2.2

Recall Thm 2 p21 from 2.1

Properties of Matrix Multiplication

Whenever both sides are calculable!

- $A(BC) = (AB)C$ Associative
- $A(B+C) = AB + AC$
- $(B+C)A = BA + CA$
- $(rA)\cdot B = r(AB)$
- If A is $m \times n$.

$$I_m A = A = A \cdot I_n$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 1 & 7 & 4 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 7 & 4 \end{bmatrix}.$$

CAUTION

False In general

- $AB \neq BA$
- $AB = AC \not\Rightarrow B = C$ or $A = 0$
- $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

means: Sometimes implication is false, and

& sometimes " is true.

INVERSE Given an $n \times n$ matrix A,

IF there is a matrix C ($n \times n$) s.t.

$A \cdot C = I_n = C \cdot A$, then

we call A invertible and $C = A^{-1}$

inverse of A.

- Not every square matrix has an inverse; Some do, Some don't.

Ex

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$$

2/19/14 (p3)

Ex

$$\underbrace{\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{0}.$$

A Can't have an inverse, since:

Suppose A has an inverse C !

Then See what will go wrong:

$$B = I \cdot B = \underbrace{(CA)}_{I} \cdot B = C\underbrace{(A \cdot B)}_{0} = C \cdot 0 = 0$$

Since B is not $0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the assumption is false.

There doesn't exist $A^{-1} (= C)$.

Defn A^{-1} exists $\Longleftrightarrow$ A invertible

$\Longleftrightarrow$ A non-singular

How do we find inverses

1x1 $\quad [2]^{-1} = [\tfrac{1}{2}]$.

$$[r][\tfrac{1}{r}] = [1] = I_1 \quad \text{if } r \neq 0.$$

2x2

Thm: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix s.t. $ad - bc \neq 0$, then A^{-1} exists and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Why: Let $\Delta = ad - bc$

Check whether $A^{-1} = \begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} = C$ is true :

Calculate $A \cdot C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{bmatrix}$

$$= \begin{bmatrix} \frac{ad}{\Delta} - \frac{bc}{\Delta} & \underbrace{\cancel{\frac{-ab}{\Delta} + \frac{ba}{\Delta}}}_{0} \\ \underbrace{\cancel{\frac{cd}{\Delta} - \frac{dc}{\Delta}}}_{0} & \frac{-bc + ad}{\Delta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Do also $C \cdot A = \dots = I_2$. So: $C = A^{-1}$.

Ex

Does $\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}^{-1}$ exist?

$$ad - bc = 7 \cdot 3 - (4)(-1) = 21 + 4 = 25 \neq 0$$

Inverse exist!

$$\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}^{-1} = \frac{1}{25}\begin{bmatrix} 3 & 1 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} \frac{3}{25} & \frac{1}{25} \\ -\frac{4}{25} & \frac{7}{25} \end{bmatrix}.$$

ALGORITHM TO FIND A^{-1} if exists, (or determine if A^{-1} $\underbrace{\text{DNE}}_{\text{does not exist}}$)

- Given A
- If A is not square: # rows $\neq$ # columns then A^{-1} Doesnot exist
- If A is square $n \times n$, then we do the following

$$n\Big\{ \underbrace{\left[\, A \;\middle|\; I_n \,\right]}_{2n} \xrightarrow[RR]{} \left[\, \mathrm{RREF}(A) \;\middle|\; C \,\right].$$

Case 1 If $RREF(A) = I_n$ then $C = A^{-1}$

Case 2 If $RREF(A) \neq I_n$ then A^{-1} DNE.

This means $RREF(A)$ has a row of 0's

You do not need to wait until the end, if a row of 0's appears on the left side:

$$\left[\begin{array}{c|c} & \\ \underbrace{0\,0\,0\,0\,0}_{\text{Then No } A^{-1}} & * * * * \end{array}\right]$$

Then No A^{-1}.

Ex # 32 of the book

a $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 5 \end{bmatrix}$ Does the inverse exist? If it does, what is it?

PTO →

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[4R_1+R_2 \to R_2]{} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ -2 & -6 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_1+R_3 \to R_3]{} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 3 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3 \to R_3]{} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -7 & -2 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 - R_3 \to R_1 \\ R_2 + R_3 \to R_2}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & -4 & -1 \\ 0 & 1 & 0 & 14 & 3 & 1 \\ 0 & 0 & 1 & 10 & 2 & 1 \end{array}\right]$$

So: $A^{-1} = \begin{bmatrix} -17 & -4 & -1 \\ 14 & 3 & 1 \\ 10 & 2 & 1 \end{bmatrix}$

Given
$$\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -17 & -4 & -1 \\ 14 & 3 & 1 \\ 10 & 2 & 1 \end{bmatrix}$$

Solve
$$\underbrace{\begin{aligned} x + 2y - z &= 0 \\ -4x - 7y + 3z &= 11 \\ -2x - 6y + 5z &= 4 \end{aligned}}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 11 \\ 4 \end{bmatrix}}_{\vec{b}}$$

$$A \cdot \vec{x} = \vec{b}$$

$$A^{-1}(A\vec{x}) = A^{-1} b \quad \text{Since } \underline{A^{-1} \text{ exists!}}$$

$$\underbrace{(A^{-1}A)}_{I} \cdot \vec{x} = A^{-1} b$$

$$\vec{x} = A^{-1} b$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} -17 & -4 & -1 \\ 14 & 3 & 1 \\ 10 & 2 & 1 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} 0 \\ 11 \\ 4 \end{bmatrix} = \begin{bmatrix} -48 \\ 37 \\ 26 \end{bmatrix}$$

$$\left.\begin{aligned} x &= -48 \\ y &= 37 \\ z &= 26 \end{aligned}\right\} \text{unique soln.}$$

<u>Thm</u> Let A be an $n \times n$ invertible matrix. For any given $\vec{b} \in \mathbb{R}^n$, the SLE

$\vec{A} \cdot \vec{x} = \vec{b}$ has the unique sol^n

$$\vec{x} = A^{-1}\vec{b}.$$

<u>Properties of</u> Inverses.

- $(A^{-1})^{-1} = A$ if A^{-1} exists.
- If A and B are invertible, then so is AB and $(AB)^{-1} = B^{-1} \cdot A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

<u>Exc: #8 p109-110</u>

P invertible & $A = PBP^{-1}$. Solve B.

Always multiply from the same side with the same matrix

$$A = PBP^{-1}$$

$$AP = (PBP^{-1})P$$

$$AP = (PB)\underbrace{P^{-1}P}_{I} = PB$$

$$AP = PB$$

$$P^{-1}(AP) = P^{-1}(PB) = (P^{-1}P)B = IB = B$$

$$\boxed{P^{-1}AP = B}$$