Lecture II

Identical particles

1-1 correspondence between states and vectors

\[ \downarrow \]

Many particle wave functions must be symmetric or antisymmetric with respect to interchange of identical particles.

Spin statistics theorem

\( \frac{1}{2} \) integer spin particles =
fermions =
antisymmetric wave function

Integer spin particles =
bosons
symmetric with respect to interchange of identical particles

2 particles

bosons

\[ |n_1, n_2\rangle = \frac{1}{\sqrt{2}} (|n_1\rangle, |n_2\rangle + |n_2\rangle, |n_1\rangle) \]

fermions

\[ |n_1, n_2\rangle = \frac{1}{\sqrt{2}} (|n_1\rangle, |n_2\rangle - |n_2\rangle, |n_1\rangle) \]
construction of symmetric - antisymmetric states.

permutation $\sigma$

1 2 3 N
1 1 1
$\sigma(n)$ $\sigma(n)$

$\sigma(n) \in \{1, \ldots, N\}$

$m \neq n \quad \sigma(m) \neq \sigma(n)$

there are $N!$ permutations of $N$
identical particles

every permutation can be expressed
as a product pairwise transposition,

$P_{ij}$

$P_{ij} \langle x_1 \ldots x_i \ldots x_j \ldots x_N | \psi \rangle =$

$\langle x_1 \ldots x_i \ldots x_j \ldots x_N | \psi \rangle$

$P_0 = P_{12} P_{13} \ldots P_{0N}$

* this decomposition is not unique.

* permutations can be uniquely
classified into being made of
an odd or even # of transpositions:

$|G| = \{ 1, 0 \}$

$P_0 = \text{odd # transpositions}$

$P_0 = \text{even # transpositions}$
* every permutation has an inverse

\[ \Pi = P_1 P_2 \cdots P_N \]

\[ P_0^\dagger = P_N P_{N-1} \cdots P_1 \]

* for Hamiltonians for systems of identical particles

\[ [P_0, H] = 0 \]

* \[ [P_0, P_0^\dagger] \neq 0 \] for \( \sigma_i \neq \sigma_i' \)

Construction of projection operators on symmetric/antisymmetric subspace of N particle Hilbert space

\[ S = \frac{1}{N!} \sum \sigma_i P_0 \]

\[ A = \frac{1}{N!} \sum (-1)^{\sigma_i} P_0 \]

where the sums are over all permutations.
To show these are orthogonal projects we need to show that

\[ S = S^* \quad S^2 = S \]
\[ A = A^* \quad A^2 = A \]

Consider \( A \)

\[ A^* = \frac{1}{n!} \sum (e^*)^i P_0^* \]
\[ = \frac{1}{n!} \sum (e^*)^i P_0 = 0 \]

as \( e \) runs over all permutations \( e^* \) also runs over all permutations

\[ P_0 \neq P_1 \quad P_0^{-1} = P_{0^{-1}} = 1 \]
\[ P_i \neq P_2 \quad \text{contradiction} \]

This means that there are \( n! \) different \( e^* \) - Also \( |e| = 16^n \)
since the inverse can be constructed from the same transpositions except in reverse order.

Let \( e'' = e^* \)

\[ A^* = \frac{1}{n!} \sum (e^*)^i P_0 = A \]
We also have

\[ A^2 = \frac{1}{(N!)^2} \sum_{\sigma} \sum_{\sigma'} (-1)^{\sigma + \sigma'} P_\sigma P_{\sigma'} = \]

\[ = \frac{1}{(N!)^2} \sum_{\sigma} \sum_{\sigma'} (-1)^{\sigma + \sigma'} P_{\sigma' \sigma} \]

Note: Since \( P_\sigma P_{\sigma'} \) are products of transpositions,

\[ 1 \sigma + 1 \sigma' = 1 \sigma' \sigma \]

\[ = \frac{1}{(N!)^2} \sum_{\sigma} \sum_{\sigma'} (-1)^{\sigma' \sigma} P_{\sigma' \sigma} \]

Let \( \sigma'' = \sigma \sigma' \) be fixed \( \sigma \).

\[ = \frac{1}{(N!)^2} \sum_{\sigma''} (-1)^{\sigma''} P_{\sigma''} \]

for each fixed \( \sigma \) if we sum over \( \sigma'' \) we get each \( \sigma' \) since \( \sigma' = \sigma'' \sigma'' \)

Since there are \( N! \) \( \sigma' \)s,

\[ = \frac{N!}{(N!)^2} \sum_{\sigma''} (-1)^{\sigma''} P_{\sigma''} \]

\[ = A \]
The proof for $S$ works the same way. 

\[ A^2 = A, \quad A = A^3 \]
\[ S^2 = S, \quad S = S^+ \]

If \( [H P_{ij}] = 0 \), \( [H S] = [H, A] = 0 \)

In addition,

\[ AS = \left( \frac{1}{N_i!} \right) \sum_{i} (-)^{i+1} P_{i,6} \]

half of the permutations are even and half are odd.

If \( 6 \neq 6' \) is even,
\[ P_{i,6} \neq P_{i,6'} \] is odd.

Which means \( \frac{N_i}{2} \) even permutation = \( \frac{N_i}{2} \) odd ones.

\[ AS = \left( \frac{1}{N_i!} \right) \sum_{i} P_{i,6} (-)^{i+1} = 0 \]

\[ AS = SA = 0 \quad \text{and} \quad [AS] = 0. \]
\[ P_{12} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \]

\[ P_{23} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \]

\[ P_{31} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \]

Claim: to get even only by multiplying by and \( P_{11} \)

\[ P_{12}^2 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \]

\[ P_{12} P_{23} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \]

\[ P_{12} P_{31} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \]

\[ P_{23} P_{12} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \]

\[ P_{23} P_{23} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \]

\[ P_{23} P_{31} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \]
a general Hamiltonian in a system of identical particles will have simultaneous eigenstates of $A, S$ (It will also have states of mixed symmetry)

\[ H |A \psi\rangle = E |A \psi\rangle \]
\[ H |S \psi\rangle = E |S \psi\rangle \]

only one of these equations gives physical eigenstates

while $H$ is the same, matrix elements of $H$ on symmetric vs antisymmetric states differ.

we found this out when we looked at matrix elements of 2 state systems.

while $A, S$ are elegant solutions to the identical particle problem, they are not easy to use.
\[ N = 3 \]

\[ A = \frac{1}{6} \left( I + P_{12} P_{23} + P_{12} P_{31} - P_{12} - P_{23} - P_{31} \right) \]

\[ S = \frac{1}{6} \left( I + P_{12} P_{23} + P_{12} P_{31} + P_{12} + P_{23} + P_{31} \right) \]

These expressions are not unique - they get very complicated for an \( N \) particle system.

A more useful basis is called the occupation number representation.

The starting point is a fixed orthonormal basis \( \{ \phi_n(x) \} \) for a single particle.

Here \( n \) includes all orbital and spin quantum numbers.

\[ I = \sum n n' |n n' \rangle \]

A general \( N \) particle state can be expressed as a linear combination of products of \( N \) of these basis functions.
all we know about each basis state is there are \( k_1 \) particles in state 1, \( k_2 \) particles in state 2, etc. - with \( \sum_{n=0}^{\infty} k_n = N \).

The way to treat this is to introduce a vector with an infinite number of indices:

\[
|M_1 M_2 \ldots M_N\rangle
\]

\( m_1 \), \# particles in state \( \phi_1 \),
\( m_2 \), \# particles in state \( \phi_2 \).

we normalize these so states with different indices are orthogonal.

For bosons \( \psi \), each \( n_i \) can be any integer.

For fermions, each \( n_i \) can be 0 or 1.

For fermions, there is a question of how to treat the phase.
To do this we begin with

\[ |0\rangle = \text{no particle state} = 1000 \cdots > \]

For each \( \phi_n \) we introduce operators

\[ a_m, a_m^+, b_m, b_m^+ \]

satisfying

\[ [a_m, a_n] = [a_m^+, a_n^+] = 0 \quad \text{and bosons} \]
\[ [a_m, a_n^+] = \delta_{mn} \]
\[ [b_m, b_n^+] = [b_m^+, b_n^+] = 0 \]
\[ [b_m, b_{m'}^+] = \delta_{mn} \]

where

\[ \{AB, C\} = AB + BA \]

we define

\[ N_n = a_n^+ a_n \quad \text{bosons} \]
\[ N_n = b_{m'}^+ b_{m'} \quad \text{fermions} \]
\[
[N_m, a_n] = [a_m^t a_m, a_n] = \\
= [a_m^t a_n] a_m + a_m^t [a_m a_n] \\
= -\delta_{mn} \\
= -\delta_{nm} a_m \\
\]

\[
[N_m, a_n^t] = [a_m^t a_n^t, a_m^t] \\
= [a_m^t a_n] a_n + a_m^t [a_m a_n^t] \\
= \delta_{nm} a_m^t \\
\]

\[
N_m (a_m^t)^N = N(a_m^t)^N + (a_m^t)^N N_m \\
\]

for the fermion:

\[
[N_m, b_n] = (b_m^t b_m b_n + b_n b_m^t b_m) \\
= -b_m^t b_n b_m + b_n b_m^t b_m \\
= - (b_m^t b_n + b_n b_m^t) b_m + b_n b_m^t b_m \\
= -\delta_{nm} b_n + b_n N_m \\
\]
Next we use these operators to define multiparticle states.

1. One particle in the state $a_n^+|0\rangle$
2. $b_n^+|0\rangle$

Note that

$$\langle 0 | a_n a_n^+ | 0 \rangle =$$

$$\langle 0 | (a_n a_n^+ - a_n^+ a_n + a_n^+ a_n) | 0 \rangle$$

$$= \langle 0 | 0 \rangle + 0$$

$$= 1$$

$$\langle 0 | b_n b_n^+ | 0 \rangle =$$

$$\langle 0 | (b_n b_n^+ + b_n^+ b_n - b_n b_n) | 0 \rangle$$

$$= \langle 0 | 0 \rangle$$

Both of these one particle states are normalized to unity.
2 particle states \( m, n, m \neq n \)

\[
\begin{align*}
\langle o | a_m^+ a_n^+ | 1 \rangle & = \\
\langle 1 | a_n^+ a_m^+ | o \rangle \\
\langle o | 1 \rangle & = 1
\end{align*}
\]

\[
\langle o | a_m^+ a_n^+ a_m^+ a_n^+ | 1 \rangle = \frac{\langle 1 | (a_m^+ a_n^+ - a_n^+ a_m^+ + a_n^+ a_n^+ a_m^+ a_n^+ - a_n^+ a_m^+ a_n^+ a_n^+) | o \rangle}{1}
\]

This state is symmetric and normalized to unity \( m = n \)

\[
\begin{align*}
\langle o | a_n^+ a_n^+ | 1 \rangle & = \\
\langle 1 | a_n^+ a_n^+ | o \rangle \\
\langle o | 1 \rangle & = 1
\end{align*}
\]

If there are \( n \) states this becomes \( n! \):

\[
\begin{align*}
\langle o | a_{n_1}^+ a_{n_1}^+ | 1 \rangle & = \sqrt{n_{n_1}} \langle n_1 | n_1 \rangle \langle 1 | n_{n_1} \rangle \\
\langle o | a_{n_1}^+ a_{n_1}^+ | 1 \rangle & = \sqrt{n_{n_1}} \langle n_1 | n_{n_1} \rangle \langle 1 | n_{n_1} \rangle
\end{align*}
\]
in general
\[ |n_1 n_2 \ldots n_\omega > = \]
\[ \frac{1}{\sqrt{n_1!}} (a_1^+)^{n_1} \frac{1}{\sqrt{n_2!}} (a_2^+)^{n_2} \ldots 1^0 > \]

we can do a similar calculation for fermions except we note
\[ b_1^+ b_2^+ = - b_2^+ b_1^+ \]
\[ (b_i^+)^2 = 0 \]

In order to have a well defined notation we need a phase convention
\[ \frac{(b_i^+)^{n_i}}{\sqrt{n_i!}} \frac{(b_{\omega}^+)^{n_\omega}}{\sqrt{n_\omega!}} |0 > = |n_1 n_\omega > \]

where \( n_i = 1 \ or \ 0 \ only. \)

In this way
\[ (b_{\omega}^+ |n_1 n_\omega > = (-)^{n_1+n_\omega+\ldots+n_{\omega-1}} |n_1 n_{\omega+1} \ldots > \]
\[ \quad \text{if } n_\omega = 0 \]
\[ \quad \text{if } n_\omega = 1 \]
\[ (b_k^+ |n_1 n_\omega > = (-)^{n_1} n_\omega \ldots (n_1 n_{\omega-1} n_{\omega+1} \ldots > \]
\[ \quad \text{if } n_k = 0 \]
\[ \quad \text{if } n_k = 1 \]