b(1+b)\bar{G}(u)}{(1+b-m_G(\gamma))^2} \sim \frac{b(1+b)}{(1+b-m_G(\gamma))^2} \frac{\mu}{D_1(r+\gamma)^2} F_c(u), \]

and thus \( \tilde{R} \in \mathcal{S}(\gamma) \) follows from Lemma 4.3. Furthermore, when \( \gamma > 0 \), from Lemma 4.4 and \( F_c \in \mathcal{S}(\gamma) \), we know that \( F \in \mathcal{S}(\gamma) \) and the relationship (4.13) can be rewritten as

\[ \tilde{R}(u) \sim \frac{b(1+b)}{D_1(r+\gamma)^2} \bar{F}(u) = \frac{b(1+b)}{D_1(r+\gamma)^2} F(u). \]

**Step 4.** The proof of items (A) and (B) of Theorem 4.7.

For item (A), recalling Equations (3.8), (4.13), (4.9), and (4.10), by Lemma 4.5, we get that

\[ \phi(u) \sim \frac{\mu}{b^2D_1^2r_2^2} \int_0^u \bar{F}_c(u-t)\varphi(t) \, dt + \frac{1}{bD_1r_2} \int_u^\infty \varphi(t) \, dt. \]

From the definition of \( b \) and Equation (3.3), it holds that \( bD_1r_2 = \rho \mu \). This proves Equation (4.5).

For item (B1), because of Equation (4.14), it can be derived analogously that

\[ \phi(u) \sim \frac{1}{(1+b-m_G(\gamma))^2D_1^2(r+\gamma)^2} \int_0^u \bar{F}(u-t)\varphi(t) \, dt \]

\[ + \left( \frac{1}{(1+b-m_G(\gamma))^2D_1^2(r+\gamma)^2} \int_u^\infty \varphi(t) \, dt \right) \]

where the equation

\[ m_R(\gamma) = \frac{bm_G(\gamma)}{1+b-m_G(\gamma)} \]

is used, which results from Equation (3.6). Applying Equation (4.1) and the definition of \( b \), we know that

\[ \frac{1}{(1+b-m_G(\gamma))^2D_1^2(r+\gamma)^2} = \frac{\gamma}{\bar{f}(\gamma)}. \]

This proves Equation (4.6).

For item (B2), from Equations (3.8), (4.14), and (4.11) and Lemma 4.5, it holds that

\[ \phi(u) \sim \left( \frac{C_1(m_H(\gamma) - 1) + C_2m_R(\gamma)}{b} \right) \bar{F}(u) \times \frac{D_2}{(1+b)D_1} \]

Substituting Equation (4.15) into the above and using Equation (4.1), we can obtain Equation (4.7).

**Step 5.** The proof of item (C) of Theorem 4.7.

When \( \varphi(x_1, x_2) = x_2^2 \), from Equation (4.11), we know that \( \bar{H}(u) \sim C_2 \bar{F}(u) \). Then \( \bar{H}(u) = o(e^{-\gamma u}) \), so that \( e^{au} \bar{H}(u) = o(e^{-(r+\gamma)u}) \). Because \( e^{-(r+\gamma)u} \) is directly integrable on \([0, \infty)\), it is trivial to see that there exists a \( \delta > 0 \) such that \( \sigma_1(\delta) < \infty \) for this function. Corresponding to this division of \([0, \infty)\), the sum \( \sigma_2(\delta) \) to the function \( e^{au} \bar{H}(u) \) must be finite. Again because \( e^{au} \bar{H}(u) \) is a bounded and non-negative absolutely continuous function, by Lemma 4.6, it must be directly Riemann...
integrable. Thus, Equation (4.2) holds according to Theorem 4.1.

As to the other case of \( \varphi(x_1, x_2) = x_1^\alpha \), by Equation (4.9), it holds that, for all sufficiently large \( u \),

\[
\tilde{H}(u) \sim \frac{1}{(r_2 + \gamma)D_2} \int_0^\infty t^\alpha \tilde{F}(t) \, dt \\
\leq \frac{1}{(1 + \alpha)(r_2 + \gamma)D_2} u^{\alpha+1} \tilde{F}(u) \\
\leq \frac{1}{(1 + \alpha)(r_2 + \gamma)D_2} u^{\alpha+1} e^{-\gamma u}.
\]

Noting that the function \( u^{\alpha+1} e^{-(\gamma-\kappa)u} \) is directly integrable and that the proof of the frontal case, Equation (4.2) also holds. This ends the proof of Theorem 4.7.

\[\square\]

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