5/7/14

p1

7.2

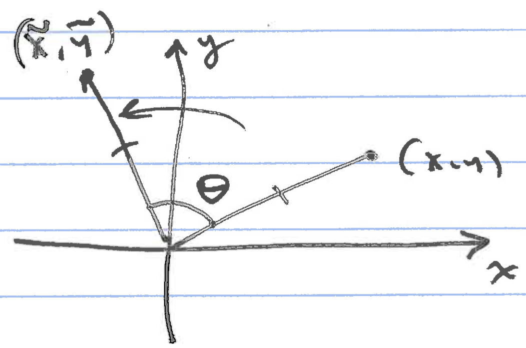
SPECTRAL THM of 7.1

$\Longrightarrow$ PRINCIPAL AXIS THEOREM

Let A be an $n \times n$ symmetric matrix. Then there is an orthogonal change of variables $\vec{x} = P\vec{y}$ that transforms the quadratic form $x^T A x$ into $y^T D y$ (D is diagonal) with no cross-product terms.

Ex

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$



$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix

Rotation Matrix

Exc #10 p.407

$Q(x_1, x_2) = 9x_1^2 - 8x_1x_2 + 3x_2^2$

$x = Py \longrightarrow$ q. form with no cross-terms

$\longrightarrow$ i.e. diagonalize (orthogonally)

$$A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$$

$$[x_1 \ x_2] \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 - 8x_1x_2 + 3x_2^2$$

Orthogonally diagonalize A.

$$\begin{vmatrix} 9-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} = (9-\lambda)(3-\lambda) - 16$$

$$= 27 - 12\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 12\lambda + 11$$

$\underset{\substack{\text{trace} \\ \| \\ \lambda_1+\lambda_2}}{12}$; $11 \to \det = \lambda_1\lambda_2$

Det $A = \begin{vmatrix} 9 & -4 \\ -4 & 3 \end{vmatrix} = 11 = \lambda_1\lambda_2$

Trace $A = 9 + 3 = 12 = \lambda_1 + \lambda_2$

$a_{11} + a_{22}$

$$= (\lambda - 11)(\lambda - 1)$$

$\lambda = 1$ $\quad A - I = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}.$

$x_1 - \frac{1}{2}x_2 = 0$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ basis for eigenspace $\lambda = 1$

$\lambda = 11$

$A - 11I = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ basis for eigenspace $\lambda = 11$.

$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ $\qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$

$Py = x$ $\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$

Rotation matrix

$\cos \theta_0 = 1/\sqrt{5}$

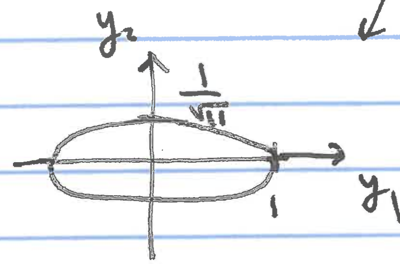
$\sin \theta_0 = 2/\sqrt{5}.$

$9x_1^2 - 8x_1x_2 + 3x_2^2 = y_1^2 + 11y_2^2$

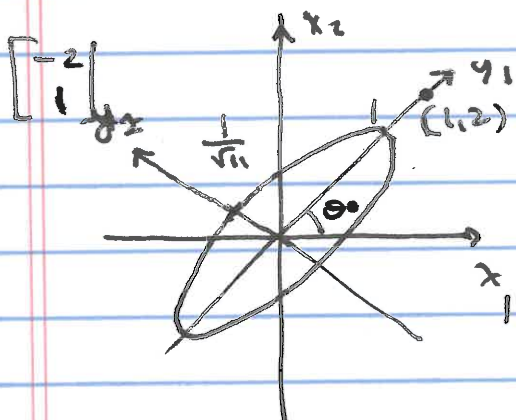
We also do more:

Graph:

(b) $9x_1^2 - 8x_1x_2 + 3x_2^2 = 1. = y_1^2 + 11y_2^2$

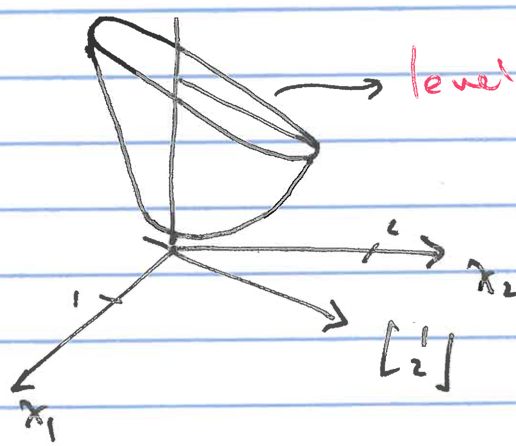


$\begin{bmatrix}1\\2\end{bmatrix}$ direction in $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ space



(c) $x_3 = 9x_1^2 - 8x_1x_2 + 3x_2^2$ plot 3-D graph

level set for $x_3 = 1 = 9x_1^2 - 8x_1x_2 + 3x_2^2$



CLASSIFYING QUADRATIC FORMS

A quadratic form is positive-definite if

$$Q(x) > 0 \quad \text{if } x \neq 0$$

A quadratic form is negative definite if

$$Q(x) < 0 \quad \text{for all } x \neq 0$$

A quadratic form is called indefinite if

$$Q(x) < 0 < Q(x')$$

for some x, x'.

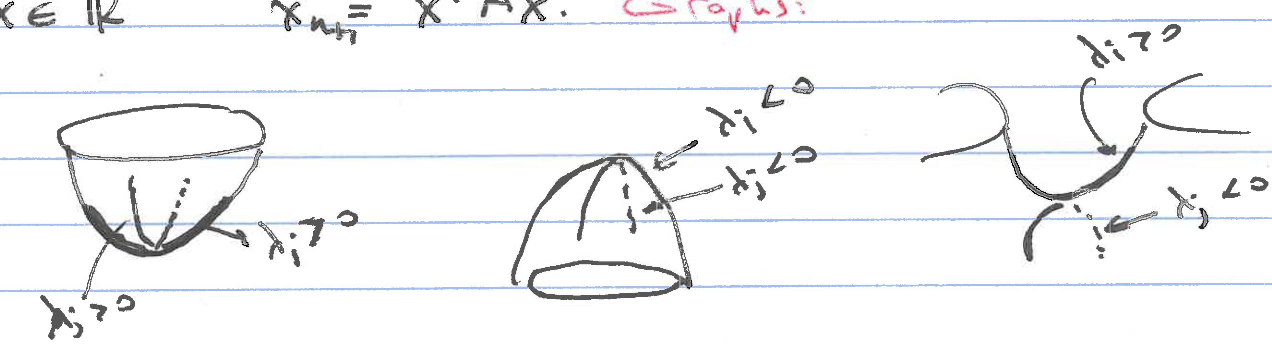
<u>Thm</u> Given $Q = x^T A x$, A symmetric

Q is $+$ definite $\iff$ All eigenvalues of A are $+$

Q is $-$ " $\iff$ " " " " " $-$

Q is indefinite $\iff$ A has at least one $(+)$ eigenvalue & at least one $(-)$ eigenvalue

$x \in \mathbb{R}^n$ $\quad x_{n+1} = x^T A x$. Graphs:



$+$ definite $\qquad\qquad$ $-$ definite $\qquad\qquad$ indefinite

Ex

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = A$$

when + definite?

− definite?

indefinite?

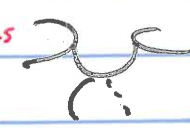
$$\lambda_1 \lambda_2 = \det = ac - b^2$$

$$\lambda_1 + \lambda_2 = \text{trace} = a + c.$$

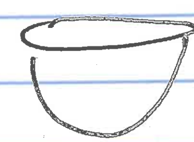
$Q(x) = x^T A x$ is indefinite if $\overbrace{ac - b^2}^{\lambda_1 \lambda_2} < 0.$

$z = ax^2 + 2bxy + cy^2$

graphs



$$\left.\begin{aligned} ac - b^2 &> 0 \\ a + c &> 0 \end{aligned}\right\} + \text{ definite}$$



$$\left.\begin{aligned} ac - b^2 &> 0 \\ a + c &< 0 \end{aligned}\right\} - \text{ definite}$$