unitary representation of the Poincaré group for a single particle of mass $m$ and spin $\frac{1}{2}$.

wavefunctions

\[
\langle (m\frac{1}{2}) \bar{\rho} \mu | \psi \rangle
\]

\[
\psi_{\frac{1}{2}} = \frac{1}{\sqrt{2}} \int \langle (m\frac{1}{2}) \bar{\rho} \mu \rangle \Omega \langle (m\frac{1}{2}) \bar{\rho} \mu | \psi \rangle
\]

\[
\psi_{\frac{1}{2}} = \frac{1}{\sqrt{2}} \sum_{\nu = -1}^{1} \langle (m\frac{1}{2}) \bar{\rho} \nu \rangle \sqrt{\frac{\omega_{m}(\nu)}{\omega_{m}(\nu)}} \cdot \bar{D}_{\nu \mu} \lambda_{c}(\mu \rho) \Lambda \lambda_{c}(\rho)
\]

\[
A_{c}(m \bar{s}) = (\sqrt{m^{2} + \bar{s}^{2}}, \bar{p})
\]

(m is arbitrary $A_{c}(p) \rightarrow A'_{c}(p) = A_{c}(p) R(p)$ but fixed)

\[
R_{\omega}(x, p) = \Lambda_{c}(\mu \rho) \Lambda \lambda_{c}(p) \text{ is a rotation called a Wigner rotation}
\]

\[
\omega_{m}(\bar{p}) = \sqrt{m^{2} + \bar{p}^{2}}
\]
\[ \langle x | \psi(t) \rangle = \] 
\[ \langle x | \mathcal{U}(t, t') | t' \rangle = \] 
\[ -\frac{i}{\hbar} \frac{\partial}{\partial t} \langle x | e^{\frac{i}{\hbar} \sqrt{m^2 c^2 + \bar{p}^2}} | t' \rangle \] 
\[ \langle x | \psi(t) \rangle \] 
\[ = -\frac{i}{\hbar} \sqrt{m^2 c^2 - \bar{p}^2} \langle x | \psi(t) \rangle \]

(here \( \bar{p} = -i\hbar \nabla \) follows because \( \bar{p} \) is the generator of translations)

We write this as

\[ i\hbar \frac{\partial}{\partial t} \langle x | \psi(t) \rangle = \sqrt{m^2 c^2 - \bar{p}^2} \langle x | \psi(t) \rangle \]

This is called the relativistic Schrödinger equation. It can be obtained by inserting

\[ H \rightarrow i\hbar \frac{\partial}{\partial t} \quad \bar{p} \rightarrow -i\hbar \nabla \]

in the relativistic relation

\[ H = \left( m^2 c^4 + \bar{p}^2 c^2 \right)^{1/2} \]
the problems with the relativistic Schrödinger equation

* It did not give the correct spectrum for the Hydrogen atom.

* It leads to a very asymmetric treatment of the space and time derivatives

Positive properties

* \[ \frac{d}{dt} \langle \Psi(t) | \Psi(t) \rangle = \]

* \[ ( \frac{d}{dt} \langle \Psi(t) | 1 \rangle \langle 1 | \Psi(t) \rangle + \langle \Psi(t) | \frac{d}{dt} 1 \rangle \langle 1 | \Psi(t) \rangle ) = \]

* \[ \frac{1}{\hbar} \langle \Psi(t) | (\hat{H} - \hat{H}) | \Psi(t) \rangle = 0 \]

* \[ \langle \Psi(t) | \hat{H} | \Psi(t) \rangle = \frac{\hbar}{2} K + \Gamma(m_1) \beta \omega \mu \sqrt{m_1 c^2 + \beta \mu} \geq 0 \]

Probability conserved - energy bounded below (i.e. positive)

Klein-Gordon and Schrödinger replaced by relativistic Schrödinger equation by considering

\[ H^2 = \frac{p^2 c^2}{\hat{T}} + m^2 c^4 \]

\[ ( - \hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \nabla^2 - m^2 c^4 ) \langle \Psi(t) | \Psi(t) \rangle = 0 \]
note that if \( \langle x|\psi(t)\rangle \) satisfies
the relativistic Schrödinger equation then

\[
(i\hbar \frac{\partial}{\partial t}) \langle x|\psi(t)\rangle =
\begin{align*}
(i\hbar \frac{\partial}{\partial t}) \sqrt{m^2c^4 - \hbar^2c^2\nabla^2} \langle x|\psi(t)\rangle &= \\
\sqrt{m^2c^4 - \hbar^2c^2\nabla^2} (i\hbar \frac{\partial}{\partial t}) \langle x|\psi(t)\rangle &= \\
(m^2c^4 - \hbar^2c^2\nabla^2) \langle x|\psi(t)\rangle &= 0 \\
(-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2c^2\nabla^2 - m^2c^4) \langle x|\psi(t)\rangle &= 0
\end{align*}
\]

so solutions of the relativistic Schrödinger equation are solutions of the Klein-Gordon equation. If we apply the same steps to

\[
i\hbar \frac{\partial}{\partial t} \langle x|\psi(t)\rangle = -\sqrt{m^2c^4 - \hbar^2c^2\nabla^2} \langle x|\psi(t)\rangle
\]

which replaces \( H' \rightarrow H \) in the relativistic Schrödinger equation it also satisfies the Klein-Gordon equation.

* The Klein-Gordon equation has positive and negative solutions
Let \( \langle \chi_1 \psi_\pm(t) \rangle \) be superpositions of positive (resp negative) energy solutions of the KG equation such that:

\[
\frac{d}{dt} \langle \psi_+(t) \rangle = -\frac{i}{\hbar} H \langle \psi_+(t) \rangle \\
\frac{d}{dt} \langle \psi_-(t) \rangle = \frac{i}{\hbar} H \langle \psi_-(t) \rangle \\
\frac{d}{dt} \langle \psi_+(t) | \psi_-(t) \rangle = \frac{i}{\hbar} \langle \psi_+(t) | H | \psi_-(t) \rangle \neq 0
\]

This shows that in this case the probability amplitudes are not generally time independent.

This is in contrast to equations of the general form

\[
i \hbar \frac{d}{dt} \langle \psi(t) \rangle = H \langle \psi(t) \rangle \\
H = H^\dagger
\]

which always satisfy

\[
\frac{d}{dt} \langle \psi_+(t) | \psi_-(t) \rangle = -\frac{i}{\hbar} \langle \psi_+(t) | (H - H^\dagger) | \psi_-(t) \rangle = 0.
\]
The Klein Gordon equation was discarded because

1. Probability amplitudes are no longer conserved
2. The energy is not bounded from below
3. Still gave wrong Hydrogen fine structure splitting

Dirac attempted to repair this by using 2 requirements:

1. The equation should only have first order space and time derivatives
2. The square of the equation should lead to the Klein Gordon equation

(As with the relativistic Schrödinger equation we still expect $H^2 = P^2 c^4 + m^2 c^4$)
\[ i \hbar \frac{\partial}{\partial t} \langle x|14 \rangle = \langle x|1H|14 \rangle \]

(tis already ensures probabilities are conserved in time)

\[ H = \vec{\alpha} \cdot \vec{p} c + \beta m c^2 \quad \Rightarrow \quad H = H^\dagger \]

The requirement that the square of the equation gives the KG equation implies:

\[ -\hbar^2 \frac{\partial^2}{\partial t^2} \langle x|14 \rangle = \left( \frac{\partial}{\partial t} - i \hbar (\alpha_i \frac{\partial}{\partial x_i} + \beta m c^2) \right) \left( \frac{\partial}{\partial x_i} - i \hbar (\alpha^*_i \beta + \beta \alpha^*_i) \frac{\partial}{\partial x_i} \right) \langle x|14 \rangle \]

\[ = \left( -\hbar^2 \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} - i \hbar m c^3 (\alpha^*_i \beta + \beta \alpha^*_i) \frac{\partial}{\partial x_i} \right) \langle x|14 \rangle \]

The first term can be expressed as:

\[ -\hbar^2 c^2 \left( \alpha_i \alpha^*_j + \alpha^*_i \alpha_j \right) \frac{\partial^2}{\partial x_i \partial x_j} \]

The above equation becomes the Klein-Gordon equation if:

\[ \alpha_i^* \alpha_j + \alpha^*_i \alpha_j = 2 \delta_{ij} \]
\[ \beta \alpha_i^* + \alpha \beta^*_i = 0 \]
\[ \beta^2 = 1 \]
clearly
(1) \( \alpha^i \beta \) cannot be numbers which satisfy
\[ \beta \alpha^i + d/\beta = 0, \quad \alpha^i \beta + \alpha^i \alpha^i = 0 \quad i \neq i \]

(2) The next simplest thing to try is matrices
since \( \alpha^i \alpha^i = \beta \beta = 1 \) may have
eigenvalues \( \pm 1 \). This implies that the dimension
of the matrices should be even.

(3) In 2 dim the Pauli matrices
are the only independent
Hermitian anticommuting
matrices. There are only 3 -
we need 4

The simplest solution is 4x4
matrices. Dirac chose

\[ \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \]

\[ \beta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \]

It is easy to check that these
matrices have the desired
properties
\[ \Sigma \left( i \hbar \frac{\partial}{\partial t} - c \bar{x} \cdot \bar{p} - \beta mc^2 \right) \psi = 0 \]

Since the Hamiltonian is a matrix, the wave functions are 4 component column vectors.

\[ \left( i \hbar \frac{\partial}{\partial t} + i \hbar c \sum \alpha_i \frac{\partial}{\partial x_i} - \beta mc^2 \right) \psi = 0 \]

It has become customary to write this as an equation with matrices on the derivatives — to do this multiply on the left by \( \beta \) and define

\[ \gamma^0 = \beta \quad \gamma^i = \beta \alpha^i \]

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \frac{1}{\beta} \begin{pmatrix} \alpha^i & 0 \\ 0 & \bar{\alpha}^i \end{pmatrix} = \begin{pmatrix} 0 & \bar{\alpha}^i \\ -\bar{\alpha}^i & 0 \end{pmatrix} \]

\[ \gamma^0 \gamma^0 = 1 \quad \gamma^i \gamma^i = \beta \alpha^i \bar{\alpha}^i = -\beta^2 \alpha^i \bar{\alpha}^i = -1 \]

\[ \gamma^{i} \gamma^{j} + \gamma^{j} \gamma^{i} = \beta^2 \alpha^{i} \alpha^{j} + \beta \alpha^{i} \bar{\alpha}^{j} = \beta^2 \alpha^{i} \alpha^{j} \]

\[ \gamma^{i} \gamma^{i} + \gamma^{j} \gamma^{j} = \beta \alpha^{i} \bar{\alpha}^{j} \bar{\alpha}^{i} + \beta \alpha^{i} \alpha^{j} \]

\[ = - (\alpha^{i} \alpha^{j} + \bar{\alpha}^{i} \bar{\alpha}^{j}) = 0 \]

\[ \{ \gamma^{i} \gamma^{j} \} = -2 \eta^{ij} \]
\[\partial_t = \left( \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x} \right) = \left( \frac{i}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) = i \]

\[(i \hbar c \left( \gamma^0 \partial_0 + 2 \gamma^i \partial_i \right) - mc^2) \psi = 0\]

\[\sum_i i \hbar c \gamma^i \partial_i - mc^2 \psi = 0\]

This is the standard textbook form of the Dirac equation. To solve it, try a plane wave solution

\[\psi = \psi_0 e^{ip.x/m} \quad px = -p^0 x - p^i x\]

Inserting this in the above equation gives

\[(i \hbar c \left( -\gamma^0 \partial_0 + i \hbar c i \gamma^i \partial_i \right) - mc^2) \psi(0) e^{ip.x/m}\]

\[c \gamma^0 p^0 - c \gamma^i \partial_i - mc^2 \psi(0) e^{ip.x/m}\]

We need to choose \( E = cp^0 \), so the matrix

\[\text{det} \left( E \gamma^0 - c \gamma^i \partial_i - mc^2 \right) = 0\]
\[
\det \begin{pmatrix}
E-mc^2 & -c\vec{p}\cdot\vec{a} \\
c\vec{p}\cdot\vec{a} & -E-mc^2
\end{pmatrix} = 0
\]

Multiply the first 2 rows by \( \frac{c\vec{p}\cdot\vec{a}}{E-mc^2} \) and subtract from the second 2 rows.

\[
\det \begin{pmatrix}
E-mc^2 & -c\vec{p}\cdot\vec{a} \\
0 & -(E+mc^2) + \frac{c^2 \vec{p}^2}{(E-mc^2)}
\end{pmatrix}
\]

\[
\det \left( \frac{(E-mc^2)}{-(E+mc^2) + \frac{c^2 \vec{p}^2}{E-mc^2}} \right) =
\]

\[
\left( \frac{-(E+mc^2)(E-mc^2) + c^2 \vec{p}^2}{E-mc^2} \right)^2
\]

\[
(E^2 - m^2 c^4 - c^2 \vec{p}^2)^2
\]

\[
E = \pm \sqrt{m^2 c^4 + c^2 \vec{p}^2}
\]

We see that the Dirac equation in a plane wave has 4 solutions—2 for positive energy and 2 for negative energy.
\[(\Sigma Y^\nu - c \Sigma p^\nu - mc^2) \Psi = 0\]

\[- (c \eta_{\mu\nu} Y^\nu p^\mu - mc^2) \Psi = 0\]

Consider

\[(c \eta_{\mu\nu} Y^\nu p^\mu - mc^2)(c \eta_{\kappa\lambda} Y^\kappa p^\lambda + mc^2) =
\]

\[c^2 \eta_{\mu\nu} \eta_{\kappa\lambda} \frac{1}{2} \varepsilon_{\mu \nu \kappa \lambda} \varepsilon_{\rho \sigma \theta \iota} p^\rho p^\sigma - m^2 c^4 =
\]

\[- c^2 \eta_{\mu\nu} \eta_{\kappa\lambda} \eta_{\rho\sigma} p^\rho p^\sigma - m^2 c^4 =
\]

\[c^2 p^\mu - c^2 \bar{p}^\mu - m^2 c^4 =
\]

\[E^2 - c^2 \bar{p}^2 - m^2 c^4 = 0\]

This means that if

\[(c \eta_{\kappa\lambda} Y^\kappa p^\lambda + mc^2) \Psi(x)\]

is not 0 \( \rightarrow \) \( p^\mu = \frac{E^2}{c^2} \) if it is a solution

\[
\begin{pmatrix}
E c^2 + mc^2 & \bar{p} c \\
- \bar{p} c & -E c^2 + mc^2
\end{pmatrix}
\]

Note when \( \bar{p} \rightarrow 0 \rightarrow E + mc^2 \rightarrow 1\)

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

\[
\bar{p} \rightarrow E + mc^2 \rightarrow 0
\]

\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]
we construct positive energy solutions:

\[
\begin{pmatrix}
\text{matrix on last page}\n
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\xi} \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

and negative energy solutions:

\[
\begin{pmatrix}
\frac{1}{\xi} \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
\xi \\
0 \\
0 \\
0
\end{pmatrix}
\]

these are the columns of the matrix

\[
\Psi_{+1} = C \begin{pmatrix} E + mc^2 \\ 0 \\ -p_2 c \\ -p_x - i p_y \end{pmatrix} \quad E = \sqrt{p^2 c^2 + m^2 c^4}
\]

\[
\Psi_{+2} = C \begin{pmatrix} 0 \\ E + mc^2 \\ -p_2 c \\ -p_x + i p_y \end{pmatrix} \quad E = -|\xi| = -\sqrt{p^2 c^2 + m^2 c^4}
\]

\[
\Psi_{-1} = C \begin{pmatrix} p_x c \\ p_x + i p_y \\ 1|\xi| + mc^2 \\ 0 \end{pmatrix}
\]

\[
\Psi_{-2} = C \begin{pmatrix} p_x - i p_y \\ -p_x c \\ 0 \\ 1|\xi| + mc^2 \end{pmatrix}
\]
The normalization constant assuming $\nu, \bar{\nu} = 1$ is

$$C = \frac{1}{\sqrt{2E(mc^2 + iE)}}$$

In all 4 cases,

$$\psi_1 = \frac{1}{\sqrt{2E(mc^2 + iE)}} \begin{pmatrix} E + mc^2 \\ 0 \\ -p_x c \\ -p_y c - ip_z c \end{pmatrix} e^{-iEt/\hbar + ip_z \phi/\hbar} \cdot \frac{1}{(2\pi)^{3/2}c}$$

$$E = \sqrt{p^2 c^4 + mc^4}$$

with the other three having similar forms.

Note that when $\bar{p} \to 0$ these become:

$$\psi_{1+} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{(2\pi)^{3/2}} e^{-imc^2t/\hbar}$$

$$\psi_{2+} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{(2\pi)^{3/2}} e^{-imc^2t/\hbar}$$

$$\psi_{1-} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{(2\pi)^{3/2}} e^{imc^2t/\hbar}$$

$$\psi_{2-} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{(2\pi)^{3/2}} e^{imc^2t/\hbar}$$
one can show that the spin in the rest frame is

\[ \gamma = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

so \( \psi_{+1} \) = positive energy spin up
\( \psi_{+2} \) = positive energy spin down
\( \psi_{-1} \) = negative energy spin up
\( \psi_{-2} \) = negative energy spin down

While the Dirac equation has conserved probabilities, it still has negative energy states with energy unbounded from below.

* \( \rho = \rho - \frac{eA}{c} \) gives correct fine structure for Hydrogen - correct electron magnetic moment.

* Solution correspond to spin \( \frac{1}{2} \).

Dirac assumed all negative energy states occupied - possible to excite bound negative energy states \( \rightarrow \) Dirac called them holes.
like electrons - but negative charge predicted existence of positrons

many important successes

electron + proton - both satisfy dirac equation - H is a boson - can have negative energy - this is the ultimate failure of the dirac equation