Example 1: one electron atom interacting with a plane electromagnetic field of frequency $\omega$.

$$H = \left( \frac{p^2}{2m} - \frac{e}{r} \right) - \frac{ze^2}{r} + e\Phi$$

$$\vec{A} = \alpha \hat{x} e^{i(\omega t - \frac{\omega}{c} \hat{z} \cdot \vec{r})}$$

$$\vec{B} = \nabla \times \vec{A} = \left( - \frac{i\omega}{c} \hat{z} \right) \times \vec{A} = \frac{i\omega}{c} \hat{y} e^{i(\omega t - \frac{\omega}{c} \hat{z} \cdot \vec{r})}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -i\omega \alpha \hat{x} e^{i(\omega t - \frac{\omega}{c} \hat{z} \cdot \vec{r})}$$

$$|\vec{E}| = \omega a \quad |\vec{B}| = \frac{\omega a}{c}$$

Here we assume $\Phi = 0$ and the field is weak, so we can ignore the $A^2$ term.

$$H = \frac{p^2}{2m} - \frac{e}{2mc} (\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}) - \frac{ze^2}{r}$$

We note $\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p} = 2 \alpha P_x e^{i(\omega t - \frac{\omega}{c} \hat{z} \cdot \vec{r})}$.

$$c_n(t) = -\frac{i}{\hbar} \int_{t_0}^{t} e^{i(\omega_n - \omega_m) t'} \left( e^{-i\frac{\omega}{c} \hat{z} \cdot \vec{r}} \right) \left( e^{i\omega t'} \right) \left| \langle n | 120 P_x e^{i(\omega_n - \omega_m) t} | m \rangle \right| e^{i\omega t} dt'$$

$$= -\frac{i}{\hbar} \left( \frac{\langle n 120 P_x e^{i(\omega_n - \omega_m) t} | m \rangle \langle m 120 P_x e^{i(\omega_n - \omega_m + \omega) t} | n \rangle}{i(\omega_n - \omega_m + \omega)} \right)$$
Squaring gives the transition probability

\[ P_{n\rightarrow m} = \frac{1}{k} 4a^2 \text{Knl} P_x e^{-i\frac{\hbar}{2}z\cdot\tau} |m\rangle |m\rangle^* 4 \sin^2 \left( \frac{\omega_n - \omega_m + \omega_l}{2} \right) \frac{\left( \omega_n - \omega_m + \omega_l \right)}{\left( \omega_n - \omega_m + \omega_l \right)^2} \]

we see immediately that the probability is enhanced when

\[ \omega = \omega_m - \omega_n \]

we also see that the size of this is controlled by the matrix element

\[ \text{Knl} P_x e^{-i\frac{\hbar}{2}z\cdot\tau} |m\rangle |m\rangle^* \]

example 2 spin systems

Consider a 2 state system in a time varying magnetic field

\[ H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} - \mu \mathbf{B} \cdot \hat{e}_x e^{i\omega t} \]

we choose \( \mathbf{B} \) in the \( x \) direction

\[ H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} - \mu \mathbf{B} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\omega t} \]
assume that the system is initially in the $E_1$ state. Find the probability of a transition to the $E_2$ state in time $t$

$$C_2 = -\frac{i}{\hbar} \int_0^t (1,0) \left( \begin{array}{cc} 0 & -\mu B \\ \mu B & 0 \end{array} \right) (1,0)^T e^{i \left( \frac{E_1 - E_2}{\hbar} + \omega \right) t} dt$$

$$= -\frac{i}{\hbar} (-\mu B) \frac{e^{i \left( \frac{E_1 - E_2}{\hbar} + \omega \right) t} - 1}{i \left( \frac{E_1 - E_2}{\hbar} + \omega \right)}$$

$$P_{1 \rightarrow 2}(t) = |C_2|^2 =$$

$$\left( \frac{\mu B}{\hbar} \right)^2 \frac{4 \sin^2 \left( \frac{E_1 - E_2 + \omega}{2 \hbar} t \right)}{(E_1 - E_2 + \omega)^2}$$

This is also maximal when $\hbar \omega$ is near the energy difference. Note this quantity is well behaved when $E_1 - E_2 + \omega = 0$

example 3 - scattering
what is the probability in a transition from an initial state corresponding to a projectile and target, and a final state of 2 scattered particles

\[ P = \left| \langle \Psi_f | \Psi_i \rangle \right|^2 \]

\[ K \Psi_f \Phi_e e^{-itH_a} | \Psi_i \rangle \]

this is independent of time - it can be evaluated at any time

\[ | \Psi_{f(\pm)} \rangle \]

looks like a free target and free projectile before the collision

\[ | \Psi_{f(\pm)} \rangle \]

looks like two free particles after the collision
\[ \langle \Phi_0(t) \rangle = \Psi_1(\mathbf{p}_1) \Psi_2(\mathbf{p}_2) e^{-i \left( \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} \right) t / \hbar} \]

(Structure of free particle wave packets)

We expect

\[ \| \Phi(t) - \Phi_0(t) \| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \]
\[ \| \Phi(t) - \Phi_0(t) \| \rightarrow 0 \quad \text{as} \quad t \rightarrow -\infty \]

We can write this as

\[ \| e^{-i H t / \hbar} \Phi(0) - e^{-i H t / \hbar} \Phi_0(0) \| = \]

by unitarity

\[ \| \Phi(0) - e^{i H t / \hbar} e^{-i H t / \hbar} \Phi_0(0) \| \]

mu,

\[ \langle \Phi_0(0) | \Phi(t) \rangle = \lim_{t \to \infty} e^{-i H(t-s) - i H s} \]

\[ \lim_{s \to -\infty} e^{-i H(t-s) - i H s} \]

We understand

\[ U(t,s) = e^{-i H(t-s) - i H s} \]

\[ U(t,s) = e^{-i H(t-s) - i H s} \]
This is the interaction picture time evolution operator

\[
\frac{dU(t)}{dt} = -\frac{i}{\hbar} e^{iH_0t/\hbar} (H - H_0) e^{-iH_0t/\hbar} U(t) \\
= -\frac{i}{\hbar} V(t) U(t)
\]

Integrating from \( s \to t \)

\[
U(t, s) = U(s) - \frac{i}{\hbar} \int_s^t V(t') U(t') dt'
\]

for a fast projectile the system feels the potential on the short time when the particles are close

\[
U(t, s) \approx I - \frac{i}{\hbar} \int_s^t V(t') I dt'
\]

\[
= I - \frac{i}{\hbar} \int_s^t e^{iH_0t'/\hbar} \left( e^{-iHt'/\hbar} V e^{iHt'/\hbar} \right) dt'
\]

Using this in the formula on the previous page with

\[ t \to \infty \quad s \to -\infty \]
\[ \langle \Phi_{f}(c) | \Phi_{i}(c) \rangle = \iint_{\mathbb{R}^{2}} \iint_{\mathbb{R}^{2}} \langle \Phi_{f}(c) | e^{-i \Delta t \hat{H} / \hbar} | \Phi_{i}(c) \rangle \, dp \, dp' \]

If we insert a complete set of plane wave states,

\[ \langle \Phi_{f}(c) | \Phi_{i}(c) \rangle = \iint_{\mathbb{R}^{2}} \langle \Phi_{f}(c) | \Phi_{i}(c) \rangle \, dp \, dp' \]

the integral over \( t \rightarrow \hbar 2 \pi \delta(E-E') \)

\[ \langle \Phi_{f}(c) | \Phi_{i}(c) \rangle = 2 \pi i \int \langle \Phi_{f}(c) | \Phi_{i}(c) \rangle \, dp \, dp' \]

\[ \times \ \delta \left( \frac{p}{2m} - \frac{p'}{2m} - \frac{p''}{2m} \right) \, dp \, dp' \]

This expresses the probability amplitude in terms of properties of the three particle wave packets that can be seen below a given collision.

The first term is the transition amplitude in the absence of interaction.
The second term only appears if $V \neq 0$. One can see that the transitions must conserve energy. If the potentials are invariant under translations then there will also be a momentum conserving function.

This shows the role played by the interaction representation in scattering — this will be developed in more detail later.

Identical particles:

One feature of quantum mechanics that measuring a complete set of observables on a particle tells everything we know about the particle.

A system where electron 1 is in state $A$ and electron 2 is in state $B$ is physically indistinguishable from one where...
electron 1 is in state B and electron 2 is in state A - this is because we can paint numbers on electrons:

\[ |AB\rangle = |A\rangle, |B\rangle, \]
\[ |BA\rangle = |B\rangle, |A\rangle. \]

These states describe the same physical system - but if 
\[ \langle A|A\rangle = \langle B|B\rangle = 1 \quad \langle A|B\rangle = 0 = \langle B|A\rangle \]

they are orthogonal.

\[ \langle AB|AB\rangle = 1 \]
\[ \langle BA|AB\rangle = 0 \]

If we let \[ |\psi\rangle = \alpha |AB\rangle + \beta |BA\rangle \]
this still describes a 2 electron state with 1 in state A and one in state B, but 
\[ \langle \psi|\psi\rangle = |\alpha|^2 \]

can take any value in the complex unit circle.
Examples

Integer spin = bosons (mesons)

\[ |4\rangle = c \left( |a\rangle_1 |b\rangle_2 + |b\rangle_1 |a\rangle_2 \right) \]

Normalization

\[ \langle 4|4 \rangle = 1 c^2 \left\{ \langle a|_1 |a\rangle_2 \langle b|_2 |b\rangle_1 + \right. \]
\[ \left. \langle b|_1 |b\rangle_2 \langle a|_2 |a\rangle_1 + \langle b|_1 |a\rangle_2 \langle a|_2 |b\rangle_1 \right\} \]
\[ c = \frac{1}{\sqrt{2}} \]

\[ |4\rangle = \frac{1}{\sqrt{2}} \left( |a\rangle_1 |b\rangle_2 + |b\rangle_1 |a\rangle_2 \right) \]

Half integer spin (electrons, protons, neutrons)

\[ |4\rangle = c \left( |a\rangle_1 |b\rangle_2 - |b\rangle_1 |a\rangle_2 \right) \]

As in the boson case it is easy to see \( c = \frac{1}{\sqrt{2}} \)

\[ |4\rangle = \frac{1}{\sqrt{2}} \left( |a\rangle_1 |b\rangle_2 - |b\rangle_1 |a\rangle_2 \right) \]
If you want quantum mechanical observables in different causally separated regions of space-time to commute, then

* states of identical particles with half integral spins must be antisymmetric under interchange of (obey the pauli principle) identical particles.
* states of identical particles with integer spins must be symmetric under interchange of identical particles.

There are no known particles that violate this symmetry.

(There are fictitious particles called ghosts that are tools used to understand symmetries that have the wrong statistics - these do not correspond to real particles.)
For 3 identical electrons there are 6 orthogonal vectors representing the same state.

The resolution:

Symbols: symmetric postulate.

States representing a system of $N$ identical particles are symmetric or antisymmetric with respect to interchange of identical particles.

In both cases there is a 1-1 correspondence between physical states and vectors (up to normalization) in Hilbert space.

How does one decide spin statistics theorem — comes from quantum field theory.
implications

1. shell structure of atoms
   
   + - l=0  s shell  \((2)\)
   
   + - l=1  m=1  o - 1  \((2x3)\)
   
   + - l=2  m=2  o - 1 - 2  \((2x5)\)

2. neutron stars
   
   will distinct states up to fermi energy

3. Bose condensation
   
   spin 0 particles can all be in lowest energy state. (could be pair of electronic coupled to spin 0)