1.4 Continue

Defn Given vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$ in $\mathbb{R}^n$

$$\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p)$$

$$= \{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \mid \underbrace{c_1, \ldots, c_p}_{\text{varying}} \in \mathbb{R} \}$$

fixed vectors

Ex Is every $\vec{b} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ in the span$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right)$? If not, what are conditions on A, B, C which make b in this span?

Soln

Suppose A, B, C are given.
Do there exist x_1, x_2 s.t.

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} ?$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = A$$
$$x_1 + 2x_2 = B$$
$$0 = C$$

$$\left[\begin{array}{cc|c} 1 & 1 & A \\ 1 & 2 & B \\ 0 & 0 & C \end{array}\right] \xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{cc|c} 1 & 1 & A \\ 0 & 1 & B-A \\ 0 & 0 & C \end{array}\right]$$

$$\xrightarrow[R_1 - R_2 \to R_1]{} \left[\begin{array}{cc|c} 1 & 0 & A-(B-A) \\ 0 & 1 & B-A \\ 0 & 0 & C \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2A-B \\ 0 & 1 & B-A \\ 0 & 0 & C \end{array}\right]$$

If $C \neq 0$, this system is inconsistent

and $\begin{bmatrix} A \\ B \\ C \end{bmatrix} \notin \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$

If $C = 0$, this sysm is consistent

and $\begin{bmatrix} A \\ B \\ 0 \end{bmatrix} \in \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$

How:

$$\begin{bmatrix} A \\ B \\ 0 \end{bmatrix} = \underbrace{(2A-B)}_{x_1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \underbrace{B-A}_{x_2} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2A - \cancel{B} + \cancel{B} - A \\ 2\cancel{A} - B + 2B - \cancel{2A} \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} A \\ B \\ 0 \end{bmatrix}.$$

(b) Is every $\vec{b} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ in the span of

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \; ? \qquad \underline{\underline{\text{YES}}}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Thm</u> Let A be an $m \times n$ matrix

For <u>all</u> $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.	$\iff$	For <u>all</u> $\vec{b} \in \mathbb{R}^m$, b is a linear combination of the columns of A
$\Updownarrow$	$\Leftrightarrow$	$\Updownarrow$
RREF of A has a leading 1 (pivot) in every row.	$\iff$	The columns of A span $\mathbb{R}^m$

<u>PTO</u> for an example.

Ex

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdots} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{RREF of } A \text{ has a leading 1 in every row}}$$

Not all steps are shown.

Conclusions

- $\mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \mathbb{R}^3$

- Every $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ can be written as equal to

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

for some c_1, c_2, c_3 (depending on A, B, C)

- SLE $\left.\begin{aligned} x_1 + 2x_3 &= A \\ 2x_1 + x_2 + x_3 &= B \\ x_1 + x_2 \quad &= C \end{aligned}\right\}$ is consistent for every choice of A, B, C

1.5 A system of Linear equations is called <u>homogenous</u> if it is of the form:

$$A \cdot \vec{x} = \vec{0}$$

Ex (a) $\left.\begin{aligned} x + 2y &= 0 \\ x - y &= 0 \end{aligned}\right)$ Homogenous

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A \cdot \vec{x} = \vec{0}$$

(b) $\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_1 + 5x_2 &= 1 \end{aligned}\right\}$ non-homogenous since there is a non-zero entry.

<u>Obs</u> Every homogenous system is consistent.

<u>Defn</u> $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is called the trivial solⁿ. Anything else is called non-trivial

<u>Prop</u> Every homogenous equation $A \cdot \vec{x} = \vec{0}$ has the trivial soln.

Also True:

If $A\vec{x} = \vec{b}$ has $\vec{x} = 0$ as a solution then $\vec{b} = \vec{0}$

Original Statement: If it rains then I go to movies

Equiv. Not equivalent

Converse: If I go to movies then it rains

Contrapositive: If I don't go to movies then it doesn't rain

Go back to LA (Linear Algebra)

Ex 8 p47 $A \cdot x = 0$

$$A = \begin{bmatrix} 1 & -3 & -8 & 5 \\ 0 & 1 & 2 & -4 \end{bmatrix},$$ want all Sol^{ns}. in Vector parametric form

$$\longrightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & -3 & -8 & 5 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{array}\right] \end{array} \xrightarrow{3R_2 + R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -7 & 0 \\ 0 & 1 & 2 & -4 & 0 \end{array}\right]$$

free free.

$$\begin{aligned} x_1 - 2x_3 - 7x_4 &= 0 \\ x_2 + 2x_3 - 4x_4 &= 0 \end{aligned} \longrightarrow \begin{aligned} x_1 &= 2x_3 + 7x_4 \\ x_2 &= -2x_3 + 4x_4 \\ x_3 &= x_3 \quad \text{free} \\ x_4 &= x_4 \quad \text{free} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + 7x_4 \\ -2x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix}}_{\text{Called Vector Parametric Solut}}$$

The collection of such solutions is

$$\text{Soln} = \text{span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right)$$

Final Remark:

(*) • The solution set of a homogenous SLE, $A \cdot \vec{x} = 0$, is always a span of some vectors. ($\vec{x} = 0$ is always a solution of a homogenous eq^n $A \cdot \vec{x} = \vec{0}$.)

• Some homogenous eq. $Ax = 0$ may have a unique soln, which must be $\vec{x} = \vec{0}$. Then: $\{\vec{0}\} = \text{Span}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

CAUTION For NON-HOMOGENOUS SYSTEMS (*), above will be modified, by a shift. (Next time)