Lecture 11

Identical particles

In quantum mechanics if we have a system of 2 identical particles in different states then

\[ |l\rangle_1 |m\rangle_2 \]
\[ |m\rangle_1 |l\rangle_2 \]

Both correspond to one particle in state $|l\rangle$ and one particle in state $|m\rangle$. This should be a single state in the theory, but

\[ \langle n | \langle m | |l\rangle_1 |m\rangle_2 = \langle n | |m\rangle_2 |m|_1 \langle l\rangle_2 = 0 \]

what we expect is that for the true physical state, if we interchange identical particles, we should get back to the same state up to phase

\[ P_{12} |lm\rangle = e^{i\Phi} |lm\rangle \]

and

\[ P_{21}^2 |lm\rangle = e^{i2\Phi} |lm\rangle = |lm\rangle \]
This means that the phase should be $1, -1$.

This means that in a 2-particle system the physical states should be symmetric or antisymmetric with respect to interchange of identical particles.

$$\frac{1}{\sqrt{2}} \left( |n\rangle_1 |m\rangle_2 - |m\rangle_1 |n\rangle_2 \right)$$

or

$$\frac{1}{\sqrt{2}} \left( |n\rangle_1 |m\rangle_2 + |m\rangle_1 |n\rangle_2 \right)$$

Again - there should be only one state with one particle in the $n^{th}$ state and one in the $m^{th}$ state.

It follows that only one of these two combinations correspond to the physical state.

It is useful to consider the case of three particles - in this case there are $6 = 3!$ states with one particle in the $n^{th}$ state, $m^{th}$, and $k^{th}$ state.
For three identical particles only one of the 6! states are possible - only one of these corresponds to the physical state.

Spin statistics theorem

* wave functions for identical half integral spin particles are antisymmetric
* wave functions for identical integral spin particles are symmetric

Elements of argument

* requires field observables associated with spacelike separated regions commute

\[ \text{theorem} \]

- proof is not complicated - but it requires building up field theory machinery, which is complicated.
\[ \begin{align*}
\frac{1}{2} \text{ integer spin particles are called fermions (proton, electron, neutrino)} \\
\times \text{ integer spin particles are called bosons (mesons)} \\
\text{the Hamiltonian acts on the full Hilbert space} \\
\begin{pmatrix}
\text{(symmetric)} \\
\text{(mixed)} \\
\text{(antisymmetric)}
\end{pmatrix}
= \begin{pmatrix}
E
\end{pmatrix}
\end{align*} \]

It is possible to find eigenstates with all of the possible symmetries, however in the physical system only a limited set of eigenstates that have wave functions of the desired symmetry are relevant.

This actually impacts the structure of the Hamiltonian - generating exchange interactions.

Consider
\[ \frac{1}{\sqrt{2}} \left( \langle n_1^\prime m_1^\prime \pm \langle m_1^\prime n_1 \rangle \right) V_{12} \frac{1}{\sqrt{2}} \left( \langle n_2^\prime m_2^\prime \pm \langle m_2^\prime n_2 \rangle \right) = \]
\[ \frac{1}{2} \left( \langle n_1 \langle m_1 \nu \mid n_1' \rangle m_2' \rangle + \langle m_1 \langle n_1 \nu l \mid m_1' \rangle n_2' \rangle \right) \]

The first and last 2 terms are the same, the 3rd and 4th terms are also the same, but they differ from the first 2 terms because there is a pairwise exchange of just the initial and final pair of particles.

This additional contribution is called an exchange interaction; it is clearly different for bosons and fermions.

Note that the correct exchange interaction appears automatically by using properly symmetrized wave functions.
we start by defining elementary transpositions

\[ P_{ij} < r_i, r_i, r_j, 1d > = \]
\[ < r_i, r_i, r_j, 1d > \]

a permutation \( \sigma \) is an invertible mapping

\[ m \neq n, \sigma(m) \neq \sigma(n) \]

for a system of \( N \) particles there are \( N! \) distinct permutations

proof

\[ N \text{ choices for particle 1} \]
\[ N-1 \text{ choices for particle 2} \]
\[ \vdots \]
\[ 1 \text{ choice for particle } N \]

every permutation is a product of elementary pairwise transpositions

this is easy to see if one imagines moving numbers a pair at a time.
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 3 & 1 & 2
\end{pmatrix}
\]

1 2 3 4 \quad P_{14}
4 2 3 1 \quad P_{23}
4 3 2 1 \quad P_{34}
4 3 1 2

* each permutation is a product of an even or odd # of transpositions

Proof: assume that a given permutation can be expressed as both an odd and even number of transpositions.

\[14 > = P_6 P_{6^{-1}} 14 >\]

\[P_6 = P_1 \quad P_{2n} = P_1' \quad P_{2n+1}'\]

\[P_{6^{-1}} = P_{2n} \quad P_1\]

If \(14 >\) is even and antisymmetric then

\[I 14 > = P_6 P_{6^{-1}} 14 > = (-1) 14 >\]

\[214 > = 0\]

Contradiction
we define

\[
|G| = \begin{cases} 
0 & \text{if } \sigma = \text{even number of transpositions} \\
1 & \text{if } \sigma = \text{odd number of transpositions}
\end{cases}
\]

Projections on symmetric + antisymmetric subspaces

\[
S = \sum_{\sigma \in \Theta(n)} \frac{1}{N!} P_\sigma
\]

\[
A = \sum_{\sigma \in \Theta(n)} (-1)^{\sigma_1} \frac{1}{N!} P_\sigma
\]

\[
S^2 = \sum_{\sigma \in \Theta(n)} \sum_{\sigma' \in \Theta(n)} \frac{1}{(N!)^2} P_\sigma P_{\sigma'} =
\]

\[
= \frac{1}{(N!)^2} \sum_{\sigma'} \sum_{\sigma''} P_{\sigma'} P_{\sigma''}
\]

Note that for each fixed \( \sigma \), as \( \sigma' \) ranges over \( \Theta(n) \), \( \sigma, \sigma' \) ranges over \( \Theta(n) \)

\[
\sigma, \sigma' = \sigma, \sigma'' \Rightarrow \sigma' = \sigma''
\quad \sigma, \sigma' \neq \sigma, \sigma'' \Rightarrow \sigma' \neq \sigma''
\]

Thus

\[
= \frac{1}{(N!)^2} \sum_{\sigma''} \sum_{\sigma} P_{\sigma''} = \frac{N!}{(N!)^2} \sum_{\sigma} P_{\sigma} = S
\]
Similarly with $A$

$$A^2 = \frac{1}{N^2} \sum_{\sigma} \sum_{\sigma'} (-1)^{\sigma + \sigma'} P(\sigma \sigma') P(\sigma' \sigma') =$$

$$= \frac{1}{N^2} \sum_{\sigma} \sum_{\sigma'} (-1)^{\sigma + \sigma'} P(\sigma \sigma')$$

For fixed $\sigma$ as $\sigma' \in \mathcal{P}(N) \rightarrow \sigma, \sigma' \in \mathcal{P}(N)$

$$\sigma \sigma' = \sigma + \sigma'$$

$$= \frac{N^2}{(N! \ln N) \sum_{\sigma} \mathcal{P}(\sigma \sigma) \mathcal{P}(\sigma' \sigma')}$$

$$= A$$

It is clear $A^2 = S^2$

For 2 particles

$$A + S = I$$

In general

$$A \cdot S = 0$$

$$AS = \frac{1}{(N! \ln N)} \sum_{\sigma} \sum_{\sigma'} (-1)^{\sigma + \sigma'} P(\sigma \sigma')$$

$$\sum_{\sigma} (-1)^{\sigma} P(\sigma \sigma') = -2 \sum_{\sigma} (-1)^{\sigma} P(\sigma \sigma')$$

If $\sigma'$ even $\sigma$ odd, so these all cancel.
example

\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 2 & 1 \\
3 & 2 & 1 \\
2 & 1 & 3
\end{pmatrix} = I
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
1 & 2 & 3 \\
3 & 2 & 1 \\
2 & 3 & 1 \\
1 & 3 & 2
\end{pmatrix} = P_{12}
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 2 \\
2 & 3 & 1 \\
1 & 3 & 2 \\
3 & 1 & 2
\end{pmatrix} = P_{13}
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
2 & 1 & 3 \\
3 & 1 & 2
\end{pmatrix} = P_{12}P_{13}
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 1 & 3 \\
1 & 2 & 3
\end{pmatrix} = P_{13}P_{12}
\]

\[
P_{c} = I P_{12} P_{13} P_{13} P_{12} P_{11} P_{12} P_{23} P_{31} \]

\[
\begin{array}{c}
\text{1C1 = 0} \\
\text{1C1 = 1}
\end{array}
\]

Note we see \( P_{12} P_{13} \neq P_{13} P_{12} \) so transpositions do not commute.

In general

\[\left\{ \begin{array}{c}
A|\psi\rangle \\
S|\psi\rangle
\end{array} \right\} \text{symmetrized}
\]

but because these are projectors the state has to be renormalized.
If $H$ is invariant with respect to exchange of identical particles, then

$$\{\Lambda H\Pi = \{\Lambda S\Pi\} = 0$$

This means that it is possible to find simultaneous eigenstates of $A$ and $H$ on $S$ and $H$.

**Occupation number representation**

Let $\{\phi_n(r)\}$ be a single particle basis

$$|n_1, n_2, n_3, \ldots \rangle$$

$n_1$ particles in state 1
$n_2$ particles in state 2

\vdots

Symmetric or antisymmetric under interchange of identical particles

$$|1000, \ldots \rangle$$

We introduce creation and annihilation
\[ [a_n a_n^+] = 1 \quad \text{bosons} \quad [a_n a_m] = 0 \]
\[ \{a_n, a_n^+\} = 1 \quad \text{fermions} \quad \{a_n, a_m\} = 0 \]