2/12/14 (p1)

1.7

**Defn** An <u>indexed</u> set $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\} \subseteq \mathbb{R}^n$ is called <u>linearly independent</u> if the equation

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = 0$ has the <u>only</u> solution: $c_1 = c_2 = c_3 = \cdots = c_p = 0$.

**Caution** $\{1, 2, 3\} = \{3, 2, 1\}$ As sets, <u>not indexed</u>

$\{1, 1\} = \{1\}$

<u>Indexed</u> means <u>ordered</u>.

**Defn** An indexed set $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_p\} \subseteq \mathbb{R}^n$ is called <u>linearly dependent</u>, if

[ there are "<u>not all zero numbers $c_1, \dots c_p$</u>" s.t.

Less ambigious:

Rewrite:

There are numbers $c_1, \dots c_p$ not all zero

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}.$$

**Ex** $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ is a linearly dependent set since:

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underbrace{2\begin{bmatrix}1\\1\end{bmatrix}+3\begin{bmatrix}0\\1\end{bmatrix}-1\begin{bmatrix}2\\5\end{bmatrix}}_{\text{linear dependence relation}}=\begin{bmatrix}0\\0\end{bmatrix}$$

$\left\{\begin{array}{l} c_1 = 2 \\ c_2 = 3 \\ c_3 = -1 \end{array}\right.$

Ex $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ linearly independent set since

$$c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Longrightarrow$$

$c_1 = c_2 = 0$ is the only soln

Ex 3

(a) Is $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}1\\-1\\4\end{bmatrix}\right\}$ linearly dependent or independent?

Solve: $$c_1\begin{bmatrix}1\\1\\2\end{bmatrix}+c_2\begin{bmatrix}1\\2\\1\end{bmatrix}+c_3\begin{bmatrix}1\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Find all solutions for $c_1, c_2, c_3$.

$$\begin{bmatrix}c_1+c_2+c_3\\c_1+2c_2-c_3\\2c_1+c_2+4c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$c_1 + c_2 + c_3 = 0$$
$$c_1 + 2c_2 - c_3 = 0$$
$$2c_1 + c_2 + 4c_3 = 0$$

skipping steps...

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is a free variable $\Longrightarrow$ there is a non-trivial sol'n.

($\Rightarrow$ linearly dependent)

(b) What about finding a linear relation?

$$\begin{aligned} c_1 + 3c_3 &= 0 \\ c_2 - 2c_3 &= 0 \end{aligned} \longrightarrow \begin{aligned} c_1 &= -3c_3 \\ c_2 &= 2c_3 \\ c_3 &= c_3 \text{ free} \end{aligned}$$

Take:

$$\left.\begin{aligned} c_1 &= -3 \\ c_2 &= 2 \\ c_3 &= 1 \end{aligned}\right\} \text{non-trivial sol'n.}$$

$$\underbrace{-3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{check!}}$$

**THM** Columns of a matrix $A$ form a linearly independent set

$\iff$ $A \cdot \vec{x} = \vec{0}$ has only the trivial sol$^n$

$\iff$ RREF of $A$ has a leading 1 in every column.

(without augmentation)

**Ex #14 p 61**

$$\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$$

determine $h$ for which these vectors are dependent (or independent)

Sol$^n$

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 1 \\ -4 & 6 & h \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 5 \\ 0 & -6 & h+8 \end{bmatrix}$$

skipping steps

$$\to \begin{bmatrix} 1 & 0 & 17 \\ 0 & 1 & 5 \\ 0 & 0 & h+38 \end{bmatrix}$$

**Case 1** $h = -38$

$$\begin{bmatrix} 1 & 0 & 17 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \text{Lin. Dependent}$$

free variable

**Case 2** $h \neq -38$

$$\cdots \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \text{Lin. Independent.}$$

## Shortcuts:

- Any set containing $\vec{0}$ is linearly dependent.

Ex. $\{\vec{0}, \vec{v}_2\}$ $\quad 1\cdot\vec{0} + 0\cdot\vec{v}_2 = 0$

$\vec{v}_2 \neq 0$

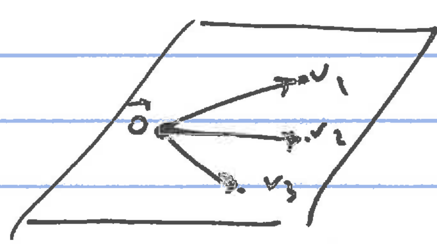
$\left.\begin{matrix} c_1 = 1 \\ c_2 = 0 \end{matrix}\right\}$ not all zero.

- Any set repeating a vector is linearly dependent.

Ex. $\{\vec{v}_1, \vec{v}\}$ $\quad c_1 = 1$, $c_2 = -1$ $\quad 1\cdot v + (-1)v = 0$.

- $\{v_1, v_2\}$ linearly independent $\Longleftrightarrow$
  - neither $v_i = 0$ and
  - $v_1$ and $v_2$ are not scalar multiples of each other

- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent $\Longleftrightarrow$ The tips of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{0}$ are coplanar



- In $\mathbb{R}^n$, any collection of $n+1$ (or more) vectors form a <u>dependent set</u>

- If $\{\vec{v}_1, \vec{v}_2 .. \vec{v}_p\}$ is <u>linearly independent</u>, and if we <u>remove</u> any of these vectors, then the new set we get is automatically <u>linearly independent</u> (unless $\emptyset$).

- If $\{\vec{v}_1 .. \vec{v}_p\}$ is linearly dependent and if we add a new vector to this set, then the new set is also lin. dependent

Not algebraic adding; we include a new element

PTO

<u>Review</u>: Let $\{\vec{v}_1 \ldots \vec{v}_p\}$ be a set of vectors in $\mathbb{R}^n$.

Let $A = \begin{bmatrix} \vec{v}_1 \ldots \vec{v}_p \end{bmatrix}$ be the $n \times p$ matrix whose columns are $\vec{v}_1 - \vec{v}_p$. Row reduce $A$ to get RREF(A).

1.5 $\{\vec{v}_1 \ldots \vec{v}_p\}$ span $\mathbb{R}^n$ $\iff$ Every row of RREF(A) has a leading 1

1.7 $\{\vec{v}_1 \ldots \vec{v}_p\}$ is linearly independent $\iff$ Every column of RREF(A) has a leading 1

$\{\vec{v}_1 \ldots \vec{v}_p\}$ is a basis of $\mathbb{R}^n$ $\iff$ Every column and every row of RREF(A) has a leading 1

<u>Defn</u> Basis of $\mathbb{R}^n$.

- Lin independent and
- Spans $\mathbb{R}^n$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & \cdots & & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ 0 & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix} = I_n$$