Continue

1.3 Linear Combinations

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

#'s vectors.

Ex Is $\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$?

Does there exist any solution of

vector eq^n

$$\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \; ?$$

$$\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 2x_1 \end{bmatrix}$$

SLE

$$5 = x_1 - x_2$$
$$-1 = x_1 + x_2$$
$$4 = 2x_1$$

$$\downarrow$$

matrix

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 1 & 1 & -1 \\ 2 & 0 & 4 \end{array}\right]$$

PTO for Sol^n.

Solve it next:

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 1 & 1 & -1 \\ 2 & 0 & 4 \end{array}\right] \xrightarrow[R_2 - R_1 \to R_2]{} \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 2 & -6 \\ 2 & 0 & 4 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_2,\ \frac{1}{2}R_3]{} \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & -1 & 5 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{array}\right] \xrightarrow[R_3 + R_2 \to R_3]{} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

Consistent

$x_1 = 2$

$x_2 = -3$

$$\underbrace{\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + (-3)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{Linear combination we want}}$$

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$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

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$$(*) \quad b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$$

↓ vector eqn

↓ matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \xrightarrow[2R_1 + R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right]$$

$$\xrightarrow[-2R_2 + R_3 \to R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free

$$x_1 = 2 - 5x_3$$
$$x_2 = 3 - 4x_3$$
$$x_3 = \text{free}$$

There are ∞-ly many ways to solve (*)

$x_1 = 2,\ x_2 = 3,\ x_3 = 0$ is a sol'n

$x_1 = -3,\ x_2 = -1,\ x_3 = 1$

1.3 END

1.4

Ex Is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\-3\\0\end{bmatrix}$, $\begin{bmatrix}2\\-1\\4\end{bmatrix}$?

vector eqn.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = \underbrace{x_1\begin{bmatrix}1\\-3\\0\end{bmatrix} + x_2\begin{bmatrix}2\\-1\\4\end{bmatrix}}_{\text{a compact way of writing}} = \begin{bmatrix}x_1 + 2x_2\\-3x_1 - x_2\\0 + 4x_2\end{bmatrix}$$

matrix Eq.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & 2\\-3 & -1\\0 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\vec{b} = A \cdot \vec{x}$$

SLE

$$\begin{aligned} x_1 + 2x_3 &= 1\\ -3x_1 - x_2 &= 2\\ 4x_2 &= 3 \end{aligned}$$

$\boxed{\text{Different ways of writing the same SLE}}$

$$\left[\begin{array}{cc|c}1 & 2 & 1\\-3 & -1 & 2\\0 & 4 & 3\end{array}\right]$$

Augmented Coeff matrix

Defn Let A be an $m \times n$ matrix
(An array of mn entries placed in m rows and n columns inside [ ])

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ then we define

$$A \cdot \vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \vec{a}_1 \end{bmatrix} + x_2 \begin{bmatrix} \vec{a}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \vec{a}_n \end{bmatrix}$$

column 1 of A column n of A

Exs

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+6-4 \\ 3+4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

Ex

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 7 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ -x_1 + 4x_2 + 7x_3 \\ 6x_1 + 4x_2 + 3x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & -4 \\ 7 & 6 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 43 \end{bmatrix}$$

THM If A is an $m \times n$ matrix with columns $\underbrace{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n}_{m\text{-vectors}}$, $b \in \mathbb{R}^m$, then

the matrix eq^n $A \cdot \vec{x} = \vec{b}$
$\left(\text{where } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)$ has the same

solution set of the equation

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b}$$

which in turn has the same solution of the SLE whose augmented matrix is

$$\left[\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_n \;\middle|\; \vec{b} \right].$$

Defn The span of the vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}$

is the set of all linear combinations of $\{\vec{v}_1, \ldots \vec{v}_p\}$

that is $= \{ c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_p \vec{v}_p \mid \underbrace{c_1, c_2, \ldots c_p}_{\text{vary}} \in \mathbb{R} \}$

$v_1 \ldots v_p$ are fixed

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As promised in class:

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Q. Solve $A\vec{x} = \vec{b}$ where $\begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix} = A$ and $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = b$.

Ans: Suffices to solve

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{array}\right] \xrightarrow[R_2+3R_1]{} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 5 & 2 & 3 & -3 \end{array}\right]$$

$$\xrightarrow[R_3-5R_1]{} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{array}\right] \xrightarrow[-\frac{1}{8}R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1-R_2\to R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_2\leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 5 \end{array}\right]$$

$$\xrightarrow[R_3-2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[\substack{R_2+R_3 \\ \downarrow \\ R_2}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Hence $x = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$ solves $A \cdot x = b$.